Quantum Cosmology of Pure Connection GR

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Chiral Connection Formulations of GR

Chiral connection formulations are reformulations of GR that take the form of *Diffeomorphism Invariant Gauge Theories*. The central field is a complex $SU(2)$ connection $A^i_\mu(x)$, à la Yang-Mills. They are derived from the Chiral Plebański formulation by integrating out variables.

Instanton Representation

$$S_{\text{Inst}}[A,M,\mu] = \frac{1}{16\pi G i}\int M^{-1}_{ij} F^i \wedge F^j + \mu\,(\mathrm{tr}\,M - \Lambda) \tag{1}$$

M^{ij} is a symmetric matrix field and μ is a complex 4-form. The field equations are

$$\Sigma^i \equiv (M^{-1})^{ij} F^j \quad : \quad d\Sigma^i + \epsilon^{ijk} A^j \wedge \Sigma^k = 0 \quad , \quad \Sigma^i \wedge \Sigma^j = \mu\,\delta^{ij}\ . \tag{2}$$

Pure Connection

Fix a nowhere vanishing 4-form ε_X and define $F^i \wedge F^j = \varepsilon_X X^{ij}$,

$$S_{\text{PC}}[A] = \frac{1}{16\pi G i\Lambda}\int \varepsilon_X \left(\mathrm{tr}\sqrt{X}\right)^2 \tag{3}$$

Where $(\sqrt{X})^{ij}$ is the square root of a complex matrix. This action has only one field equation,

$$\Sigma^i \equiv \Lambda^{-1}\,\mathrm{tr}\sqrt{X}\,(X^{-\frac{1}{2}})^{ij} F^j \quad : \quad d\Sigma^i + \epsilon^{ijk} A^j \wedge \Sigma^k = 0\ . \tag{4}$$

Constructing a Metric

In order to recover the equations of GR, we must construct a metric tensor from the variables present in our theories.

The Urbantké Metric

$$g_{\mu\nu}\sqrt{-g} = \frac{i}{12}\,\epsilon_{ijk}\,\epsilon^{\alpha\beta\gamma\delta}\,\Sigma^i_{\mu\alpha}\,\Sigma^j_{\nu\beta}\,\Sigma^k_{\gamma\delta}\ . \tag{5}$$

The Σ^i 2-forms satisfy $\Sigma^i \wedge \Sigma^j = \frac{1}{3}\Sigma^k \wedge \Sigma_k\,\delta^{ij}$, so there exists a tetrad of complex 1-forms e^I such that

$$\Sigma^i = ie^0 \wedge e^i - \frac{1}{2}\epsilon^{ijk}\,e^j \wedge e^k \quad \text{and} \quad g_{\mu\nu} = e^I_\mu e^J_\nu\,\eta_{IJ}\ . \tag{6}$$

The Urbantké metric satisfies the Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$

Reality Conditions I

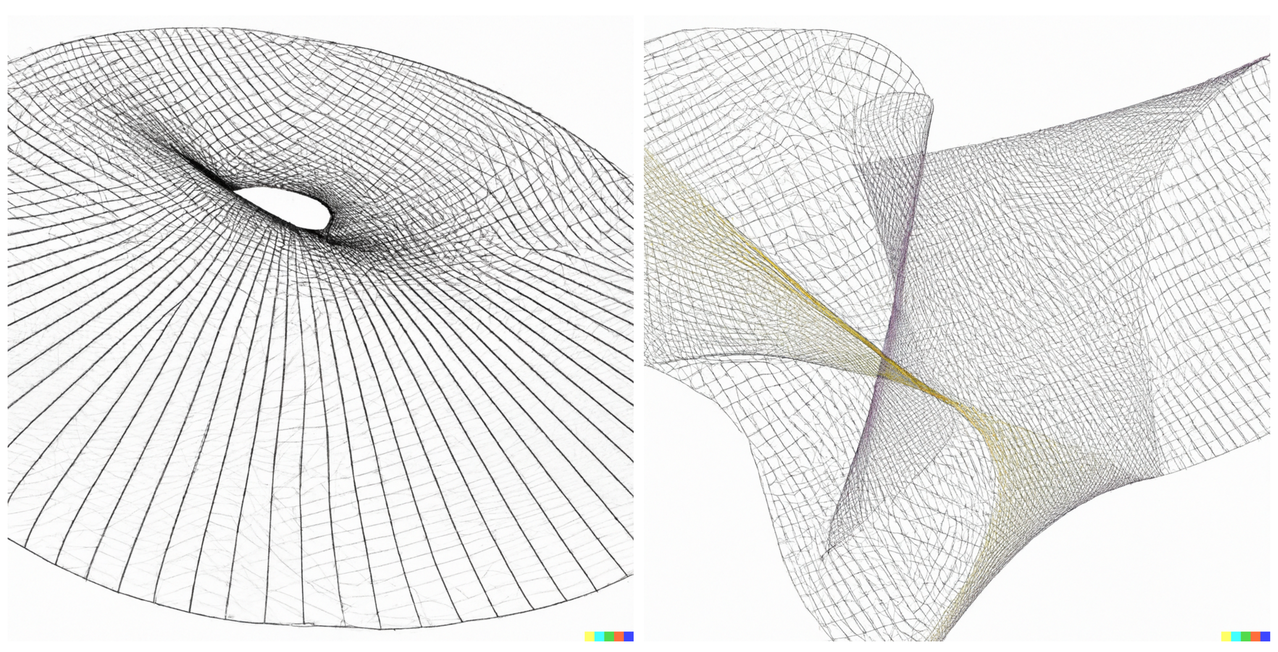
In general, the Urbantké metric is complex valued. We'd like to restrict to connections whose Urbantké metrics are *Real Lorentzian*. We need *Reality Conditions*,

$$F^i \wedge \overline{F^j} = 0 \quad , \quad \mathrm{Re}\left(\Sigma^i \wedge \Sigma_i\right) = 0\ . \tag{7}$$

These are extra conditions that must be satisfied on top of the field equations. They guarantee that the Urbantké metric is *Real Lorentzian* up to a possibly imaginary conformal factor.

