Semi-classical gravity as a characteristic initial value problem

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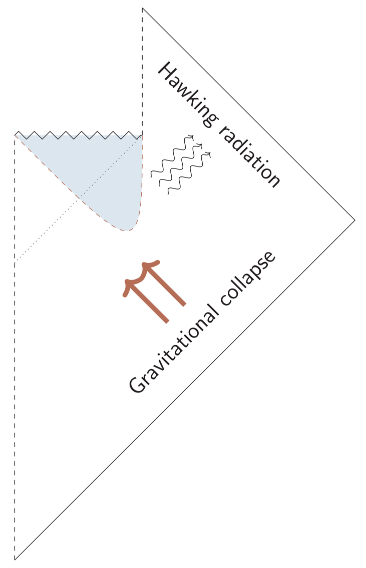
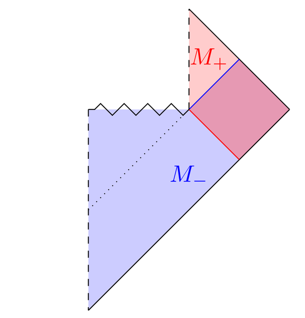
1. Evaporating black holes

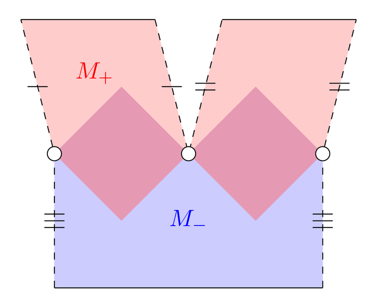
- Black hole evaporation, introduced in *(Hawking, 1975)*, has been studied extensively. Yet we lack a satisfactory understanding of the local dynamics and relation to thermodynamics & information theory.
- Black hole formation, both in the sense of formation of a horizon and a singularity, is poorly understood in semi-classical gravity.
- Can one analyze horizon dynamics beyond adiabatic+Unruh state approximation?
- Does semi-classical gravity admit black hole solutions in the first place?
- Can the full lifetime of semi-classical black holes be studied via an initial value problem?



A typical picture of black hole formation and evaporation



(a) A time ordered cover of a BHE space-time



(b) A time ordered cover of a topology changing space-time

- Can the causal structure of black hole evaporation (BHE) space-times be consistent with quantum field theory? Compare to other semi-globally hyperbolic space-times (see Ref. *(Janssen, 2022)*), such as topology changing space-times.

2. The semi-classical Einstein equations

$$G_{\mu\nu} = \kappa \langle : T_{\mu\nu} : \rangle . \qquad (1)$$

Classical dynamical background
Einstein tensor of a (globally hyperbolic) (3+1 D) space-time

Quantum field theory
Renormalized stress energy tensor of
- Linear scalar field (our toy model)
- Standard model
- ...

3. Renormalization in QFT on curved backgrounds

Nonlinear observables for a linear scalar field need to be renormalized. For example, the vacuum polarization $\langle : \Phi^2 : \rangle$ for a state with (Hadamard) 2-pt function Λ is given by

$$\langle : \Phi^2 : \rangle = \lim_{x' \to x} (\Lambda - H)(x, x') + \text{ren.fr.},$$
$$H \sim \frac{U}{\sigma + i0^+ \Delta t} + V \ln(\sigma + i0^+ \Delta t), \qquad (2)$$

σ Synge world function, Δt a time difference regulator, U, V smooth (for 3+1 dim).

