# Carrollian limits and ModMax

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## Causal structures and limits

Carrollian and Galilean geometries arise as possible kinematical structures which allows boosts, rotations translations and obey criteria of causality. These causal structures can be visualized by varying the speed of light in the Lorentzian case.

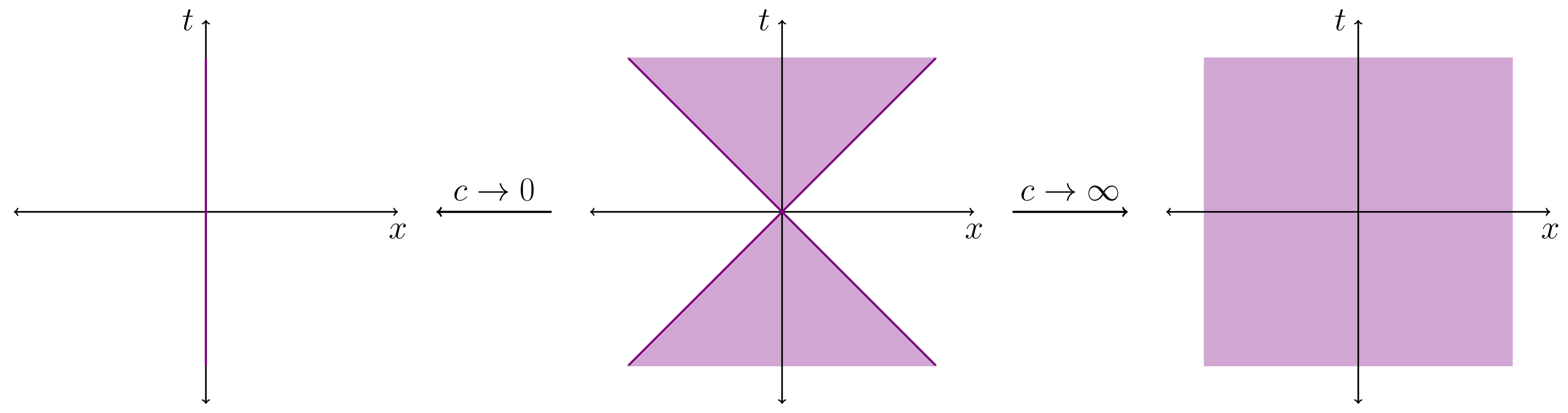


Figure 1. Causally connected regions in Carrollian, Lorentzian and Galilean manifolds.

The Galilean limit is also known as the non-relativistic limit, giving the wrong impression that there's no group structure. Less popular until now is the Carrollian limit, useful in the description of null surfaces and the event horizon of black holes.

### Carrollian structure

A Carroll manifold is a quadruple $(C, g, \xi, \nabla)$, where $C$ is a $(d+1)$-smooth manifold, $g$ is a rank 2 degenerate metric tensor field, $\xi$ is a nowhere vanishing complete vector field which spans $\ker(g)$, $\nabla$ is a symmetric affine connection that parallel transports both $g$ and $\xi$. A Carroll group is the group of automorphisms of a Carroll structure.

The standard flat Carroll structure is given by the choice

$$C^{d+1} = \mathbb{R}\times\mathbb{R}^d \qquad g = \delta_{AB}dx^A\otimes dx^B \qquad \xi = \frac{\partial}{\partial s} \qquad \Gamma^i_{jk} = 0$$

Given our choices, representatives of $Carr(d+1)$ and $C^{d+1}$ are of the form

$$a_C = \begin{pmatrix} R & 0 & \mathbf{c} \\ -\mathbf{b}^T R & 1 & f \\ 0 & 0 & 1\end{pmatrix} \in Carr(d+1) \qquad x = \begin{pmatrix} s \\ \mathbf{x}\end{pmatrix} \in C^{d+1}$$

Which allows us to define the action

$$\triangleright_C : Carr(d+1)\times C^{d+1} \to C^{d+1}$$
$$(a,x) \to a\triangleright x := \begin{bmatrix} R\mathbf{x}+\mathbf{c} \\ s - \mathbf{b}^T R\mathbf{x} + f\end{bmatrix}$$

### Galilean structure

A Newton-Cartan manifold is a quadruple $(N, \gamma, \theta, \nabla_N)$, where $N$ is a $(d+1)$-smooth manifold, $\gamma$ is a twice-symmetric, contravariant, positive tensor field, $\theta$ is a closed nowhere-vanishing 1-form that generates $\gamma$'s kernel, $\nabla$ is a symmetric affine connection that parallel transports both $\gamma$ and $\theta$.

