# Second Order Geometry and the Quantum Foam

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## Quantum Foam

**Aim:** construct a covariant theory on the quantum foam (fluctuating spacetime).

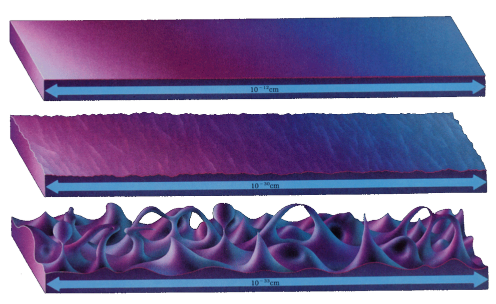

Fig. 1: Illustration of spacetime on the Microscopic, Mesoscopic and Macroscopic scale

(source: pngegg.com)

We consider a mesoscopic scale. Motivation:

- There exist many different microscopic models, but these can be grouped in a smaller set of mesoscopic universality classes.
- Near future experiments cannot distinguish between microscopic models, but may observe the mesoscopic effects.
- Microscopic scale: discrete path integrals
  Mesoscopic scale: continuous path integrals
  Typically, discrete models are more suited for numerical studies, while analytical studies are easier to perform in the continuum limit.

**Problem:** path integrals seem to break general covariance.
**Old solution (DeWitt):** consider a non-trivial geometry on the configuration space of fields (jet bundle).
**New solution (Second Order Geometry):** extend the (co)tangent bundle to a second order (co)tangent bundle and the jet bundle to a second order jet bundle.

## From Path Integrals to Itô Integrals

- The Feynman-Kac theorem provides a mathematical equivalence between the Euclidean path integral and an Itô integral along the Wiener process (a.k.a. Brownian motion).
- This equivalence can be extended to Lorentzian path integrals by considering complex extensions of the Wiener process.

Thus, a path integral over the path space $L^2(\Omega)$ can be written as an Itô integral over the sample space $\Omega$, i.e.

$$\int_{L^2(\Omega)} f(X)\, e^{-S(X(\tau))}\, DX(\tau) = \int_\Omega f(X)\, \mathrm{d}X(\tau,\omega)\,. \tag{1}$$

Right hand side (Itô integral): $X:[0,T]\times\Omega\to\mathcal{M}$ is a stochastic process that extremizes the action $\mathbb{E}[S(X)] = \int_\Omega S(X(\omega))\,\mathrm{d}\mathbb{P}(\omega)$ subjected to the Wiener measure $\mathrm{d}\mathbb{P}:\Omega\to[0,1]$.

Left hand side (path integral): For every $\omega\in\Omega$, one obtains a path $X(\cdot,\omega):[0,T]\to\mathcal{M}$. The paths are weighted with the induced measure $\mathrm{d}\mu: L^2(\Omega)\to[0,1]$, which is given by $\mathrm{d}\mu_X = \mathrm{d}(\mathbb{P}\circ X^{-1}) = e^{-S(X(\tau))}\,DX(\tau)$.

**Consequence:** in the standard path integral formulation, the individual paths $X(\cdot,\omega)$ are continuous, but neither differentiable nor causal.
Only the probabilistic average $\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$ over all paths is causal.
**Remarks**

- The non-differentiability of the paths is related to the non-commutativity of phase space.
- The acausality of the paths is related to the presence of negative normed eigenstates.
- The causality of the process ensures that the causality violations of the paths are exponentially suppressed on proper time intervals larger than the Compton wavelength.

## Stochastic Calculus and Quadratic Variation

In the Itô formulation differential calculus is modified:

$$\mathrm{d}f = \partial_\mu f\,\mathrm{d}X^\mu + \frac{1}{2}\partial_\nu\partial_\mu f\,\mathrm{d}[X^\mu,X^\nu]\,, \tag{2}$$
$$\mathrm{d}[f,g] = \partial_\mu f\,\partial_\nu g\,\mathrm{d}[X^\mu,X^\nu]\,, \tag{3}$$
$$\mathrm{d}(h\circ f) = (h'\circ f)\,\mathrm{d}f + \frac{1}{2}(h''\circ f)\,\mathrm{d}[f,f]\,, \tag{4}$$
$$\mathrm{d}(f\,g) = f\,\mathrm{d}g + g\,\mathrm{d}f + \mathrm{d}[f,g]\,, \tag{5}$$

where the bracket $\mathrm{d}[.,.]$ is called the quadratic variation.
Deterministic theories: $\mathrm{d}[X^\mu,X^\nu]=0$; stochastic/quantum theories: $\mathrm{d}[X^\mu,X^\nu]\neq 0$.
**Euclidean Quantum Theory:** the quadratic variation satisfies the structure relation

$$\mathrm{d}[X^\mu,X^\nu](\tau) = \hbar\, g^{\mu\nu}_{\rm Eucl}(X(\tau))\,\mathrm{d}\tau\,. \tag{6}$$

