Polyhedron 2.0: higher structures in 4d quantum geometry

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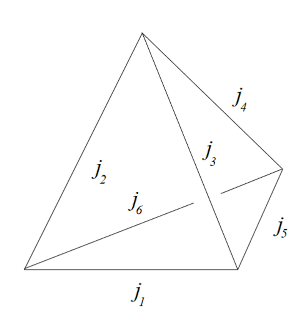
Higher-dimensional geometry from categorification

We study the geometry of 4-dimensional piece-wise linear manifolds by applying a categorification procedure to the 3-dimensional scenario. We propose that the fundamental geometric variables in 4-dimensions are given by elements of a *2-group* — more precisely, surface holonomies of a 2-connection — and we construct the graded Poisson structure of these decorated surfaces. This leads to the flux-holonomy phase space relevant for 4d geometry.

3d quantum geometry: Ponzano-Regge [1, 2]

$$Z = \int D[A]D[e]e^{i\int e\wedge F} \sim \sum_{\text{\Tetrahedron}} \prod_{\triangle} W, \qquad W = \int_G d[h_\triangle]\delta(\prod_{\text{edges}} h)$$

1. 1d links & edges: connections to holonomies
$$\begin{cases} A \mapsto h \in SU(2) & \text{dual complex} \\ e \mapsto X \in \mathbb{R}^3 & \text{triangulation} \end{cases}, \qquad (X, h) \in T^*SU(2)$$
2. 2d triangles: **spinning top** phase space, modulo closure constraint,
$$\mathcal{P}_\triangle = (T^*SU(2))^3//C, \qquad C = \sum_{/\in\partial\triangle} X_/.$$
   - Quantization: $SU(2)$-intertwiners
$$\mathcal{P}_\triangle \to \text{Hom}_{\text{Rep(SU(2))}}(j_1 \otimes j_2 \otimes j_3, 1)$$
3. 3d tetrahedron: $6j$-symbols
$$Z_{\text{\Tetrahedron}}(j_1, \dots, j_6) = \begin{pmatrix} j_1 j_2 j_3 \\ j_4 j_5 j_6 \end{pmatrix}.$$

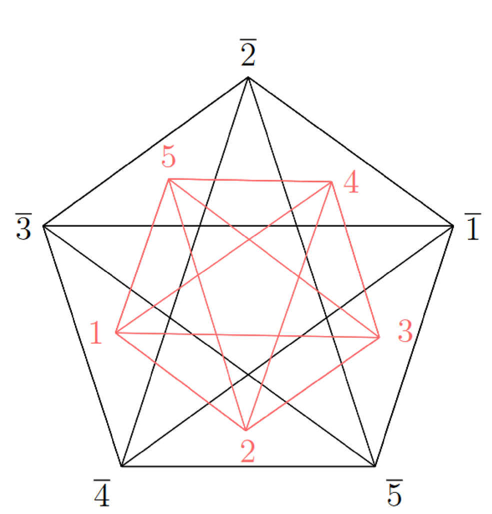


Result: Evaluation of quantum amplitudes W as state sum over $6j$-symbols.

4d quantum geometry: topological BF-BB [3, 4]

$$Z = \int D[A]D[B]e^{i\int B\wedge F - B\wedge B} \sim \sum_{\text{4-simplices}} \prod_{\text{\Tetrahedron}} W, \qquad W = ?$$

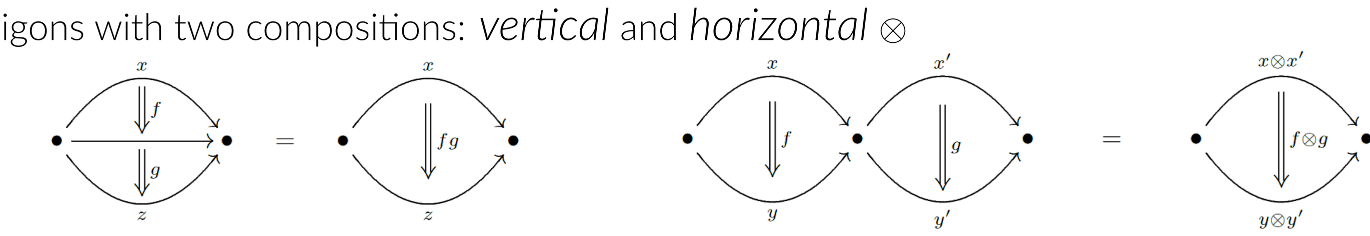
2-gauge shift symmetry ($L \in \Lambda^1$) and equations of motion,
$$\begin{cases} A \mapsto A - tL \\ B \mapsto B + d_A L \end{cases}, \qquad \begin{cases} \text{Fake-flatness:} \quad F = B \\ \text{2-flatness:} \quad d_A B = 0 \end{cases}.$$



