# Diffeomorphism Covariant Dynamics in Quantum Kantowski-Sachs

July 10 - 14
Quantum Gravity 2023
Nijmegen, Netherlands

Rafael G. Dias (rdias2019@fau.edu) Ian W. Bornhoeft Jonathan S. Engle

## Why?

- Diffeomorphism covariance (or background independence) is a key feature of the formulation of General Relativity.
- Loop Quantum Gravity (LQG) is a non-perturbative proposal to a quantum theory of gravity, based on the principle of of diffeomorphism covariance.
- Loop Quantum Cosmology (LQC) applies quantization techniques analogous to LQG to symmetry-reduced models, but does not require diffeomorphism covariance a priori.
- Requiring diffeomorphism covariance to a LQC model can help reducing ambiguities in its construction[1,2].

## Kantowski-Sachs Framework

- Homogeneous model with spatial section of topology $S^2 \times \mathbb{R}$, with geometry described by pairs $(b, p_b)$ and $(c, p_c)$, such that

$$\{b, p_b\} = G\gamma \quad \text{and} \quad \{c, p_c\} = 2G\gamma,$$

$$ds^2 = -N^2 d\tau^2 + \frac{p_b^2}{|p_c| L_0^2} dx^2 + |p_c| d\Omega^2,$$

$$V = 4\pi |p_b| \sqrt{|p_c|}.$$

- Ashtekar-Barbero variables:

$$A_a^1 = -b \sin\theta \partial_a \phi, \qquad E_1^a = -\frac{p_b}{L_0}\phi^a$$

$$A_a^2 = b \partial_a \theta, \qquad E_2^a = \frac{p_b}{L_0} \sin\theta \theta^a$$

$$A_a^3 = \frac{c}{L_0}\partial_a x + \cos\theta \partial_a \phi, \qquad E_3^a = p_c \sin\theta x^a$$

- Hamiltonian Constraint (with lapse $N_{V_n} = \lambda V^n$)

$$H_{c\ell}[N_{V_n}] = -\frac{\lambda V^{n+1}}{8\pi G \gamma^2} \operatorname{sgn} p_b \left[\frac{b^2 + \gamma^2}{p_c} + \frac{2bc}{p_b}\right]. \tag{1}$$

## Diffeomorphism Covariance

- Residual diffeomorphisms: group of transformations preserving the form of $(A, E)$:

$$\mathscr{L}_{\vec{v}} A_a^i(t) = \dot{A}_a^i = \frac{\partial A_a^i}{\partial b}\dot{b}(t) + \frac{\partial A_a^i}{\partial c}\dot{c}(t)$$

$$\mathscr{L}_{\vec{v}} E_i^a(t) = \dot{E}_i^a = \frac{\partial E_i^a}{\partial p_b}\dot{p}_b(t) + \frac{\partial E_i^a}{\partial p_c}\dot{p}_c(t)$$

- Subgroup with non-trivial action generated by $\{x\vec{x}\}$.
- Flow equations result in

$$\dot{b} = 0 \quad , \quad \dot{p}_b = p_b \quad , \quad \dot{c} = c \quad , \quad \dot{p}_c = 0 \tag{2}$$

- Flow of the Hamiltonian:

$$\dot{H}_{c\ell} = (n+1) H_{c\ell}.$$

We seek to require the quantum Hamiltonian to follow a quantization of this condition (*quantum covariance*).

## Quantization Procedure

- Standard procedure corresponding to canonical transformations: turn quantities into operators, Poisson brackets into commutators, choose an ordering for quantizing products

$$\dot{F} = \{\Lambda, F\} \quad \Rightarrow \quad \dot{\hat{F}} = \frac{1}{i\hbar}[\hat{F}, \hat{\Lambda}] \quad \Rightarrow \quad \hat{F}(t) = e^{\frac{t}{i\hbar}\hat{\Lambda}} \hat{F}(0) e^{-\frac{t}{i\hbar}\hat{\Lambda}}.$$

Flows in (2) are non-canonical, but they can be cast in a related form

$$\dot{F} = \omega_1\{\Lambda_1, F\}(b, p_b) + \omega_2\{\Lambda_2, F\}(c, p_c) = \frac{p_b}{\gamma G}\{b, F\} - \frac{c}{2\gamma G}\{p_c, F\}.$$

