Beyond Semiclassical Time: Dynamics and Quantum Diffeomorphism Invariance

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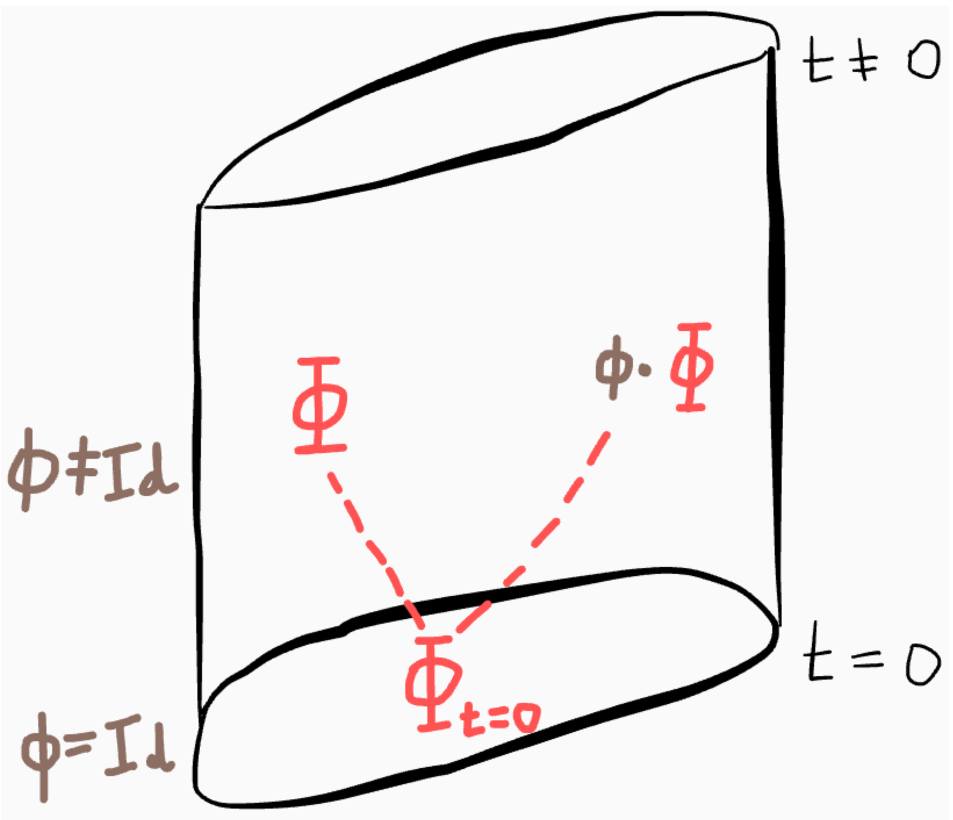
I. Problem of Time

In general relativity, there is no preferred choice of time variable. As a consequence, the bulk Hamiltonian is constrained to be zero. Canonical quantisation à la Dirac leads to the Wheeler-DeWitt equation

$$\hat{H}\Psi = 0 \Rightarrow \Psi = \sum_{\mathbf{k}} \Psi(\mathbf{k}) \left| E = 0, \mathbf{k} \right\rangle ,$$

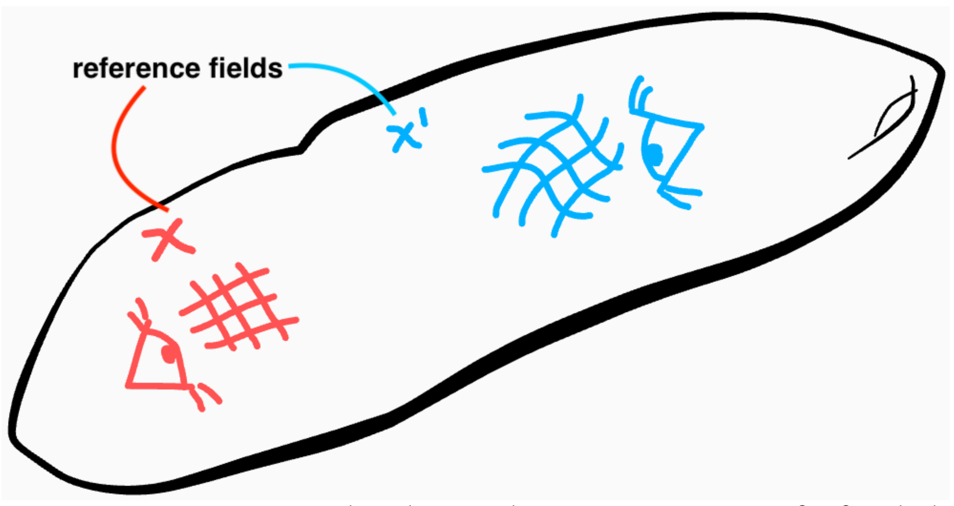
which implies that the wave function of the universe does not depend on the label time t. In this case, how to understand dynamics?