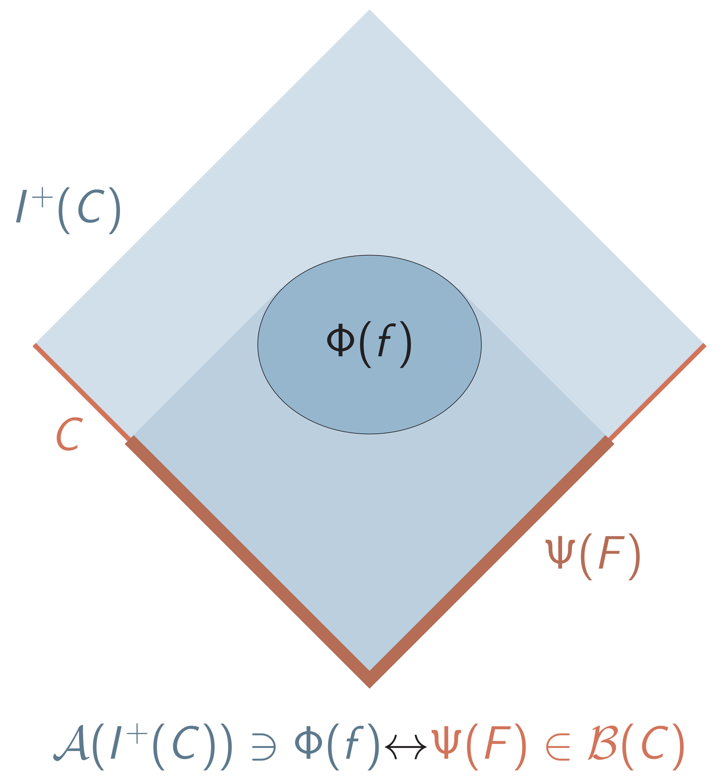
H a priori only locally defined, difficult to calculate explicitly beyond local expansions, how to control (coincidence limit of) $\Lambda - H$, and hence $\langle : \Phi^2 : \rangle$, $\langle : T_{\mu\nu} : \rangle$ globally?

- **Lower dimensional models** *(Unruh 1976, Juárez-Aubry & Louko, 2018)*
  Conformal flatness in 1+1 dim significantly simplifies analysis.
- **Weak field approximation** *(Horowitz, 1980, Flanagan & Wald, 1996)*
  Linear response of $\langle : T_{\mu\nu} : \rangle$ to metric perturbations around flat background.
- **Euclidean methods** *(Howard & Candelas, 1984, Taylor et al., 2022)*
  Applicable to static settings, such as Hartle-Hawking state on Schwarzschild BH.
- **Pragmatic mode sum method** *(Levi & Ori, 2016)*
  Numerical method, been used to calculate Unruh state stress-tensor on Kerr black holes. Highly computationally expensive beyond stationary setting.
- **Adiabatic renormalization** *(Parker & Fulling, 1974, Junker & Schrohe, 2003)*
  Scheme adapted to 3+1 split into Cauchy slices, often applied in cosmological setting. Generally requires pseudo-differential operator theory.

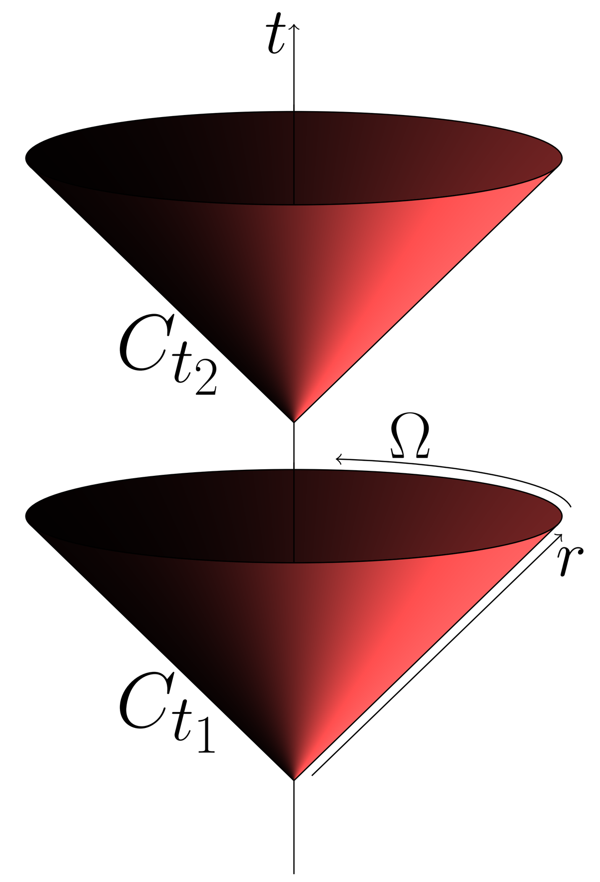
4. A characteristic approach

For M globally hyperbolic & spherically symmetric, $C \subset M$ characteristic cone s.t. $I^+(C) \subset M$ globally hyperbolic, one can consider boundary observables $\mathcal{B}(C) \cong \mathcal{A}(I^+(C))$ for a linear scalar QFT $\mathcal{A}(M)$.

