The chaotic behavior of the Bianchi IX model under the influence of quantum effects

M. Bojowald1, D. Brizuela2, P. Calizaya-Cabrera3, S.E. Uria2
1Institute for Gravitation and the Cosmos, The Pennsylvania State University, University Park, USA
2Department of Physics and EHU Quantum Center, University of the Basque Country, Bilbao, Spain
3Department of Physics and Astronomy, Louisiana State University, Baton Rouge, USA
www.fermatdos@ehu.eus

According to the Belinski-Khalatnikov-Iliashenko conjecture, close to a generic spacelike singularity different special points decouple, and the evolution of each of them can be described in terms of the Bianchi IX model, one of the most general spatially homogeneous but anisotropic spacetimes. This dynamics, among other properties, is known to be chaotic. However, close to the singularity, quantum effects must be considered; thus, in this work we aim to answer:

Do quantum effects modify the chaotic behavior of the Bianchi IX model?

Classical model:

Hamiltonian constraint

\[ C = \frac{1}{2}e^{-3\alpha}(-p_0^2 + p_1^2 + p_2^2) + e^\alpha V(\beta_1, \beta_2) = 0 \]

\[(a, b, c) = (\beta_1, \beta_2, \beta_3)^{-1}\]

"Planckian"

\[ \alpha \rightarrow \text{Spatial volume} \]

Physical Hamiltonian

\[ H = -p_4 = |p_4^2 + p_2^2 + 2e^{2\alpha}V(\beta_1, \beta_2)|^{1/2} \]

Equations of motion

\[ \frac{dp_4}{dt} = \frac{\partial H}{\partial p_4} = \frac{p_4}{|p_4^2 + p_2^2 + 2e^{2\alpha}V(\beta_1, \beta_2)|^{1/2}} \]

Sequence of free dynamics + bounces:

1. When \( V(\beta_1, \beta_2) \) is negligible: free dynamics
2. When \( V(\beta_1, \beta_2) \) is relevant: bounce against the potential walls of Fig. 1

Along the 3 symmetry semiaxes of Fig. 1 no potential wall is encountered, and no bouncing will happen: these directions define the 3 exits of the system.

The system is analogous to a point particle with coordinates \((\beta_1, \beta_2)\) moving in a potential well with 3 exits

Final state of the trajectories

1. Infinite sequence of bounces
2. "Escape" through one of the exits

A slight difference in the initial conditions might imply a different final state:

Chaotic motion

How to prove that the system is chaotic? 2 main methods

1. Lyapunov exponent:


- Each trajectory on the phase space is isomorphic to a geodesic on a Riemannian manifold with metric:
  \[ ds^2 = E^2 \frac{d\xi^2}{(\xi^2 - 1)} + (\xi^2 - 1)d\theta^2 \]

- The geodesic deviation equation shows that initially close geodesics exponentially diverge → Lyapunov exponent \( \lambda > 0 \)

2. Fractal methods:


Classify each point of the space of initial data in terms of its final state (exit):

- Exit 1
- Exit 2
- Exit 3

Fractal structure → Chaos

Study of chaos in the quantum model:

Apply the above formalism to the Bianchi IX model

Semiclassical approximation

1. Small relative fluctuations of the Hamiltonian: \( H_0 = \langle H \rangle \approx \langle H^2 \rangle^{1/2} \)
2. Gaussian-like state all along the evolution:

\[ \langle \Delta(\beta_1)^2(\beta_2)^2 \rangle = \frac{e^{2\alpha}2^m(2m)!2(2m)!}{2^m m! m!} \]

\[ \langle \Delta(\beta_1)^2(\beta_2)^2 \rangle = 0 \text{ otherwise} \]

Transformation to canonical variables \((n_1, n_2) = (u_j, j = 1, 2)\):

\[ \Delta(p_1) = p_1^2 + \frac{U_1}{s_1}, \Delta(p_2) = p_2^2 + \frac{U_2}{s_2} \]

with \(U_1, U_2\) constants

Hamiltonian of the quantum model

Thus

\[ H_0 = \frac{1}{s_1} + \frac{1}{s_2} + \frac{U_1}{s_1} + \frac{U_2}{s_2} + 2e^{2\alpha/2}V(\beta_1, \beta_2, s_1, s_2) \]

In this way, we get a finite phase space, which is an extension of the classical one.

Apply the previous methods to compare with the classical results:

1. Lyapunov exponent:

- As in the classical case, each trajectory on the phase space is isomorphic to a geodesic on a Riemannian manifold with metric:

\[ ds^2 = E^2 \frac{d\xi^2}{(\xi^2 - 1)} + (\xi^2 - 1)d\theta^2 + (\xi^2 - 1)^2 \sin^2 \theta d\sigma^2 + \frac{(\xi^2 - 1)^2 \sin^2 \theta d\phi^2}{\xi^2 - 1} \]

- The geodesic deviation equation shows that initially close geodesics exponentially diverge → Lyapunov exponent \( \lambda > 0 \)

The quantum model is still chaotic

2. Fractal methods:

Less fractal → Less chaotic

Conclusion: quantum effects reduce the level of chaos of the Bianchi IX model

Description of the quantum dynamics in terms of moments:


Information of the quantum state

Wave function

Infinite set of quantum moments

\[ x = \langle x \rangle, p = \langle p \rangle \]

The dynamics of the quantum moments are governed by the following Hamiltonian:

\[ H_0 = \langle H(x, p, t) \rangle = H + \sum_{m \in \mathbb{N}} \frac{1}{m! m!} \frac{\partial^{m+n}(H(x, p, t))}{\partial x^m \partial p^n} \frac{\partial^n H(x, p, t)}{\partial x^m \partial p^n} \]

Equations of motion

\[ \frac{dx}{dt} = [x, H_0] = \frac{\partial H_0}{\partial p} \frac{dp}{dt} = [p, H_0] = \frac{\partial H_0}{\partial x} \frac{dx}{dt} = \frac{\partial H_0}{\partial p} \frac{dp}{dt} = [x, p] \]

Completely equivalent to the evolution given by the Schrödinger equation

Fig. 1: Division of the quantum space of initial conditions according to the exit through which the system escapes. Each color represents one exit. The fractal structure is clearly smoothed with respect to Fig. 2.