II. Gauge Indeterminism



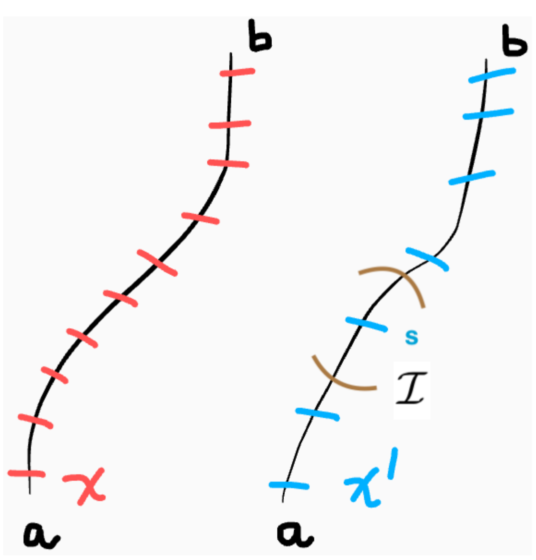
Determinism is restored by considering diffeomorphism invariants (or equivalence classes) $\Rightarrow$ **Observables** (invariant under diffeos $\phi : \mathcal{M} \to \mathcal{M}$)

III. Relational Observables



Observers record the dynamics of fields Φ in local regions using reference fields χ ('generalized clocks and rods') that define 'generalized reference frames'. The outcomes of their experiments are the values of Φ relative to $\chi \Rightarrow$ 'relational observables'. They completely encode the dynamics in the region of the experiment. E.g., on $\mathbf{0+1}$ spacetime dimensions:

$$\mathcal{O}[f|\chi = s] = \frac{\int_{\mathcal{I}} \mathrm{d}\tau\, \delta(\chi(\tau) - s) f(\tau)}{\int_{\mathcal{I}} \mathrm{d}\tau\, \delta(\chi(\tau) - s)} \qquad (1)$$



These are diffeo-invariant extensions of tensor fields on the **worldline** and satisfy $\delta_{\epsilon(\tau)}\mathcal{O} \propto \{\mathcal{O}, H\}|_{H=0} = 0$; i.e., they are invariant under infinitesimal diffeos. A model-independent quantization reads [1]

$$\hat{\mathcal{O}}[f] := \pi\hbar \sum_{E} \hat{P}_E[\hat{f}(\tau), \hat{P}_{t=s}]_+ \hat{P}_E\, , \qquad (2)$$

