The chaotic behavior of the Bianchi IX model under the influence of quantum effects

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According to the *Belinskii-Khalatnikov-Lifshitz* conjecture, close to a **generic spacelike singularity** different special points decouple, and the evolution of **each** of them can be described in terms of the *Bianchi IX* model, one of the most general spatially homogeneous but anisotropic spacetimes. This dynamics, among other properties, is known to be **chaotic.** However, close to the singularity, **quantum effects** must be considered; thus, in this work we aim to answer:

Do quantum effects modify the chaotic behavior of the Bianchi IX model?

Classical model:

Hamiltonian constraint

$$C=\frac{1}{2}e^{-3\alpha}(-p_\alpha^2+p_+^2+p_-^2)+e^{\alpha}\,V(\beta_+,\beta_-)=0$$

$\beta_+,\beta_- \longrightarrow$ *Shape parameters (encode the spatial anisotropy)*
$\alpha \longrightarrow$ *Spatial volume*
"Potential"

$\{\alpha,p_\alpha\}=\{\beta_+,p_+\}=\{\beta_-,p_-\}=1$

α can be chosen as the **internal time variable** (spatial **singularity** at $\alpha\to-\infty$):

- *Solve $C=0$ for p_α:*

Physical Hamiltonian $\quad H:=-p_\alpha=[p_+^2+p_-^2+2e^{4\alpha}\,V(\beta_+,\beta_-)]^{1/2}$

Equations of motion

$$\frac{d\beta_\pm}{d\alpha}=\{\beta_\pm,H\}=\frac{\partial H}{\partial p_\pm}=\frac{p_\pm}{H},\qquad \frac{dp_\pm}{d\alpha}=\{p_\pm,H\}=-\frac{\partial H}{\partial\beta_\pm}=-\frac{e^{4\alpha}}{H}\frac{\partial V}{\partial\beta_\pm}$$

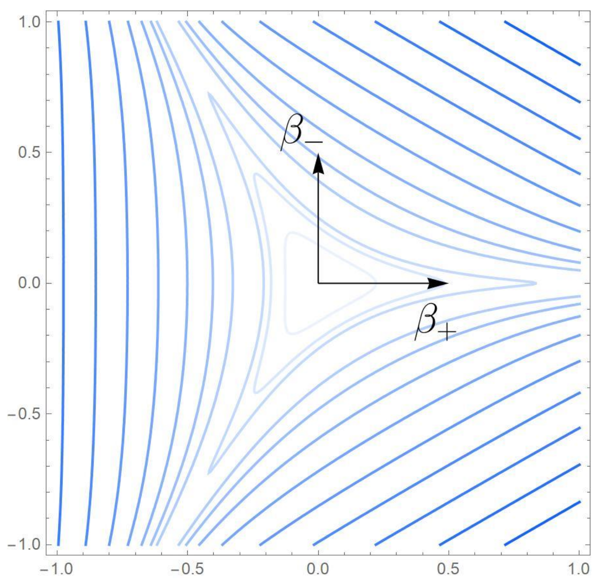
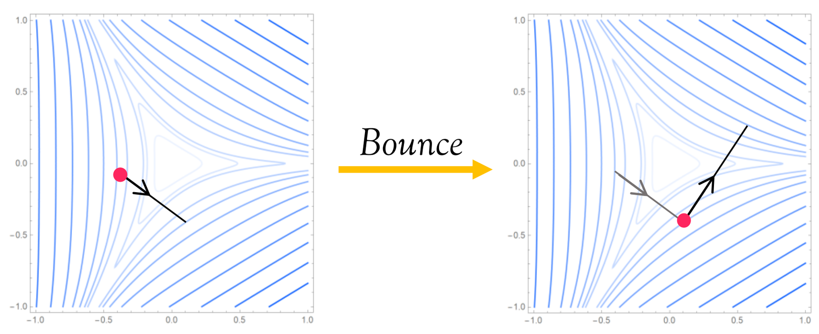


Fig. 1: Equipotential lines of $V(\beta_+,\beta_-)$.

