Minimal-Length Quantum Mechanics: why, how, what?

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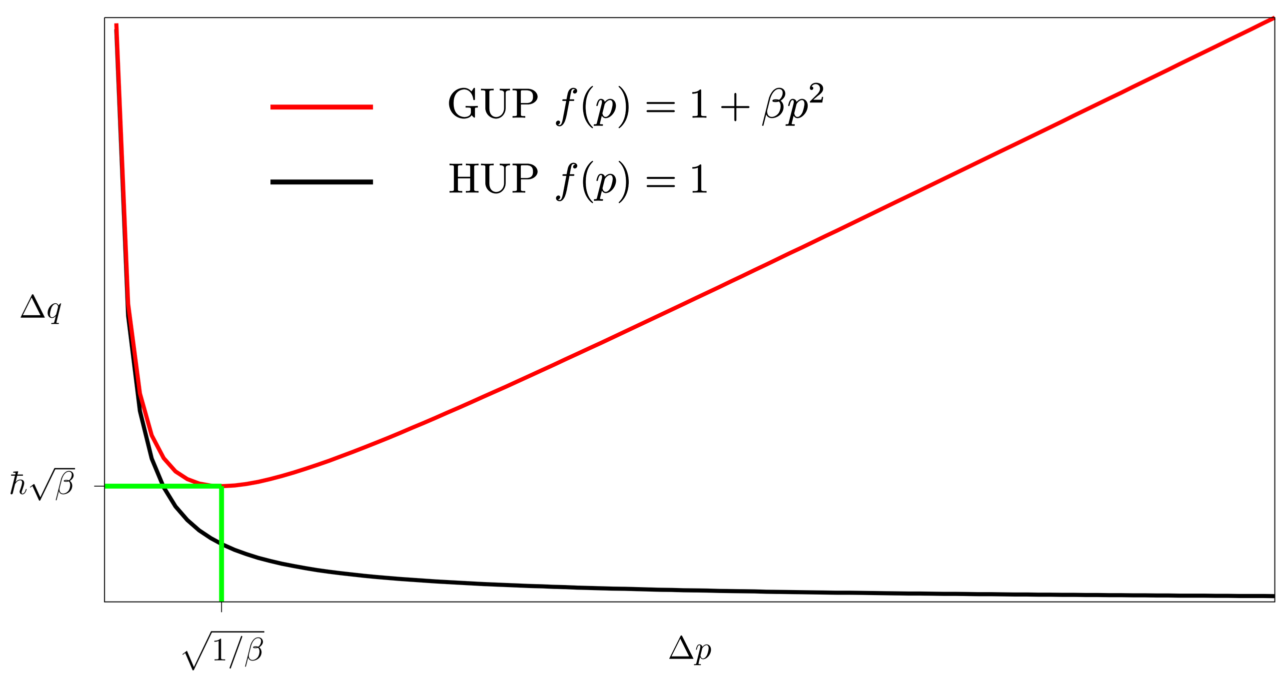
Why?

Several theories of quantum gravity introduce the possibility of some sort of **minimal length at the Planck scale**. Such a feature is introduced within **phenomenological models** of quantum gravity as a modification of the ordinary quantum theory entailing a minimal uncertainty in position.

How?

The **Generalized Uncertainty Principle** (GUP) represents a typical approach to such models in which the ordinary position-momentum commutation relation is modified to accommodate a minimal unceraity in position.

$$[q,p] = i\hbar f(p) \qquad \Rightarrow \qquad \Delta q \Delta p \geq \frac{\hbar}{2}|\langle f(p)\rangle|$$



These modifications present a rich structure and imply several effects [1-3]. However, *not all such models imply a minimal length*. A useful tool to highlight the presence of a minimal length is the **wavenumber** operator, defined as the conjugate momentum to position [2-6].