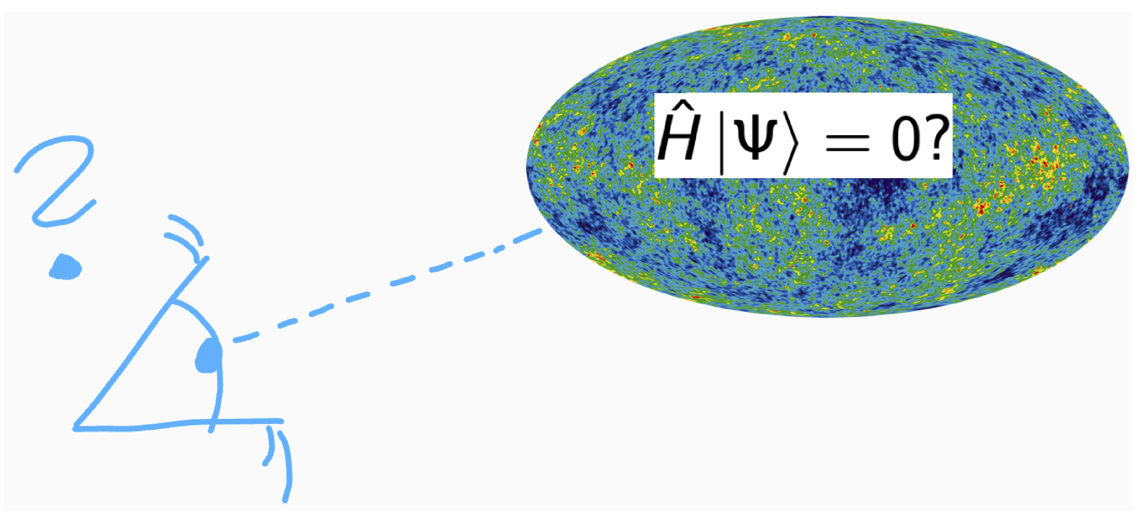
$\hat{P}_t \propto \sum_{\mathbf{k}} |t, \mathbf{k}\rangle\langle t, \mathbf{k}|$ and $|t, \mathbf{k}\rangle \propto \sum_E \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Et} |E, \mathbf{k}\rangle$ [3].

IV. General Quantum Dynamics

- Fixing the gauge = choosing a time coordinate (clock). No preferred classical choice.
- In the quantum theory, define (Rieffel) inner product $\langle\Psi_2|\Psi_1\rangle := \sum_{\mathbf{k}} \Psi_2^*(\mathbf{k})\Psi_1(\mathbf{k})$.
- Operators given by (2) that are self-adjoint define diffeo-invariant observables.
  - If the energy spectrum spans the real line, they reduce to $\hat{\mathcal{O}}[f] = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\tau\, \hat{f}(\tau)\hat{P}_{t=s-\tau} + \text{h.c.}$, which is analogous to the classical expression (1).
  - For a general spectrum, we have $\hat{\mathcal{O}}[1] = \hat{1}$ [Faddeev-Popov (FP) resolution of the identity], and the observables obey the correct dynamics (as in the classical theory):
  $$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}s}\hat{\mathcal{O}}[f] = \hat{\mathcal{O}}\left[\mathrm{i}\hbar\frac{\partial f}{\partial s} + [f, H]\right] .$$
- Time is defined intrinsically from the energy spectrum, and the evolution is unitary as long as a complete set of self-adjoint observables are defined.