The standard flat Galilean structure is given by the choice

$$N^{d+1} = \mathbb{R}\times\mathbb{R}^d \qquad \gamma = \delta^{AB}\frac{\partial}{\partial x^A}\otimes\frac{\partial}{\partial x^B} \qquad \theta = dt \qquad \Gamma^i_{jk} = 0$$

Given our choices, representatives of $Gal(d+1)$ and $N^{d+1}$ are of the form

$$a_G = \begin{pmatrix} R & \mathbf{v} & \mathbf{c} \\ 0 & 1 & e \\ 0 & 0 & 1\end{pmatrix} \in Carr(d+1) \qquad x = \begin{pmatrix} s \\ \mathbf{x}\end{pmatrix} \in C^{d+1}$$

Which allows us to define the action

$$\triangleright_G : Carr(d+1)\times C^{d+1} \to C^{d+1}$$
$$(a,x) \to a\triangleright x := \begin{bmatrix} R\mathbf{x}+\mathbf{v}t+\mathbf{c} \\ t+e\end{bmatrix}$$

## ModMax

### The theory

Modified-Maxwell (ModMax) theory the unique nonlinear $U(1)$ gauge theory for electrodynamics that possesses the same symmetries as Maxwell, i.e. Lorentz invariance, conformal symmetry and duality invariance in the absence of sources. ModMax Lagrangian is constructed from two different Lorentz-invariant quantities obtained from the Faraday tensor and its Hodge dual

$$S = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \qquad P = -\frac{1}{4}F^{\mu\nu}\bar{F}_{\mu\nu}$$

With these a one parameter family of Lagrangians is born, namely $\mathcal{L}_\gamma = \cosh\gamma\, S + \sinh\gamma\sqrt{S^2+P^2}$. With $\gamma\in\mathbb{R}^+$. Such that $\lim_{\gamma\to 0}\mathcal{L}_\gamma = S$.

### The equations of motion

The equations of motion coming from the ModMax Lagrangian are

$$\begin{aligned} 0 &= \partial_\mu\left[\frac{\partial\mathcal{L}_\gamma}{\partial F_{\mu\nu}}\right] \\ &= \cosh\gamma\,\partial_\mu F^{\mu\nu} + \sinh\gamma\left[\frac{S\partial_\mu F^{\mu\nu} + P\partial_\mu\bar{F}^{\mu\nu}}{\sqrt{S^2+P^2}} - \frac{SP\left(\partial_\mu P\, F^{\mu\nu} + \partial_\mu S\,\bar{F}^{\mu\nu}\right)}{\left(S^2+P^2\right)^{3/2}}\right] \end{aligned} \tag{1}$$

This is used to explicitly construct the limits of the equations of motion.

### Carrollian limits of ModMax equations

From vacuum ModMax electrodynamics it is possible to construct two nonequivalent limits which are Carroll-covariant, namely the so called electric and magnetic limits. For the electric limit we re-scale $\boldsymbol{E_e} = \boldsymbol{E}/c$, $s = (cC)\,t$ and $\boldsymbol{B_e} = (cC)\,\boldsymbol{B}$ in (1), then take the limit $C\to\infty$.

$$\nabla\times\boldsymbol{E_e} + \frac{\partial\boldsymbol{B_e}}{\partial s} = 0 \qquad \nabla\cdot\boldsymbol{B_e} = 0$$
$$(\cosh\gamma+\sinh\gamma)\frac{\partial\boldsymbol{E_e}}{\partial s} = 0 \qquad (\cosh\gamma+\sinh\gamma)\,\nabla\cdot\boldsymbol{E_e} = 0$$

This electric limit is equivalent to its Maxwell counterpart and proves to be Carroll invariant, with transformations under boosts given by:

$$\boldsymbol{E_e}(\boldsymbol{x},s)\to\boldsymbol{E'_e}(\boldsymbol{x},s) = \boldsymbol{E_e}(\boldsymbol{x}, s-\boldsymbol{b}\cdot\boldsymbol{x})$$
$$\boldsymbol{B_e}(\boldsymbol{x},s)\to\boldsymbol{B'_e}(\boldsymbol{x},s) = \boldsymbol{B_e}(\boldsymbol{x}, s-\boldsymbol{b}\cdot\boldsymbol{x}) - \boldsymbol{b}\times\boldsymbol{E_e}(\boldsymbol{x}, s-\boldsymbol{b}\cdot\boldsymbol{x})$$

