Non-Hausdorff Geometries:
Their Theory and Potential Applications

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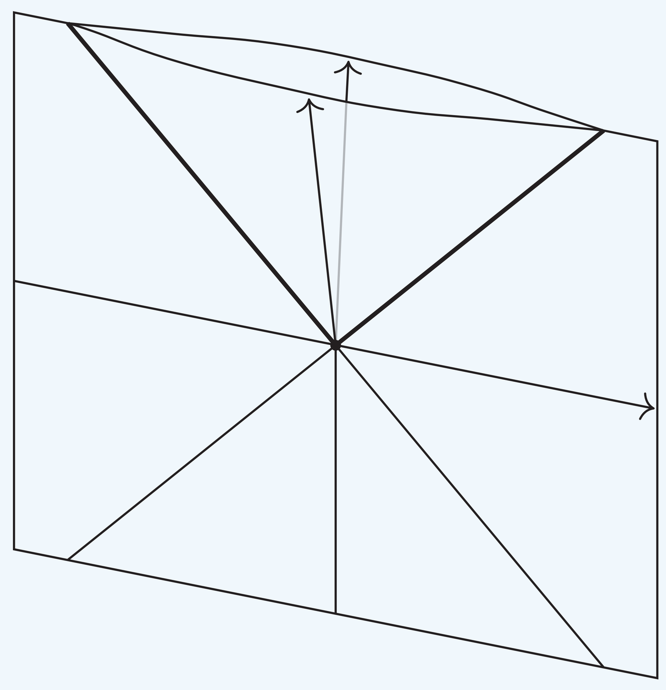
Introduction

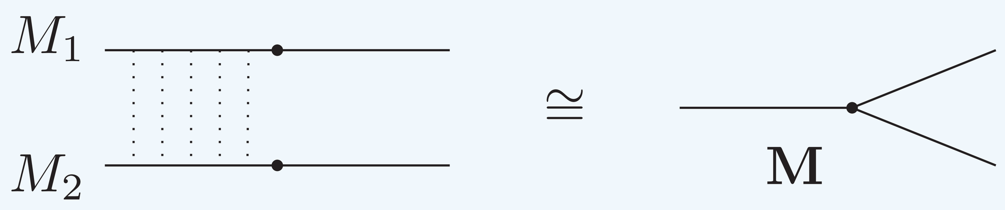
All manifolds are typically defined to be Hausdorff, meaning that any pair of points can be separated by disjoint open sets. Conversely, non-Hausdorff manifolds negate this property by allowing points to be "doubled" on top of each other.

- Although they are harder to work with, they still display lots of the familiar features of differential geometry. In particular, non-Hausdorff manifolds can be equipped with a Lorentzian metric, effectively turning them into spacetimes that exhibit some form of topology change.
- On the right is a non-Hausdorff Minkowski spacetime. Here there are two copies of the future lightcone of the origin, and the Hausdorff-violating points occur along the future nullcones.
- Given how much structure non-Hausdorff manifolds possess, there is thus a question of whether these unusual spacetimes can be meaningfully included in path integrals that sum over topologies.



Topology and Geometry[1,2,3]

All non-Hausdorff manifolds can be realised by gluing Hausdorff manifolds along open sets:



Hausdorff violation will then occur at the boundary of these open sets.

Differential geometry can be performed on these spaces:

- Non-Hausdorff manifolds are still locally-Euclidean, so we can define tangent spaces everywhere.
- All vector bundles can be constructed by gluing together Hausdorff bundles, and all sections can be constructed from sections of Hausdorff bundles.
- Integration follows an alternating sum formula:

$$\int_{\mathbf{M}} \omega = \int_{M_1} \omega_1 + \int_{M_2} \omega_2 - \int_{\overline{A}} \omega_A$$

  where here A denotes the open set that is glued along.

There are some important technical differences:

- No arbitrary partitions of unity (similar, but slightly better than the case of complex manifolds)
- No unique global flows of vector fields
- No Stoke's Theorem

Cohomology Theories[2,3]

Any non-Hausdorff manifold is covered by Hausdorff submanifolds. We may use the cohomologies of these subspaces to deduce the overall cohomology of the non-Hausdorff space. Depending slightly on the particular theory, these cohomologies are typically related with a Mayer-Vietoris sequence:

$$\cdots \to H^{q-1}(A) \xrightarrow{\delta^*} H^q(\mathbf{M}) \xrightarrow{\Phi^*} H^q(M_1) \oplus H^q(M_2) \xrightarrow{\iota^* - f^*} H^q(A) \xrightarrow{\delta^*} H^{q+1}(\mathbf{M}) \to \cdots$$

The case for a covering by three or more Hausdorff submanifolds is given by a spectral sequence.