Sequence of free dynamics + bounces:

- When $V(\beta_+,\beta_-)$ is **negligible**: free dynamics
- When $V(\beta_+,\beta_-)$ is **relevant**: bounce against the potential walls of *Fig. 1*



✦ Along the **3 symmetry semiaxes** of *Fig. 1* **no potential wall** is encountered, and **no bouncing** will happen: these directions define the **3 exits** of the system

The system is **analogous** to a point particle with coordinates (β_+,β_-) moving in a **potential well** with **3 exits** — *Similar to the billiard picture*

Final state of the trajectories
1. **Infinite** sequence of **bounces**
2. **"Escape"** through one of the **exits**

✦ A **slight difference** in the **initial conditions** might imply a **different final state:**

Chaotic motion

How to prove that the system is chaotic? *2 main methods*

1. **Lyapunov exponent:**

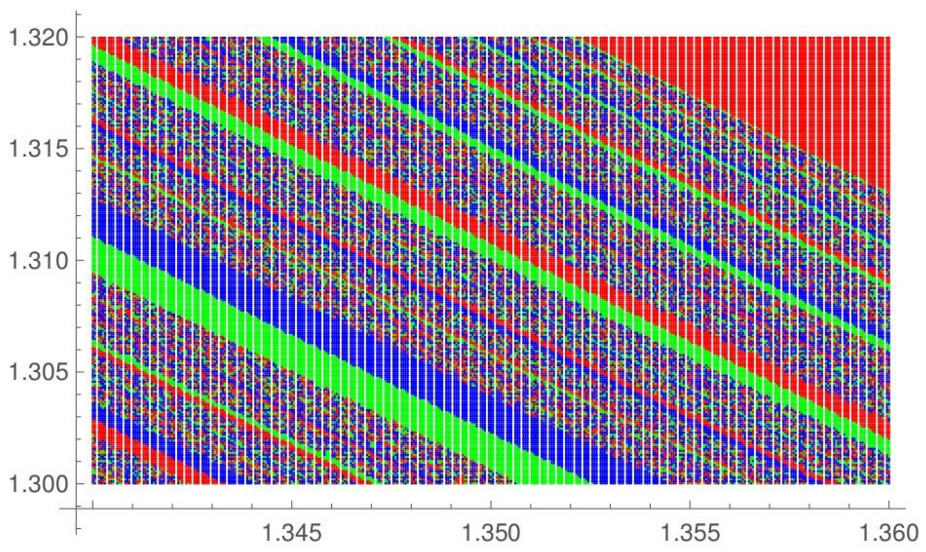
G. P. Imponente and G. Montani, *On the Covariance of the Mixmaster Chaoticity*, Phys. Rev. D **63**, 103501 (2001)

- **Each trajectory** on the phase space is **isomorphic** to a **geodesic** on a Riemannian manifold with metric $\; ds^2=E^2\left[\frac{d\xi^2}{\xi^2-1}+(\xi^2-1)d\theta^2\right]$
- The geodesic deviation equation shows that **initially close** geodesics **exponentially diverge** → Lyapunov exponent $\lambda>0$ ⟷ **Chaos**

2. **Fractal methods:**

N. J. Cornish and J. J. Levin, *The Mixmaster universe: A Chaotic Farey Tale*, Phys. Rev. D **55**, 7489 (1997)

Classify each point of the space of **initial data** in terms of its **final state (exit)**



Fractal structure ⟷ **Chaos**

Fig. 2: Division of the space of initial conditions according to the exit through which the system escapes. The dimension of the boundary between regions with different color is fractal, which gives a measure of the level of chaos.

Description of the quantum dynamics in terms of moments:

M. Bojowald and A. Skirzewski, *Effective equations of motion for quantum system*, Rev. Math. Phys. **18**, 713 (2006)

Information of the quantum state

$$\psi(x,t)\xleftrightarrow{\text{equivalent}}\{\Delta(x^m p^n):=\langle(\hat{x}-x)^m(\hat{p}-p)^n\rangle\}_{m,n=0}^{+\infty}$$

Wave function — *Infinite set of quantum moments*

$x:=\langle\hat{x}\rangle,\; p:=\langle\hat{p}\rangle$

The **dynamics** of the **quantum moments** are governed by the following Hamiltonian:

$$H_Q:=\langle\hat{H}(\hat{x},\hat{p},t)\rangle=H+\sum_{m+n=2}^{+\infty}\frac{1}{m!\,n!}\frac{\partial^{m+n}H(x,p,t)}{\partial x^m\partial p^n}\Delta(x^m p^n)$$

Hamiltonian operator — *Classical Hamiltonian*

Equations of motion $\quad \{\langle\hat{f}\rangle,\langle\hat{g}\rangle\}=\frac{1}{i\hbar}\langle[\hat{f},\hat{g}]\rangle$

$$\frac{dx}{dt}:=\{x,H_Q\}=\frac{\partial H_Q}{\partial p},\quad \frac{dp}{dt}:=\{p,H_Q\}=-\frac{\partial H_Q}{\partial x},\quad \frac{d\Delta(x^m p^n)}{dt}:=\{\Delta(x^m p^n),H_Q\}$$