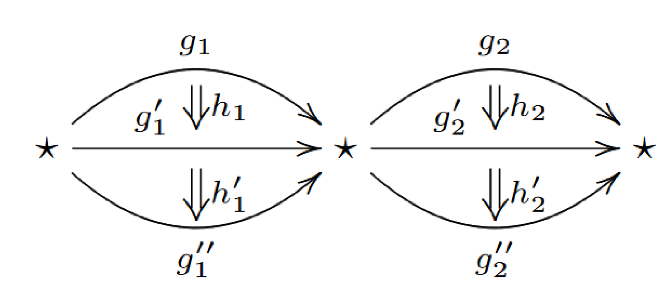
Goal: find tetrahedral phase space $\mathcal{P}_{\text{\Tetrahedron}}$.

Categories, groupoids and all that [5, 6, 7]

- Groupoid $\Gamma \rightrightarrows M$.
  1. Morphisms with associative invertible composition ($a, b, c \in M$):
$$a \xrightarrow{g} b \xrightarrow{h} c = a \xrightarrow{gh} c, \qquad \begin{cases} a \xrightarrow{g} b \xrightarrow{g^{-1}} a = a \xrightarrow{1_a} a \\ b \xrightarrow{g^{-1}} a \xrightarrow{g} b = b \xrightarrow{1_b} b \end{cases}.$$
  2. Eg. groupoid of paths $PM \rightrightarrows M$; describing *holonomies*
$$\text{Hol}^G : PM = \begin{pmatrix} \text{paths} \\ M \end{pmatrix} \to \begin{pmatrix} G \\ \bullet \end{pmatrix}, \qquad \gamma \mapsto P\exp\int_\gamma A = g_\gamma \in G.$$
- **2-group** $\Gamma \rightrightarrows M$: base itself is a group.
  1. Bigons with two compositions: *vertical* and *horizontal* $\otimes$

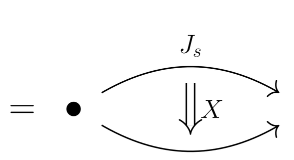


  2. **Interchange law**:



  3. 2-groups describe *2-holonomies*
$$2\text{Hol}^\Gamma : \begin{pmatrix} \text{surfaces} \\ \text{paths} \\ M \end{pmatrix} \to \begin{pmatrix} \Gamma \\ M \\ \bullet \end{pmatrix}, \qquad \begin{pmatrix} S \\ \gamma \end{pmatrix} \mapsto \begin{pmatrix} S\exp\int_S B \\ P\exp\int_\gamma A \end{pmatrix} = \begin{pmatrix} h_S \\ g_\gamma \end{pmatrix} \in \Gamma.$$
     - Surface exponential $S\exp$ constructed in [8].
  4. Infinitesimally: Lie 2-algebra/L_2-algebra.
- **3+1d spinning geometry phase space from Poisson structure on 2-groups.**

Poisson 2-groups [9, 10, 11]

1. BF-BB theory has *2-gauge symmetry*; identify the Lie 2-(bi)aglebra,
$$\underbrace{\mathfrak{d}}_{\text{2-form}} \xrightarrow{d=\text{id}} \underbrace{\mathfrak{d}}_{\text{1-form}}, \qquad \mathfrak{d} = \mathbb{R}^3 \rtimes \mathfrak{su}_2.$$
2. Integrate: *2D face-link holonomies* on triangulation:
$$(X, J) \in \mathbb{R}^3 \times \mathbb{R}^3, \qquad \Gamma = \mathbb{R}^6 \rightrightarrows \mathbb{R}^3.$$
   - Bigon representation: $(X, J_s) =$ 