Choosing the Weyl ordering for quantizing products, $\hat{A} \star \hat{B} := \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A})$, yields the covariance equation for $\hat{H}$,

$$(n+1)\hat{H} = \frac{1}{2i\gamma\ell_p^2}\left\{\hat{p}_b [\hat{b}, \hat{H}] + [\hat{b}, \hat{H}]\hat{p}_b\right\} - \frac{1}{4i\gamma\ell_p^2}\left\{\hat{c}[\hat{p}_c, \hat{H}] + [\hat{p}_c, \hat{H}]\hat{c}\right\}. \tag{3}$$

- Only exponentials of $b$ and $c$ are properly defined in the Bohr-Hilbert space arising from loop quantization, so first find the general solution for (3) in the standard Schrödinger representation, with later imposition of preservation of the Bohr-Hilbert space.
- Find the general solution for the matrix elements $\langle p_b'', p_c'' | \hat{H} | p_b', p_c' \rangle$, and use completeness of momentum basis to obtain the action of the Hamiltonian on a general state $|p_b', p_c'\rangle$,

$$\hat{H}|p_b', p_c'\rangle = \int |p_b'', p_c''\rangle \langle p_b'', p_c''|\hat{H}|p_b', p_c'\rangle dp_b'' dp_c''.$$

By changing variables, rewrite the action of $\hat{H}$ in terms of shifts and an unconstrained parameter function $\alpha : \mathbb{R}^3 \to \mathbb{C}$,

$$\hat{H}|p_b', p_c'\rangle = \left[\int e^{\frac{iA}{2}\hat{b}} e^{\frac{iB}{2}\frac{\hat{c}}{|\hat{p}_b|}} |\hat{p}_b|^{n+1} \alpha(A, B, \hat{p}_c, \operatorname{sgn} p_b) e^{\frac{iB}{2}\frac{\hat{c}}{|\hat{p}_b|}} e^{\frac{iA}{2}\hat{b}} dA dB\right] |p_b', p_c'\rangle.$$

Define the ordering prescription, for a general function $f(p_b, p_c)$ as $\widehat{f(p_b, p_c) e^{i\left(Ab + B\ell_p^2 \frac{c}{p_b}\right)}} := e^{\frac{iA}{2}\hat{b}} e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} f(\hat{p}_b, \hat{p}_c) e^{\frac{iB}{2}\frac{\hat{c}}{\hat{p}_b}} e^{\frac{iA}{2}\hat{b}}$, and thus

$$\hat{H} = \int \widehat{|p_b|^{n+1} \alpha(A, B, p_c, \operatorname{sgn} p_b) e^{i\left(Ab + B\frac{c}{|p_b|}\right)}} \, dA dB.$$

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- *Preservation of Bohr-Hilbert space*: for any $p_b', p_c'$ there must be at most countable $p_b'', p_c''$ such that the matrix elements $\langle p_b'', p_c'' | \hat{H} | p_b', p_c' \rangle$ are non-zero. Require $\alpha(A, B, p_c, \operatorname{sgn} p_b) = \sum_k \alpha_k(p_c, \operatorname{sgn} p_b)\delta(A - A_k(p_c))\delta(B - B_k(p_c))$, then

$$\hat{H} = \sum_k \widehat{|p_b|^{n+1} \alpha_k(p_c, \operatorname{sgn} p_b) e^{i\left(A_k(p_c) b + B_k(p_c)\frac{c}{|p_b|}\right)}}.$$

## Discrete Symmetries

- Define the *classical analogue* of operator as the preimage under quantization map,

$$H = \sum_n p_b \alpha_n(p_c, \operatorname{sgn} p_b) e^{i\left(A_n(p_c) b + B_n(p_c)\ell_P r \frac{c}{p_b}\right)}. \tag{4}$$

As a consequence of the ordering prescription for quantization, discrete symmetries that are left can be easily checked directly in the classical analogue.