A *boundary two point function* $\lambda \in \mathcal{D}'(C^2)$ satisfying a particular microlocal condition induces a Hadamard two-point function $\Lambda \in \mathcal{D}'(I^+(C)^2)$ *(Gerard & Wrochna, 2016)*.



(a) Bulk to (null) boundary correspondence



(b) Foliation by nullcones

We consider a (spherically symmetric) 'foliation' of M by nullcones. A (global) two-point function $\Lambda \in \mathcal{D}'(M^2)$ induces a family of (formal) boundary two-point function λ_t associated with states on $\mathcal{B}(C_t)$ via formal relation

$$\lambda_t(r, \Omega; r', \Omega') \sim \partial_r \partial_{r'} r r' \Lambda\restriction_{C_t^2}(r, \Omega; r', \Omega'). \qquad (3)$$

The evolution of a bulk 2-pt function Λ can be studied through the evolution of λ_t as a function in time t.

5. Characteristic Hadamard states

A (global) two point function λ on M is Hadamard (i.e. locally $\Lambda = H + W$) if and only if the corresponding family of boundary two-point functions λ_t on $C_t \cong \mathbb{R}^+ \times S^2$ satisfies $\lambda_t = h_t + w_t$ with $w_t \in \mathcal{E}(C_t^2)$

$$h_t \sim \frac{u_t \delta_{S^2}(\Omega, \Omega')}{(r - r' - i0^+)^2} + v_t \ln(r r'(1 - \cos(\theta(\Omega, \Omega')))), \qquad (4)$$

$r, r' \in \mathbb{R}^+$ radial (null) coordinate, $\Omega, \Omega' \in S^2$ angular coordinate, u_t, v_t smooth on C_t^2.

u_t, v_t can be derived independently of U, V *(Janssen & Verch, 2023)*.

Note that $h_t \in \mathcal{D}'(C_t^2)$ is defined on the entire cone, hence w_t globally defined (i.e. for each $t \in \mathbb{R}$ and $p, p' \in C_t$, one has a naturally defined $w_t(p, p')$).

6. The characteristic regularized two-point function

Via the formal relation $\lambda_t \leftrightarrow \Lambda\restriction_{C_t^2}$, w_t induces a $\tilde{W}_t \in \mathcal{E}(C_t^2)$ such that locally $Q_t = W\restriction_{C_t^2} - \tilde{W}_t$ smooth and state-independent. Nonlinear observables can be expressed in terms of $\tilde{W}_t$:

$$\langle : \Phi^2 : \rangle(t, r, \Omega) = \tilde{W}_t(r, \Omega; r, \Omega) + Q_t(r, \Omega; r, \Omega) + \text{ren.fr.}. \qquad (5)$$

$\tilde{W}_t$ is defined globally and satisfies a (sourced) dynamical equation

$$D^3 \tilde{W}_t = S_t(u_t, v_t), \qquad (6)$$

with $D^3 = \partial_t \partial_r \partial_{r'} + \ldots$ a third order differential operator.

Through analysing this dynamical equation (approximate solution, estimates, etc...), one gains control on nonlinear observables.