Completely **equivalent** to the evolution given by the **Schrödinger equation**

Study of chaos in the quantum model:

Apply the above formalism to the Bianchi IX model

Semiclassical approximation

1. **Small** relative **fluctuations** of the **Hamiltonian**: $H_Q:=\langle\hat{H}\rangle\approx\langle\hat{H}^2\rangle^{1/2}$
2. **Gaussian-like** state all along the evolution
$$\begin{cases}\Delta(\beta_+^{2n}\beta_-^{2m})=\frac{s_1^{2n}}{2^n}\frac{s_2^{2m}}{2^m}\frac{(2n)!}{n!}\frac{(2m)!}{m!}, & \begin{pmatrix}n,m\in\mathbb{N},\\ s_1,s_2\in\mathbb{R}^+\end{pmatrix}\\ \Delta(\beta_+^n\beta_-^m)=0 & \text{otherwise}\end{cases}$$

✦ Transformation to **canonical variables** $\{s_i,p_{s_j}\}=\delta_{ij}\;(i,j=1,2)$:

$$\Delta(p_+^2)=p_{s_1}^2+\frac{U_1}{s_1^2},\quad \Delta(p_-^2)=p_{s_2}^2+\frac{U_2}{s_2^2},\quad \text{with } U_1,U_2 \text{ constants}$$

Hamiltonian of the quantum model

Thus

$$H_Q\approx\left[p_+^2+p_-^2+p_{s_1}^2+\frac{U_1}{s_1^2}+p_{s_2}^2+\frac{U_2}{s_2^2}+2e^{4\alpha}\,V_Q(\beta_+,\beta_-,s_1,s_2)\right]^{1/2}$$

In this way, we get a **finite phase space**, which is an **extension** of the classical one

Apply the previous methods to compare with the classical results:

1. **Lyapunov exponent:**

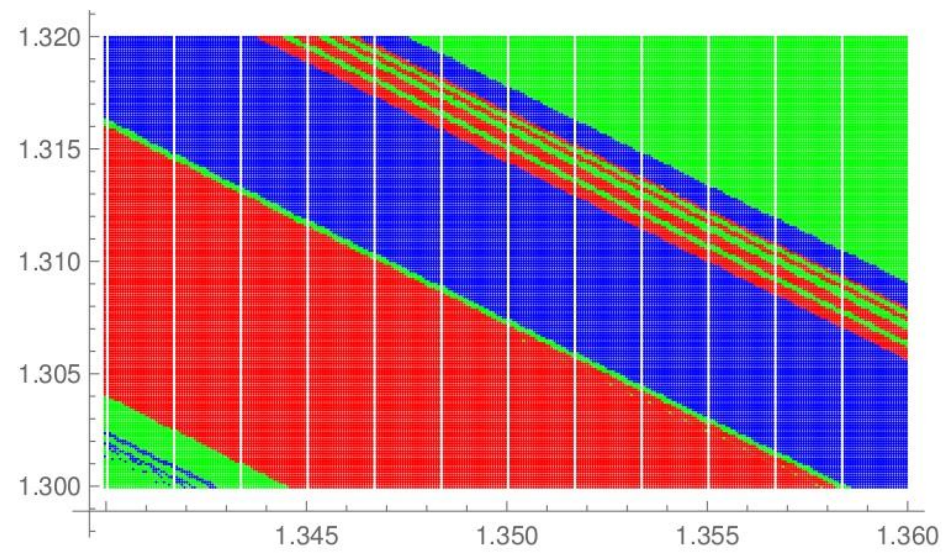
- As in the classical case, **each trajectory** on the phase space is **isomorphic** to a **geodesic** on a Riemannian manifold with metric

$$ds^2=E^2\left[\frac{d\xi^2}{\xi^2-1}+(\xi^2-1)d\theta^2+(\xi^2-1)\sin^2\theta\,d\sigma^2+(\xi^2-1)\sin^2\theta\sin^2\sigma\,d\phi^2\right]$$

Extension of the classical one

- The geodesic deviation equation shows that **initially close** geodesics **exponentially diverge** → Lyapunov exponent $\lambda>0$ ⟷ *The quantum model is* ***still chaotic***

2. **Fractal methods:**



Less fractal ⟷ **Less chaotic**

Conclusion: quantum effects **reduce** the **level of chaos** of the Bianchi IX model

Fig. 3: Division of the quantum space of initial conditions according to the exit through which the system escapes. Each color represents one exit. The fractal structure is clearly smoothed with respect to *Fig. 2*.