**Lorentzian Quantum Theory:** the quadratic variation satisfies the structure relation

$$\mathrm{d}[Z^\mu,Z^\nu](\tau) = \mathrm{i}\,\hbar\, g^{\mu\nu}(X(\tau))\,\mathrm{d}\tau\,, \tag{7}$$

where $Z = X + \mathrm{i}\,Y$ and $Y$ is an auxiliary path.
**Consequence:** Ordinary (first order) differential geometry fails in quantum theories!

**Properties of Quadratic Variation**

- Symmetry: $\mathrm{d}[X^\mu,X^\nu] = \mathrm{d}[X^\nu,X^\mu]$
- Bilinearity: $\mathrm{d}[a\,X^\mu + b\,X^\nu, X^\rho] = a\,\mathrm{d}[X^\mu,X^\rho] + b\,\mathrm{d}[X^\nu,X^\rho]$
- Positivity: $\mathrm{d}[X^\mu,X^\nu]$ is positive semi-definite
- Closure: $\mathrm{d}[[X^\mu,X^\nu],X^\rho] = 0$ if $X(\tau)$ is continuous.

## Second Order Geometry

**Geometric Perspective:** Given a (pseudo-)Riemannian manifold $\mathcal{M}$. For any $x\in\mathcal{M}$, the (co)tangent space $T_x^{(*)}\mathcal{M}$ is extended to the second order (co)tangent space $\tilde{T}_x^{(*)}\mathcal{M}$. Vectors $v\in\tilde{T}_x\mathcal{M}$ and forms $\omega\in\tilde{T}_x^*\mathcal{M}$ have the canonical representation

$$v = v^\mu\,\partial_\mu + \frac{1}{2}v_2^{\mu\nu}\,\partial_\mu\partial_\nu \qquad \text{and} \qquad \omega = \omega_\mu\,\mathrm{d}_2x^\mu + \frac{1}{2}\omega_{\mu\nu}\,\mathrm{d}[x^\mu,x^\nu] \tag{8}$$

with $\omega_{\mu\nu} = \partial_{(\nu}\omega_{\mu)}$ and a duality pairing given by $\langle v,\omega\rangle = \omega_\mu v^\mu + \frac{1}{2}\omega_{\mu\nu}v_2^{\mu\nu}$.
Second order vectors can be interpreted as velocities along stochastic/quantum trajectories:

$$v^\mu(x,\tau) = \lim_{\mathrm{d}\tau\to0}\mathbb{E}\left[\frac{\mathrm{d}X^\mu(\tau)}{\mathrm{d}\tau}\,\middle|\,X(\tau)=x\right], \tag{9}$$
$$v_2^{\mu\nu}(x,\tau) = \lim_{\mathrm{d}\tau\to0}\mathbb{E}\left[\frac{\mathrm{d}[X^\mu,X^\nu](\tau)}{\mathrm{d}\tau}\,\middle|\,X(\tau)=x\right]. \tag{10}$$

Representations $(v^\mu, v_2^{\nu\rho})$ of vectors $v$ and $(\omega_\mu,\omega_{\nu\rho})$ of forms $\omega$ don't transform covariantly!
However, there exist covariant representations $(\hat{v}^\mu,\hat{v}_2^{\nu\rho})$ and $(\hat{\omega}_\mu,\hat{\omega}_{\nu\rho})$, where

$$\hat{v}^\mu = v^\mu + \frac{1}{2}\Gamma^\mu_{\nu\rho}v_2^{\nu\rho}, \qquad \hat{\omega}_\mu = \omega_\mu,$$
$$\hat{v}_2^{\mu\nu} = v_2^{\mu\nu}, \qquad \hat{\omega}_{\mu\nu} = \omega_{\mu\nu} - \Gamma^\rho_{\mu\nu}\omega_\rho\,.$$