- *Hermiticity*: $\hat{H} = \hat{H}^\dagger = \hat{\bar{H}} \Rightarrow \bar{H} = H$,
- *b-parity*: $\Pi_b : (b, p_b) \mapsto (-b, -p_b)$, equivalent to an internal gauge rotation of $\pi$ around the 3-axis. $\hat{\Pi}_b \hat{H} \hat{\Pi}_b = \widehat{\Pi_b^* H}$ and $H$ must satisfy the covariance equation $\Pi_b^* H = -H$, satisfied by (1).
- *c-parity*: $\Pi_c : (c, p_c) \mapsto (-c, -p_c)$, equivalent to antipodal map $(\theta, \phi) \mapsto (\pi - \theta, \phi + \pi)$ + internal parity under 3-axis. $H$ must follow $\Pi_c^* H = -H$, satisfied by (1).
- Physical assumption: quantization is resultant from the fact that holonomies can only be shrank to a minimum area $\Delta$, which is dependent only on the absolute values of the momentum variables, and not on their sign – require that $A = A(|p_c|)$ and $B = B(|p_c|)$
- The general form for $H$ is then

$$H = |p_b|^{n+1}\left\{a_0^e \operatorname{sgn}(p_b p_c) - 2\sum_{k \neq 0}\left[a_k^o \operatorname{sgn} p_b \cos(A_k b)\cos\left(\frac{B_k c}{|p_b|}\right)\right.\right.$$
$$\left.\left. + a_k^e \sin(A_k b)\sin\left(\frac{B_k c}{|p_b|}\right) + b_k^e \operatorname{sgn} p_b \cos(A_k b)\sin\left(\frac{B_k c}{|p_b|}\right) - b_k^o \sin(A_k b)\cos\left(\frac{B_k c}{|p_b|}\right)\right]\right\}, \tag{5}$$

where $a_k, b_k$ are real and imaginary parts of $\alpha_k$, and the superscripts ${}^{e}, {}^{o}$ refers to the even and odd parts of each coefficient.

## Classical Asymptotic Behavior

- Expanding (5) for the limit of low curvatures $(b, c \to 0)$, and matching terms of same order with (1), result in a system of equations to find a family of Hamiltonians, depending on the parameter $N$ chosen

$$\mathscr{O}(1): \quad -\frac{\lambda(4\pi)^n}{2G\gamma^2}\gamma^2 |p_c|^{\frac{n-1}{2}} \operatorname{sgn} p_c = a_0^e \operatorname{sgn} p_c + \sum_{k=1}^{N} 2 a_k{}^o$$

$$\mathscr{O}(b): \quad 0 = \sum_{k=1}^{N} b_k^o A_k$$

$$\mathscr{O}(c): \quad 0 = \sum_{k=1}^{N} b_k{}^e B_k$$

$$\mathscr{O}(bc): \quad \frac{\lambda(4\pi)^n}{2G\gamma^2}|p_c|^{\frac{n+1}{2}} \operatorname{sgn} p_b = \sum_{k=1}^{N} a_k^e A_k B_k$$

$$\mathscr{O}(b^2): \quad -\frac{\lambda(4\pi)^n}{2G\gamma^2}|p_c|^{\frac{n-1}{2}} \operatorname{sgn} p_c = \sum_{k=1}^{N} a_k{}^o A_k^2$$

$$\mathscr{O}(c^2): \quad 0 = \sum_{k=1}^{N} a_k{}^o B_k^2$$

## Minimality

- Require Hamiltonian to have a minimum number of terms $(N = 2)$, which results in

$$H = -\frac{\lambda}{2G\gamma^2}\frac{V^n |p_b|}{|p_c|^{\frac{1}{2}}}\operatorname{sgn}(p_b p_c)\left[\gamma^2 + 2p_c \operatorname{sgn} p_b \frac{\sin(A_1 b)}{A_1}\frac{\sin\left(B_1 \frac{c}{|p_b|}\right)}{B_1} + \frac{4\sin^2\left(\frac{A_2}{2}b\right)}{A_2^2}\right]. \tag{6}$$

- Selecting the $\bar{\mu}$ prescription, by choosing $A_1 = \sqrt{\frac{\Delta}{p_c}}, B_1 = \sqrt{p_c \Delta}$, $A_2 = 2A_1$, (6) matches Chiou[3] for $n = 1$, and Joe and Singh[4] for $n = 0$.

## Discussion

- Although avoiding choosing a particular quantization prescription, the results force to a kind of $\bar{\mu}$-prescription, since $c$ can only appear in the shift coefficients in the form $\frac{c}{|p_b|}$ to guarantee covariance under residual diffeomorphisms.
- Requiring minimality – an Occam's razor assumption – matches the result with others previously presented in the literature, reached by the standard quantization method in LQC.
- However, it is worth to stress that minimality is not a physical requirement, and has the weakness that it selects a unique result and does not allow different possible dynamics of the full theory to be represented.

## References

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