Finiteness of PL quantum gravity with matter

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Let M be a smooth 4-dimensional manifold and let $T(M)$ be a PL (piecewise linear) manifold corresponding to a regular triangulation of M (the dual one simplex is a connected 5-valent graph).

Let $\{L_\epsilon | \, \epsilon \in T_1(M)\}$ be a set of the edge lengths such that $L_\epsilon^2 \in \mathbf{R}$, i.e. $L_\epsilon \in \mathbf{R}_+$ (spacelike edge) or $L_\epsilon \in i\mathbf{R}_+$ (timelike edge).

A metric on $T(M)$, which is flat in each 4-simplex σ of $T(M)$, is given by

$$G_{\mu\nu}(\sigma) = L_{0\mu}^2 + L_{0\nu}^2 - L_{\mu\nu}^2 \,, \tag{1}$$

where the five vertices of σ are labeled as $0,1,2,3,4$ and $\mu,\nu = 1,2,3,4$ (Cayley-Menger metric).

CM metric is not dimensionless and hence it is not diffeomorphic to

$$g_{\mu\nu}(\sigma) = diag(-1,1,1,1) \,.$$

This can be corrected by using a dimensionless PL metric

$$g_{\mu\nu}(\sigma) = \frac{G_{\mu\nu}(\sigma)}{|L_{0\mu}||L_{0\nu}|} \,. \tag{2}$$

The Einstein-Hilbert (EH) action on M is given by

$$S_{EH} = \int_M \sqrt{|\mathrm{det} g|} \, R(g) \, d^4x \,, \tag{3}$$

where $R(g)$ is the scalar curvature associated to a metric g. On $T(M)$ the EH action becomes the Regge action

$$S_R(L) = \sum_{\Delta \in T(M)} A_\Delta(L) \, \delta_\Delta(L) \,, \tag{4}$$

when the edge lengths correspond to a Eucledean PL geometry. A_Δ is the area of a triangle Δ, while the deficit angle δ_Δ is given by

$$\delta_\Delta = 2\pi - \sum_{\sigma \supset \Delta} \theta_\Delta^{(\sigma)} \,, \tag{5}$$

where a dihedral angle $\theta_\Delta^{(\sigma)}$ is defined as the angle between the 4-vector normals associated to the two tetrahedrons that share the triangle Δ.

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In the case of a Lorentzian geometry, a dihedral angle can take complex values, so that it is necessary to modify the formula (4) such that the Regge action takes only the real values. This can be seen from the formula

$$\sin\theta_\Delta^{(\sigma)} = \frac{4}{3}\frac{v_\Delta v_\sigma}{v_\tau v_{\tau'}}\,, \tag{6}$$

where $v_s = V_s \geq 0$, if the CM determinant is positive, while $v_s = \mathrm{i}V_s$ if the CM determinant is negative. Consequently, $\sin\theta_\Delta^{(\sigma)} \in \mathbf{R}$ or $\sin\theta_\Delta^{(\sigma)} \in \mathrm{i}\mathbf{R}$. This implies that a Lorentzian dihedral angle can take complex values so that the Regge action (4) will give a complex number when the spacelike triangles are present. One can modify the Regge action as

$$S_R^*(L) = Re\left(\sum_{\Delta(s)} A_{\Delta(s)}\,\frac{1}{\mathrm{i}}\,\delta_{\Delta(s)}\right) + \sum_{\Delta(t)} A_{\Delta(t)}\,\delta_{\Delta(t)}\,, \tag{7}$$

where $\Delta(s)$ denotes a spacelike triangle, while $\Delta(t)$ denotes a timelike triangle, so that it is always real and corresponds to the Einstein-Hilbert action on $T(M)$.

Consequently

$$Z(T(M)) = \int_D \prod_{\epsilon=1}^{N} dL_\epsilon\,\mu(L)\,e^{\mathrm{i}S_R^*(L)/l_P^2}\,,$$

where $dL_\epsilon = d|L_\epsilon|$ and $\mu(L)$ is a mesure that ensures the finiteness and gives the effective action with a correct semiclassical expansion, see [1, 3]. The integration region D is a subset of $\mathbf{R}_+^N$, consistent with a choice of spacelike and timelike edges.