**Algebraic Perspective:** The structure group of the tangent bundle $\tilde{T}\mathcal{M} = \bigsqcup_{x\in\mathcal{M}}\tilde{T}_x\mathcal{M}$ is the Itô group $G_n^I = \mathrm{GL}(n,\mathbb{R})\times\mathrm{Lin}(\mathbb{R}^n\otimes\mathbb{R}^n,\mathbb{R}^n)$ with binary operation

$$(g',\Gamma')\,(g,\Gamma) = (g'\,g,\; g'\circ\Gamma + \Gamma'\circ(g\otimes g))$$

and left action on the fibers $\tilde{T}_x\mathcal{M}\cong\mathbb{R}^n\times\mathrm{Sym}(T\mathbb{R}^n\otimes T\mathbb{R}^n)$

$$(g,\Gamma)\,(v,v_2) = (g\,v + \Gamma\,v_2,\; (g\otimes g)\,v_2)$$

for all $g,g'\in\mathrm{GL}(n,\mathbb{R})$, $\Gamma,\Gamma'\in\mathrm{Lin}(\mathbb{R}^n\otimes\mathbb{R}^n,\mathbb{R}^n)$, $v\in\mathbb{R}^n$ and $v_2\in\mathrm{Sym}(T\mathbb{R}^n\otimes T\mathbb{R}^n)$.
**Consequence:** Consider an arbitrary vector field $v(x)$ with canonical (first order) representation $v^\mu = e^\mu_a\,v^a$. Then, if $v$ is Lorentz invariant, $v^a$ is invariant under the Lorentz group $\mathrm{SO}^+(3,1)$. However, since the polyads $e^\mu_a$ transform under $G_n^I$ instead of $\mathrm{GL}(n,\mathbb{R})$, the representation $v^\mu$ is invariant under a deformed Lorentz symmetry.
These deformations vanish in two limits:

$$G\to0\Rightarrow\Gamma\to0 \qquad \text{and} \qquad \hbar\to0\Rightarrow v_2\to0\,. \tag{11}$$

## Further Results

**Embeddings in higher dimension:**

- Given a $n$-dimensional (pseudo-)Riemannian manifold equipped with second order geometry, there exists a bijective mapping to a $n$-dimensional brane embedded in a $\frac{n(n+3)}{2}$-dimensional (pseudo-)Riemannian manifold equipped with first order geometry.
- Given a $(3+1)$-dimensional Lorentzian manifold, there exists a bijective mapping to a $(3+1)$-dimensional brane embedded in a $(10+4)$-dimensional Riemannian manifold equipped with first order geometry.
- Using the Bianchi identities, one can reduce the dimensions of the embedding space from $\frac{n(n+3)}{2}$ to $\frac{n(n+1)}{2}$ and from $(10+4)$ to $(9+1)$.

**Killing equation:** The second order Killing equation is

$$\nabla_{(\mu}k_{\nu)} = e^{\mathrm{i}\,\phi}\,\hbar\,\mathcal{R}_{\mu\nu}\,. \tag{12}$$

Lorentzian theory: $\phi=\frac{\pi}{2}$; Euclidean theory: $\phi=0$.
Thus, on the quantum foam we obtain an $O(l_p^2)$ deviation from the classical Killing equation.

## Outlook

Second order geometry enables the construction of a covariant (continuous) path integral.
Quantum gravity may require path integrals along paths that are not continuous.
The typical example is a path with Poisson distributed jumps.
The characteristic structure relation of a compensated Poisson process is

$$\mathrm{d}[X,X](\tau) = \beta^2\,\gamma\,\mathrm{d}\tau + \beta\,\mathrm{d}X(\tau)\,, \tag{13}$$

where $\gamma$ is the jump rate and $\beta$ the jump size.
More generally, one can consider theories characterized by a structure relation

$$\mathrm{d}[X^\mu,X^\nu](\tau) = A^{\mu\nu}\,\mathrm{d}\tau + B^{\mu\nu}_\rho\,\mathrm{d}X^\rho\,. \tag{14}$$

Discrete theories, characterized by $B\neq0$, can be related to infinite derivative theories and theories on non-commutative spacetimes. (Stay tuned!)

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