A Euclidean Gauss-Bonnet Theorem[3]

We can use a Mayer-Vietoris sequence for singular homology to deduce a Gauss-Bonnet Theorem for non-Hausdorff manifolds. Roughly speaking, both the Euler characteristic and integration are additive, so we may apply the Hausdorff Gauss-Bonnet theorem to get:

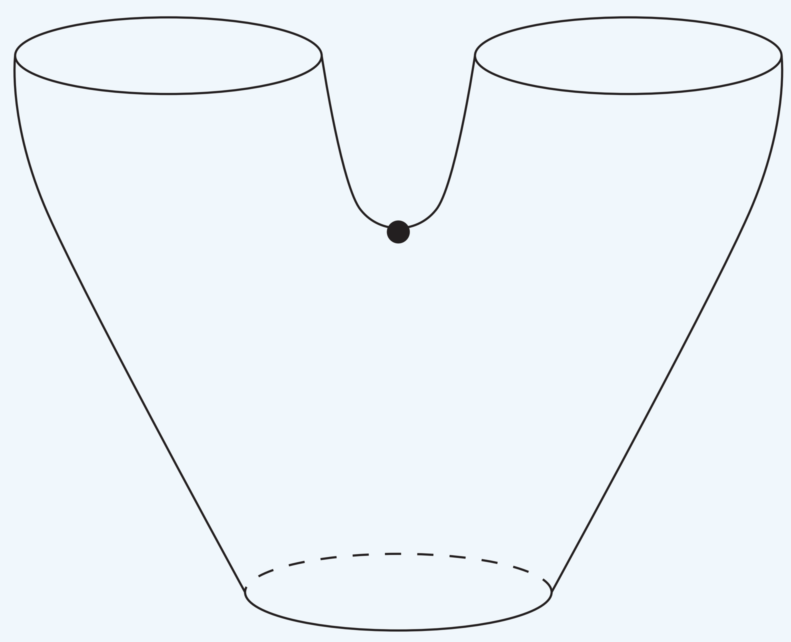
$$\begin{aligned}\chi(\mathbf{M}) &= \chi(M_1) + \chi(M_2) - \chi(\overline{A}) \\ &= \left(\frac{1}{4\pi}\int_{M_1} d^2x\sqrt{g}R\right) + \left(\frac{1}{4\pi}\int_{M_2} d^2x\sqrt{g}R\right) - \left(\frac{1}{4\pi}\int_{\overline{A}} d^2x\sqrt{g}R\right) - \left(\frac{1}{2}\int_{\partial A}\kappa_g dx\right) \\ &= \frac{1}{4\pi}\int_{\mathbf{M}} d^2x\sqrt{g}R - \frac{1}{2}\int_{\partial A}\kappa_g dx.\end{aligned}$$

The extra term is a Gibbons-Hawking boundary term for the gluing region A, which in this context is the geodesic curvature of the Hausdorff-violating codim$=1$ submanifold in $\mathbf{M}$.

Hausdorff Topology Change[4]

Results by Louko/Sorkin suggest that topology changing spacetimes such as the Trousers space (pictured) are suppressed in a sum over topologies. One finds that:

- The curvature of the Trousers space takes contribution from a delta function at the vertex of topology change (pictured).
- Interestingly, in Lorentzian signature the strength of this delta function becomes complex, and is equal to $2\pi i$. By an application of the Gauss-Bonnet theorem, this yields a contribution of -2π in the exponent of a path integral.



Current Work: Non-Hausdorff Topology Change in Lorentzian Signature

For now, I am working on including the non-Hausdorff manifolds into Lorentzian path integrals in $2d$. There are two interesting regimes:

- **An analogue of Louko/Sorkin**. There should be a Lorentzian analogue of the non-Hausdorff Gauss-Bonnet theorem. In this case, the additional boundary term in the Gauss-Bonnet theorem can be made to be null, and probably there is a delta-like singularity at the splitting of adjacent manifolds. In principle, this suggests a similar result to Louko/Sorkin is possible for non-Hausdorff manifolds as well.
- **The path integrals of bosonic string theory**. There should be a similar analysis for the path integrals of bosonic strings, except that here the more involved coupling constant might have an influence. I am most interested in considering the inclusion of non-Hausdorff manifolds that split along nullcones, since this might contribute to a geometric understanding for the Lorentzian theory of interacting strings.

References

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3. D. O'Connell, "Non-Hausdorff de Rham Cohomology", *Forthcoming*
4. J. Louko and R. Sorkin, "Complex actions in two-dimensional topology change", Classical and Quantum Gravity 14.1 (1997): 179