$Z(T(M))$ is convergent for the measure

$$\mu(L) = e^{-V(M)/L_0^4}\prod_{\epsilon=1}^{N}\left(1 + \frac{|L_\epsilon|^2}{l_0^2}\right)^{-p}\,, \tag{8}$$

where $p > 1/2$, see [3].

When the SM matter is added, we have

$$S_m = S_H + S_{YM} + S_f + S_Y = \int_M d^4x\,\sqrt{g}\,(\mathcal{L}_H + \mathcal{L}_{YM} + \mathcal{L}_f + \mathcal{L}_Y)\,,$$

where

$$\mathcal{L}_H = \frac{1}{2}D^\mu\phi^\dagger D_\mu\phi - \lambda_0^2(\phi^\dagger\phi - \phi_0^2)^2\,,\quad \mathcal{L}_{YM} = -\frac{1}{4}Tr\,(F^{\mu\nu}F_{\mu\nu})\,,$$

$$\mathcal{L}_f = \sum_{k=1}^{48}\epsilon^{abcd}e_b\wedge e_c\wedge e_d\,\bar\psi_k\,(\mathrm{i}\gamma_a(\mathrm{d} + \mathrm{i}\omega + \mathrm{i}g_0A))\,\psi_k\,,$$

$$\mathcal{L}_Y = \sum_{k,l} Y_{kl}\,\langle\bar\psi_k\psi_l\phi\rangle\,,\quad D_\mu\phi = (\partial_\mu + \mathrm{i}\,(g_0A)_\mu)\phi\,,$$

and

$$g_0A = g_{01}A_1 + g_{02}A_2 + g_{03}A_3 \in \text{Lie alg}\,(U(1)\times SU(2)\times SU(3))\,.$$

On $T(M)$ we have

$$\tilde S_H = \sum_\sigma V_\sigma(L)\,s_{HK} + \sum_\pi V_\pi^*(L)\,s_{HP}\,,$$

where $\pi \in T_0(M)$,

$$s_{HK} = g_\sigma^{\mu\nu}\left(\frac{\phi(\pi_\mu) - \phi(\pi_0)}{|L_{0\mu}|} + \mathrm{i}g_0A_\mu(\pi_0)\phi_{\pi_0}\right)^\dagger\left(\frac{\phi(\pi_\nu) - \phi(\pi_0)}{|L_{0\nu}|} + \mathrm{i}g_0A_\nu(\pi_0)\phi_{\pi_0}\right)$$

and

$$s_{HP} = \lambda_0^2 \left(\phi^\dagger(\pi)\phi(\pi) - \phi_0^2\right)^2 .$$

The fermion action on $T(M)$ is given by

$$\tilde{S}_f = \sum_\epsilon V_\epsilon^*(L)\, s_f + \sum_\pi V_\pi^*(L)\; s_{YMf} ,$$

where

$$s_f = \sum_k \epsilon^{abcd}\, B_{abc}(p)\, \bar\psi_k(\pi)\, \mathrm{i}\gamma_d \left(|L_\epsilon|\, \mathrm{i}\omega_\epsilon(L)\, \psi_k(\pi') + \psi_k(\pi') - \psi_k(\pi)\right) ,$$

$$s_{YMf} = \sum_k \bar\psi_k(\pi)\, g_0 \gamma^\mu(\pi) A_\mu(\pi)\, \psi_k(\pi) , \qquad (9)$$

and

$$\gamma^\mu(\pi) = e_a^\mu(\pi)\gamma^a , \quad e_a^\mu(\pi) = \frac{1}{n_\sigma(\pi)} \sum_{\sigma;\, \pi\in\sigma} e_a^\mu(\sigma) .$$

The Yukawa action on $T(M)$ is given by

$$\tilde{S}_Y = \sum_\pi V_\pi^*(L)\, s_Y ,$$

where

$$s_Y = \sum_{k,l} Y_{kl} \langle \bar\psi_k(\pi)\psi_l(\pi)\phi(\pi)\rangle .$$

Therefore the gravity plus matter path integral will be given by

$$Z = \int_D d^N L\, \mu(L)\, e^{\mathrm{i}S_R^*(L)/l_P^2}\, Z_m(L) , \qquad (10)$$

where

$$Z_m(L) = \int_{\mathbf{R}^n} \prod_\alpha d^n\varphi_\alpha\, e^{\mathrm{i}S_m(\varphi,L)/\hbar} ,$$

and φ is a collection of matter fields φ_α and n is the number of vertices in $T(M)$.