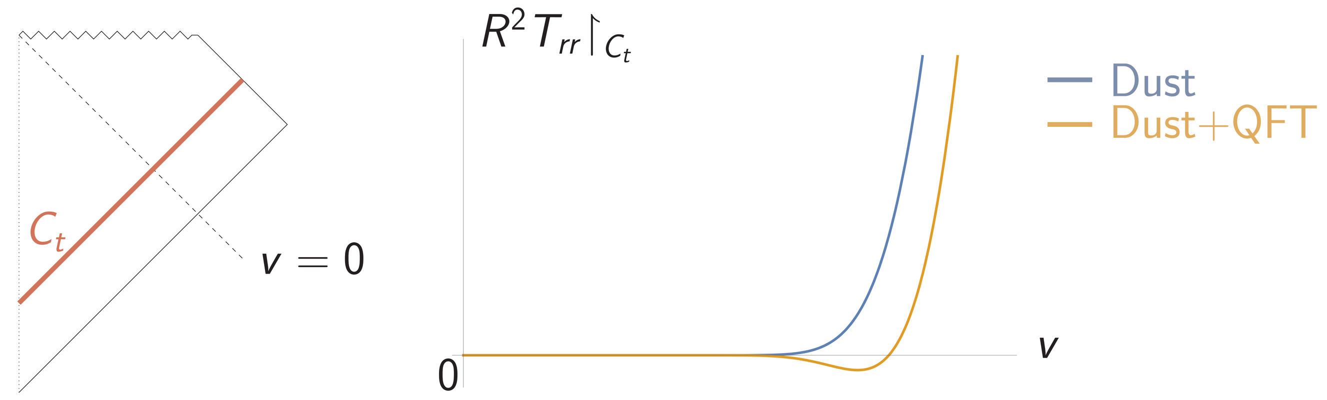
7. A weak field sanity check

Solving the dynamical equation $\tilde{W}_t$ in weak field approximation $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}$, one regains results of *(Horowitz, 1980)* adapted to spherical symmetry.

Approximation applicable for 'small v' region of ingoing Vaidya space-time

$$ds^2 = -\left(1 - \frac{R_h(v)}{R}\right) dv^2 + 2 dv dR + R^2 d\Omega^2, \qquad (7)$$

with massless scalar quantum field in vacuum state at $\mathscr{I}^-$.



One can find the dominant contribution to $\langle : T_{rr} : \rangle$ near $v = 0$

$$\langle : T_{rr} : \rangle \sim \frac{1}{120\pi^2 R^3} \int_0^v ds \ln(v - s) R_h'''(s) \qquad (8)$$

Dominates over classical stress-tensor for v sufficiently small, violation of (classical) energy conditions. More detailed bounds on $\langle : T_{rr} : \rangle$ required to estimate backreaction near horizon and implication on singularity.

8. Towards semi-classical black hole solutions

Einstein-QFT system for conformally coupled massless scalar field

$$\mathfrak{R} = -\kappa\,(\text{trace anomaly}), \quad R_{rr} = \kappa\left(T_{rr}^{reg}(\tilde{W}_t) + \text{geom.contr.}\right),$$
$$D^3 \tilde{W}_t = S(g_{\mu\nu}\restriction_{C_t}, \partial_t g_{\mu\nu}\restriction_{C_t}). \qquad (9)$$

Initial data required at C_{t_0}:
- $g_{\mu\nu}\restriction_{C_{t_0}}$, $\partial_t g_{\mu\nu}\restriction_{C_{t_0}}$ and sufficiently regular $\tilde{W}_{t_0}$,
- S.c.Einst.eq. impose (non-trivial) constraints

$T_{rr}^{reg}(\tilde{W}_t)$ can be expressed in terms of $\tilde{W}_{t_0}$ and $g_{\mu\nu}$, R_{rr} equation resembles semi-classical cosmology of *(Meda et al., 2021)*. Can their analysis be adapted to this setting?

9. Selected publications

- D.W. Janssen, Quantum fields on semi-globally hyperbolic space–times. *Commun. Math. Phys.* **391**, 669–705 (2022).
- D.W. Janssen & R. Verch, Hadamard states on spherically symmetric characteristic surfaces, the semi-classical Einstein equations and the Hawking effect. *Class. Quantum Grav.* **40**, 045002 (2023).

PhD thesis:
- D.W. Janssen, "Semi-classical aspects of black hole formation and evaporation", Leipzig (Mar. 2023)

10. Link to papers

The papers listed above are available online
Take a picture to download the full papers

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