The magnetic limit is obtained from re-scaling $\boldsymbol{E_m} = C\boldsymbol{E}/c$, $\boldsymbol{B_m} = c\boldsymbol{B}$ and $s = (cC)\,t$ in (1) and then taking the limit $C\to\infty$.

$$\frac{\partial\boldsymbol{B_m}}{\partial s} = 0 \qquad \nabla\cdot\boldsymbol{B_m} = 0$$
$$e^{-\gamma}\left(\nabla\times\boldsymbol{B_m} - \frac{\partial\boldsymbol{E_m}}{\partial s}\right) = 0 \qquad e^{-\gamma}\,\nabla\cdot\boldsymbol{E_m} + 4\sinh\gamma\,\frac{\boldsymbol{B_m}\cdot\boldsymbol{E_m}}{B_m^4}\left(\boldsymbol{B_m}\cdot\nabla\right)B_m^2 = 0$$

In this case the only nonlinear terms that survives modifies Gauss equation. This equations are invariant under Carrollian boosts with field transformations given by:

$$\boldsymbol{B_m}(\boldsymbol{x},s)\to\boldsymbol{B'_m}(\boldsymbol{x},s) = \boldsymbol{B_m}(\boldsymbol{x}, s-\boldsymbol{b}\cdot\boldsymbol{x})$$
$$\boldsymbol{E_m}(\boldsymbol{x},s)\to\boldsymbol{E'_m}(\boldsymbol{x},s) = \boldsymbol{E_m}(\boldsymbol{x}, s-\boldsymbol{b}\cdot\boldsymbol{x}) + \boldsymbol{b}\times\boldsymbol{B_m}(\boldsymbol{x}, s-\boldsymbol{b}\cdot\boldsymbol{x})$$

### Galilean limits of ModMax's equations

The Galilean electric limit of ModMax theory is obtained from (1) by re-scaling both electric and magnetic fields $\boldsymbol{E_e} = \boldsymbol{E}/c$ and $\boldsymbol{B_e} = c\boldsymbol{B}$ then taking the limit $c\to\infty$. By so doing we arrive at

$$\nabla\times\boldsymbol{E_e} = 0 \qquad \nabla\cdot\boldsymbol{B_e} = 0$$
$$e^{\gamma}\left(\nabla\times\boldsymbol{B_e} - \frac{\partial\boldsymbol{E_e}}{\partial t}\right) - 4\sinh\gamma\,\frac{\boldsymbol{E}\cdot\boldsymbol{B}}{E^4}\left(\boldsymbol{E}\times\nabla\right)E^2 = 0 \qquad e^{\gamma}\,\nabla\cdot\boldsymbol{E_e} = 0$$

Which is invariant under Galilean boost with field transformations given by

$$\boldsymbol{E_e}(\boldsymbol{x},t)\to\boldsymbol{E'_e}(\boldsymbol{x},t) = \boldsymbol{E_e}(\boldsymbol{x}-\mathbf{b}t, t)$$
$$\boldsymbol{B_e}(\boldsymbol{x},t)\to\boldsymbol{B'_e}(\boldsymbol{x},t) = \boldsymbol{B_e}(\boldsymbol{x}-\mathbf{b}t, t) + \boldsymbol{b}\times\boldsymbol{E_e}(\boldsymbol{x}-\mathbf{b}t, t)$$

For the Galilean magnetic limit we take the limit $c\to\infty$ directly from (1)

$$\nabla\times\boldsymbol{E_m} + \frac{\partial\boldsymbol{B_m}}{\partial t} = 0 \qquad \nabla\cdot\boldsymbol{B_m} = 0$$
$$e^{-\gamma}\,\nabla\times\boldsymbol{B_m} = 0 \qquad e^{-\gamma}\,\nabla\cdot\boldsymbol{E_m} + 4\sinh\gamma\,\frac{\boldsymbol{B_m}\cdot\boldsymbol{E_m}}{B_m^4}\left(\boldsymbol{B_m}\cdot\nabla\right)B_m^2 = 0$$

This proves to be invariant under Galilean boosts under with the following field transformations

$$\boldsymbol{B_m}(\boldsymbol{x},t)\to\boldsymbol{B'_m}(\boldsymbol{x},t) = \boldsymbol{B_m}(\boldsymbol{x}-\mathbf{b}t, t)$$
$$\boldsymbol{E_m}(\boldsymbol{x},t)\to\boldsymbol{E'_m}(\boldsymbol{x},t) = \boldsymbol{E_m}(\boldsymbol{x}-\mathbf{b}t, t) - \boldsymbol{b}\times\boldsymbol{B_m}(\boldsymbol{x}-\mathbf{b}t, t)$$

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