Bounded wavenumber

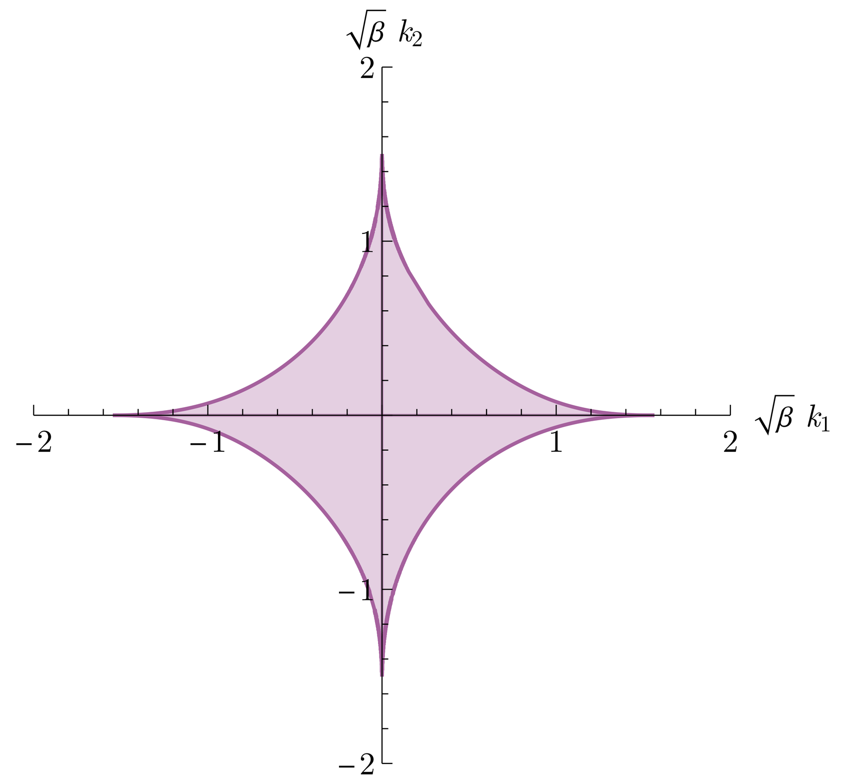
The wavenumber operator k incapsulates all the relevant information regarding the minimal length (if any) and is defined according to

$$[q,k] = i, \qquad \frac{\mathrm{d}p}{\mathrm{d}k} = \hbar f(p)$$

While the momentum p is in general unbounded, the existence of a minimal length is signaled by a **bounded wavenumber** k [5]. For example, for the typical model first introduced in [1], we find

$$f(p) = 1 + \beta p^2$$

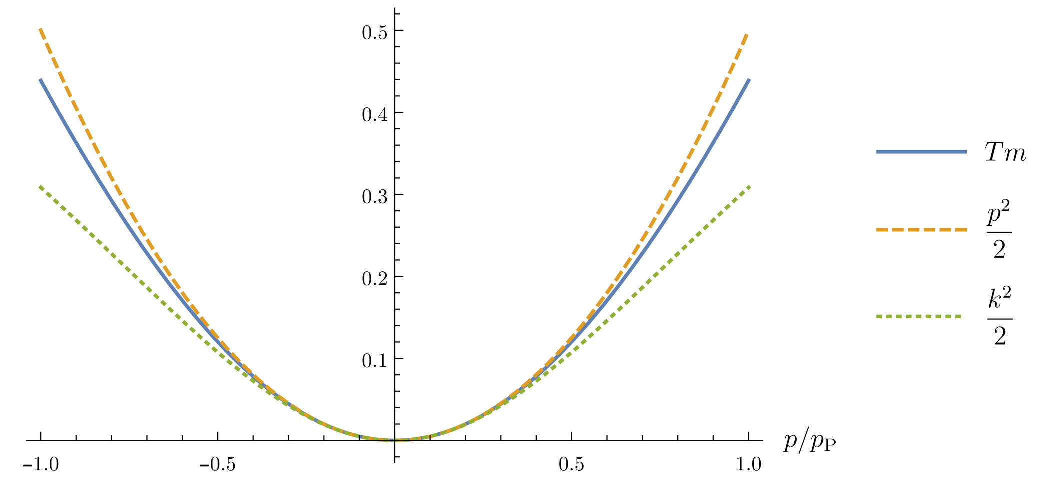
$$|k| < \frac{\pi}{2\sqrt{\beta}\hbar} \qquad \Delta q > \sqrt{\beta}\hbar$$



Given its features, the wavenumber k represents the ideal tool to describe models characterized by a minimal uncertainty in position. Moreover, it suggests deeper and alternative perspectives [5]. For example, when Galilean relativity is considered, the wavenumber is of fundamental importance, acquiring key physical roles.

Galilean Relativity

In models of quantum mechanics with a minimal length emplying Galilean relativity, the free Hamiltonian is of the form $H{=}k^2/2m$ [4]. Furthermore, defining the energy as the work done by a force, **Hamiltonian and energy are not identical**, not even for a free system.



Conclusions

Phenomenological descriptions of a minimal measurable length can be cast in terms of a bounded wavenumber. The dynamical and kinematical aspects of such models are highly non-trivial and ofter overlooked [6].

References

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[5] P. Bosso, L. Petruzziello, F. Wagner, "Minimal length: A cut-off in disguise?", PRD **107**, 12, 126009 (2023).

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