Since the convergence of Z_m is not guaranteed, we pass to a Eucledean geometry defined by the edge lengths

$$\tilde{L}_\epsilon = |L_\epsilon| ,$$

so that all the Eucledean edge lengths are positive real numbers. This is equvalent to a Wick rotation where $\tilde{L}_\epsilon = L_\epsilon$ if ϵ is a spacelike edge and $\tilde{L}_\epsilon = (-\mathrm{i})L_\epsilon$, if ϵ is a timelike edge.

Then we will consider the integral

$$\tilde{Z}_m(\tilde{L}) = \int_{\mathbf{R}^n} \prod_\alpha d^n\varphi_\alpha\, e^{-\tilde{S}_m(\varphi,\tilde{L})/\hbar} , \qquad (11)$$

where $\tilde{S}_m$ is the Euclidian matter action. Since $\tilde{S}_m(\varphi,\tilde{L})$ is a positive function of φ, and

$$\tilde{S}_m(\varphi,\tilde{L}) \to +\infty , \quad \text{for } |\varphi_\alpha| \to +\infty ,$$

then the integral $\tilde{Z}_m$ will be convergent. Hence we will define

$$Z_m(L) = \tilde{Z}_m(\tilde{L})\Big|_{\tilde{L}=w(L)} , \qquad (12)$$

where w is the Wick rotation.

In the case of the SM, it can be shown that Z is absolutely convergent for $p > 46{,}5$ [5].

When $M = \Sigma \times I$, a non-perturbative effective action $\Gamma(L, \Phi)$ on $T(M)$ is determined by the EA equation

$$e^{i\Gamma(L,\Phi)/\hbar} = \int_{D(L)} d^N l \, d^{cn}\varphi \, \mu(L+l) \, e^{iS(L+l,\Phi+\varphi)/\hbar - i\sum_\epsilon \Gamma'_\epsilon(L,\Phi) l_\epsilon/\hbar - i\sum_\pi \Gamma'_\pi(L,\Phi)\varphi_\pi/\hbar} ,$$

where c is the number of components of the matter fields ($c = c_f + c_{gh} + c_b = 96 + 24 + 52 = 172$ for the SM) and $S = S_R^*/G_N + S_m$. This equation will be only defined if the gravity plus matter path integral is finite, which is the case for $p > 46{,}5$.

The smooth-manifold approximation is given by $N \to \infty$ and $|L_\epsilon| \to 0$, so that

$$\Gamma(L, \Phi) \approx \Gamma_K[g_{\mu\nu}(x), \varphi(x)] , \quad x \in M ,$$

where Γ_K is the QFT effective action for the cutoff $K = 2\pi/\bar{L}$, $\bar{L}$ is the average edge length and

$$g_{\mu\nu}(x) \approx g_{\mu\nu}(\sigma) , \quad \varphi_\alpha(x) \approx \Phi_\alpha(v) \quad \text{for } x \in \sigma \text{ and } v = \text{dual vertex} \in \sigma .$$

We also have

$$\Gamma_K(g, \varphi) = S_{EH}(g) + S_m(g, \varphi) + l_P^2 \, \Gamma_K^{(1)}(g, \varphi) + l_P^4 \, \Gamma_K^{(2)}(g, \varphi) + \cdots ,$$

for $|L_\epsilon| \gg l_P$ and small φ, where $\Gamma_K^{(n)}$ is the n-loop QFT effective action for GR coupled to matter, see [2].

Note that one can also add the cosmological constant (CC) term to the Regge action, so that

$$S_R^*(L) \to S_R^*(L) + \Lambda_c V_4(L) .$$

Then the condition for the semi-classical expansion of the EA, $|L_\epsilon| \gg l_P$ and $L_0 \gg l_P$, is substituted by

$$|L_\epsilon| \gg l_P , \quad L_0 \gg \sqrt{l_P L_c} ,$$

where $|\Lambda_c| = 1/L_c^2$ [2]. One can then show that the observed value of the CC belongs to the spectrum of the CC in PLQG, provided the path integral for gravity + matter is finite, see [2, 4]. Since the PI is finite for $p > 46{,}5$ then the proof given in [4] is now complete.

Hence the PLQG defined by the PI measure (8) is the first example of a simple and mathematically complete theory of quantum gravity with the SM matter.

References

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