V. Weak-Coupling Expansion: Unitary Gauge Theory

- What if we cannot solve $\hat{H}\left|\Psi\right\rangle = 0$ exactly? Perturbation theory!
- Important case: the Hamilton-Jacobi Hamiltonian constraint is
$$\frac{\kappa}{2}G^{ab}(Q)\frac{\partial \mathrm{W}}{\partial Q^a}\frac{\partial \mathrm{W}}{\partial Q^b} + \frac{1}{\kappa}V(Q) + \mathsf{h}\left(Q; \frac{\partial \mathrm{W}}{\partial q}, q\right) = 0\, ,$$
where κ is a small coupling constant (e.g., $\kappa = 4\pi G/3$ or $\kappa = 1/c^2$) and W is the on-shell action. This models a heavy system Q with mass given by $M\sqrt{\kappa} \propto 1$ (gravity) coupled to a light system q (matter).
- WKB perturbative solution: $\mathrm{W}(Q, q) = \frac{1}{\kappa}\sum_{n=0}^{\infty} \mathrm{W}_n(Q, q)\kappa^n =: \frac{1}{\kappa}\mathrm{W}_0(Q) + \mathsf{S}(Q; q)$.
- Lowest-order (no-coupling limit): $\frac{1}{2}G^{ab}(Q)\frac{\partial \mathrm{W}_0}{\partial Q^a}\frac{\partial \mathrm{W}_0}{\partial Q^b} + V(Q) = 0$, induces a foliation on config. space $Q \mapsto x(Q) = (t(Q), x^i(Q))$ where $G^{ab}\frac{\partial \mathrm{W}_0}{\partial Q^a}\frac{\partial t}{\partial Q^b} = 1$ is the proper time (conjugate to H) in the no-coupling limit.
- Higher orders: $-\frac{\partial \mathsf{S}}{\partial t} = \mathsf{h} - \frac{\kappa}{4V}\mathsf{h}^2 + \frac{\kappa}{2}g^{ij}\frac{\partial \mathsf{S}}{\partial x^i}\frac{\partial \mathsf{S}}{\partial x^j} + \mathcal{O}(\kappa^2) \Rightarrow t$ is the time that orders the evolution of matter and of x: a choice of gauge that is "preferred" in this expansion. The associated FP determinant is $|\Delta| = 1 + \kappa\frac{\mathsf{h}}{2V} + \mathcal{O}(\kappa^2)$.
- Just as in the classical case, the WDW constraint $\hat{H}\Psi = -\frac{\kappa}{2}\nabla^2\Psi + \frac{1}{\kappa}V(Q)\Psi + \hat{\mathsf{h}}\Psi = 0$, can be solved perturbatively via $\Psi(Q, q) = \exp\left[\mathrm{i}\frac{1}{\kappa}\mathrm{W}_0(Q, q)\right]\psi(Q; q)$. *[Born-Oppenheimer or semiclassical approach]* Upon adopting the foliation with the "no-coupling" proper time t, we find the corrected Schrödinger equation
$$\mathrm{i}\frac{\partial\tilde{\psi}}{\partial t} = \left[\hat{\mathsf{h}} - \frac{\kappa}{4V}\hat{\mathsf{h}}^2 + \frac{\kappa}{2}\hat{\Pi}_i g^{ij}\hat{\Pi}_j^\dagger + \frac{\mathcal{Q}}{M} + \mathcal{O}(\kappa^2)\right]\tilde{\psi}\, , \qquad (3)$$
where $\mathcal{Q}$ is a quantum correction to the potential and $\tilde{\psi} := |2Vg|^{\frac{1}{4}}\left(1 + \frac{\kappa}{2V}\hat{\mathsf{h}}\right)\psi + \mathcal{O}(\kappa^2)$. Eq. (3) leads to a unitary dynamics w.r.t. the inner product
$$(\Psi_2|\Psi_1) = \int \mathrm{d}Q\mathrm{d}q\sqrt{|2Vg|}\left|\frac{\partial x}{\partial Q}\right| \Psi_2^*(x, q)\,\delta(t(Q) - s)\widehat{|\Delta|}\,\Psi_1(x, q) + \mathcal{O}(\kappa^2)\, ,$$
where $\widehat{|\Delta|} := 1 + \frac{\kappa}{2V}\hat{\mathsf{h}}$ is the quantization of the (absolute value of the) FP determinant.



- No unitarity violation (as previously thought): unitary gauge theory that can be applied to cosmology.
- de Sitter background + perturbations:
$$\mathrm{i}\frac{\partial\tilde{\psi}}{\partial\eta} = \left(\hat{H} - \kappa\frac{H_0^2\eta^4}{2}\hat{H}^2\right)\tilde{\psi} + \mathcal{O}(\kappa^2)$$

- Towards phenomenology: $\mathcal{P}_{S,T}(k) \simeq \mathcal{P}_{S,T;0}(k)\left\{1 + \kappa H_0^2\left(\frac{k_\star}{k}\right)^3[2.85 - 2\log(-2k\eta)]\right\}$

Future Work

Quantum gravity phenomenology via unitary corrections to (and dynamical renormalization of) slow-roll models; singularity resolution via invariant observables; cosmology as unitary gauge theory, quantum foundations with relational observables

References

[1] L. Chataignier, *Timeless Quantum Mechanics and the Early Universe*, Springer Theses; Cham, Switzerland: Springer (2022).
[2] L. Chataignier, Z. Naturforsch. A **77** 805 (2022).
[3] P. A. Hoehn, A. R. H. Smith and M. P. E. Lock *Phys. Rev.* D **104** 06600 (2021)
and references therein.