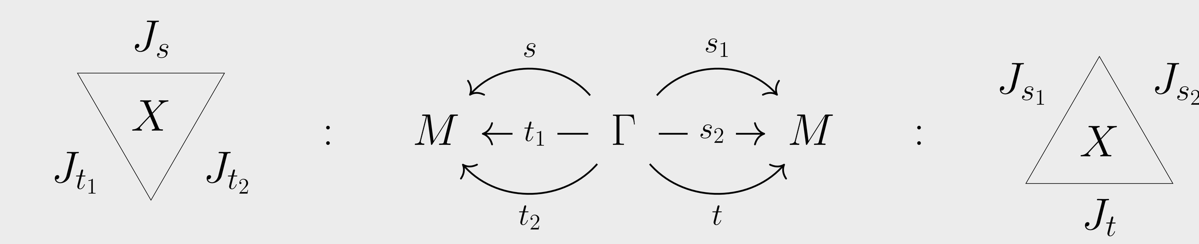
   - Poisson 2-group Γ; *graded* Poisson bracket
$$\{(X, J_s), (X', J_s')\} = \{J_s, X'\} + \{X, J_t'\} + \{J_s, J_s'\}. \tag{1}$$
3. Wish to impose closure constraints:
$$\mathcal{C} = \begin{cases} \text{Fake-flatness:} \quad d(X_\triangle) = \sum_{/\in\partial\triangle} J_/ \\ \text{2-flatness:} \quad \sum_{\triangle\in\partial\text{\Tetrahedron}} X_\triangle = 0 \end{cases}, \qquad \mathcal{P}_{\text{\Tetrahedron}} = (\Gamma)^4//\mathcal{C}.$$

Problem: Γ contains bigons, we need triangles.

Poisson bracket on triangles

Idea: **fine-grain** bigons $\to$ triangles & extend Poisson 2-group structure.

1. Introduce additional source/targets



2. Fit Poisson bracket to $\Gamma_\triangle$:
$$\{(J_{s_1}, J_{s_2}), (J_{s_1}', J_{s_2}')\} = \{J_{s_1}, J_{s_1}'\} + \{J_{s_2}, J_{s_2}\},$$
$$\{(J_{s_1}, J_{s_2}), X\} = \{J_{s_1}, X\} + \{J_{s_2}, X\},$$
   - $\Gamma_\triangle$ is *not* a groupid.
   - Vertical composition gives squares. $\triangle \circ \nabla = \square \notin \Gamma_\triangle$.
3. **Stability** of triangles.
$$\{\triangle, \triangle\} = \triangle, \qquad \{\nabla, \nabla\} = \nabla.$$
4. **Polygonal decomposition** into triangles.
$$\{\square, \square\} = \{\triangle \circ \nabla, \triangle \circ \nabla\} = \{\triangle, \triangle\} \circ \{\nabla, \nabla\}$$

Result: tetrahedral phase space
$$\mathcal{P}_{\text{\Tetrahedron}} = (\Gamma_\triangle)^4//\mathcal{C} \ \textcircled{\simeq}\ \text{2-coadjoint orbits in } \Gamma_\triangle.$$

Will prove this!

- Poisson structure reduced on orbits is *graded* symplectic.

Future Work

Quantization: formula for 4-simplex amplitude W,
$$W \sim \int_\Gamma d[(h, g)_{\text{\Tetrahedron}}]\delta(\prod_\triangle (h, g)) \to SU(2) \text{ "2-intertwiners".}$$

- Baez conjecture [12]: BF-BB amplitude = Barrett-Crane?
$$SU(2) \text{ "2-intertwiners"} = \begin{cases} 10\text{-}j? & ;\text{2-rep. theory [13]} \\ 15\text{-}j? & ;\text{Crane-Yetter [14]} \\ \text{both?} \end{cases}.$$
- Applications to 4d quantum gravity/spin-foams.

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