Further Reading

A comprehensive introduction to chiral connection formulations can be found in [3]. More on the pure connection formalism can be found in [4]. The following serves as a summary of some of the main results from our paper [2]



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Homogeneous and Isotropic Connections

We consider spacetimes of the form $\mathcal{M} = \mathbb{R}\times\mathcal{N}$ where $\mathcal{N}$ is either $\mathbb{R}^3$ or S^3. Our ansatz for a *Homogeneous and Isotropic Connection*,

$$A^j = \frac{iC(t)\,\omega^j}{3V^{1/3}} \quad : \quad d\omega^j = -\sqrt{\kappa}\,\epsilon^{jkl}\,\omega^k \wedge \omega^l\ . \tag{8}$$

ω^j is a basis of 1-forms on $\mathcal{N}$ that are invariant under a certain group of diffeomorphisms, and $V = \int_{\mathcal{V}} \omega^1 \wedge \omega^2 \wedge \omega^3$ is a fiducial 3-volume. The Urbantké metric is homogeneous and isotropic,

$$g = -N(t)^2\,dt \otimes dt + a(t)^2\,\delta_{ij}\,\omega^i \otimes \omega^j\ . \tag{9}$$

Minisuperspace Models

We construct a *Minisuperspace Model* by substituting our ansatz (8) into the pure connection action (3)

$$S_{\text{MS}}[C] = \frac{1}{3\ell_P^2\Lambda}\int_{t_i}^{t_f} dt\,C(C+2iK)\dot{C} = \frac{1}{3\ell_P^2\Lambda}\left[\frac{C^3}{3} + iKC\right]_{t_i}^{t_f} \tag{10}$$

The resulting action is a *Surface Term*. We canonically extend the action by adding a conjugate momentum P

$$S_{\text{CE}}[C,P,\rho] = \int dt\left(\dot{C}P - \rho\left(\ell_P^2\Lambda P - \frac{1}{3}C(C+2iK)\right)\right) \tag{11}$$

This action generates a constrained Hamiltonian system.

The Urbantké Metric,

$$g = -\frac{\ell_P^2\,\rho^2}{P^3}\,dt \otimes dt + \frac{\ell_P^2 P}{V^{2/3}}\left(\sum_{i=1}^{3}\omega^i \otimes \omega^i\right)\ . \tag{12}$$

Reality Condtions II

The reality conditions have two solution branches

$$C = c - iK \quad , \quad P = p \ \text{ or } \ P = ip \tag{13}$$

For real P, we must also have a real Λ. But for imaginary P, we must have an imaginary Λ.

Quantum Cosmology

The canonically extended action for a real FLRW type connection,

$$S_R[c,p,\lambda] = \int dt\left(\dot{c}p - \lambda\left(\ell_P^2\Lambda\,p - \frac{c^2+K^2}{3}\right)\right)\ . \tag{14}$$

We use a *Phase Space Path Integral* of the real theory between initial and final connection states to compute a transition amplitude

$$\langle c_f | c_i \rangle = \int_{c_i}^{c_f} \mathcal{D}c\,\mathcal{D}p\,\mathcal{D}\lambda\,\exp iS_R[c,p,\lambda]\ . \tag{15}$$

A similar calculation was carried out in [1] using metric boundary conditions. We find

$$\langle c_f | c_i \rangle = \exp\frac{i}{3\ell_P^2\Lambda}\left(\frac{1}{3}(c_f^3 - c_i^3) + K^2(c_f - c_i)\right) \tag{16}$$

This result was obtained independently by alternative methods in [5]. We have a product of *Chern-Simons Wave Functions*,

$$\langle c_f | c_i \rangle = \psi_{CS}(c_f)\,\overline{\psi_{CS}}(c_i) \quad : \quad \psi_{CS}(c) = \exp\frac{i}{3\ell_P^2\Lambda}\left(\frac{c^3}{3} + Kc\right)\ . \tag{17}$$

References

[1] Job Feldbrugge, Jean-Luc Lehners, and Neil Turok. "Lorentzian quantum cosmology". In: *Physical Review D* 95.10 (2017), p. 103508.

[2] Steffen Gielen and Elliot Nash. "Quantum cosmology of pure connection general relativity". In: *Classical and Quantum Gravity* 40.11 (2023), p. 115009.

[3] Kirill Krasnov. *Formulations of General Relativity: Gravity, Spinors and Differential Forms*. Cambridge University Press, 2020.

[4] Kirill Krasnov. "Pure connection action principle for general relativity". In: *Physical review letters* 106.25 (2011), p. 251103.

[5] João Magueijo. "Real Chern-Simons wave function". In: *Physical Review D* 104.2 (2021), p. 026002.

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