Emergent Universe Model from Modified Heisenberg Algebra

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Abstract

The Emergent Universe is a non-singular, asymptotically Einstein-static model with a cosmological constant, a radiation fluid and positive curvature, that is obtained classically by the implementation of fine-tuning and of some constraints on the cosmological and matter parameters. Here we provide an Emergent Universe picture in which the fine-tuning on the initial conditions is replaced by cut-off physics, implemented on a semiclassical level when referred to the Universe dynamics and on a purely quantum level for the quantum fluctuations of the inflaton field. The adopted cut-off physics is inspired by Polymer Quantum Mechanics but expanded in the limit of a small lattice step, resulting for the Hamiltonian Universe dynamics in an algebra similar to a Generalized Uncertainty Principle representation. The calculation of the modified primordial inflaton spectrum is then performed by treating new physics as a small correction on the standard Hamiltonian of each Fourier mode of the gauge-invariant variable associated to the inflaton field. We provide a new paradigm for a non-singular Emergent Universe, associated to a precise fingerprint on the temperature distribution of the cosmic microwave background.

Classical Emergent Universe (EU)

The EU has gained attention recently due to the new Planck data favouring a positive curvature [1].

- The EU is a FLRW model with $\rho_\Lambda = \text{const.}$, $\rho_\gamma = \overline{\rho_\gamma}\, v^{-4/3}$ and positive curvature $K > 0$.
- It is obtained by imposing a constraint on the initial conditions for energy densities: $4\overline{\rho_\gamma}\rho_\Lambda = 9K^2$.
- Result: asymptotically Einstein-static model with minimum volume $v_i > 0$ and inflation [2].

$$H^2 = \frac{\rho_\gamma + \rho_\Lambda}{3} - \frac{K}{v^{2/3}}, \qquad v(t) = v_i\left(1 + e^{\pm\frac{\sqrt{2K}}{v_i^{1/3}}t}\right)^{\frac{3}{2}}$$

- To stop inflation at a finite time, a scalar field ϕ with a specific potential is introduced.
- When ϕ is on the plateau it acts as the cosmological constant and reproduces inflation.
- Later, ϕ falls in potential well and inflation ends at a finite time with finite $e-$folds.
- If the minimum of the potential is not zero, it could represent late-time Dark Energy.

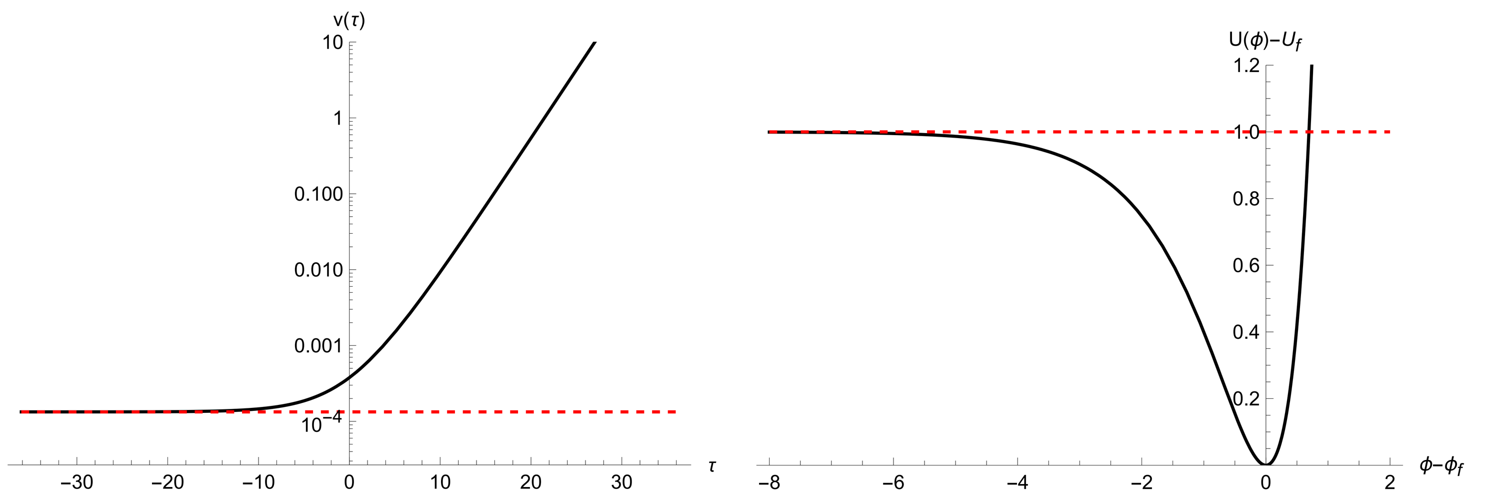

Figure 1: Volume $v = a^3$ as function of synchronous time t (left); potential $U(\phi)$ for the scalar field (right).

This model needs fine-tuning: it requires a particular choice of v_i, ρ_Λ and the initial kinetic energy to reproduce observational constraints and to ensure that the slow-roll approximation holds.

The Modified Algebra and the new Emergent Universe

Here we introduce a modified Heisenberg algebra inspired by PQM (LQC) and the GUP (Strings) [3].

$$[v, p_v] = i\,(1 - \mu^2 p_v^2), \qquad \{v, p_v\} = 1 - \mu^2 p_v^2$$

- In the Hamiltonian formulation we obtain a modified Friedmann equation for a generic density.

$$H^2 = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_\mu}\right)^2, \qquad \rho_\mu = \frac{3}{4\mu^2}, \qquad \left(\frac{v(t)}{v_i}\right)^{\frac{1+w}{2}} - \operatorname{arctanh}\left(\left(\frac{v(t)}{v_i}\right)^{\frac{1+w}{2}}\right) = \pm 3\,t\,\sqrt{\rho_\mu}$$

- The constant regularizing density ρ_μ naturally implements a non-zero minimum volume.
- Result: asymptotic behaviour without any constraint on the initial values of densities.

We choose to introduce positive curvature and integrate numerically three phases.

- Radiation dominated phase near the classical singularity.
- Scalar field-Cosmological Constant dominated phase to implement inflation.
- Radiation dominated phase after inflation to reproduce late-time Friedmann evolution.

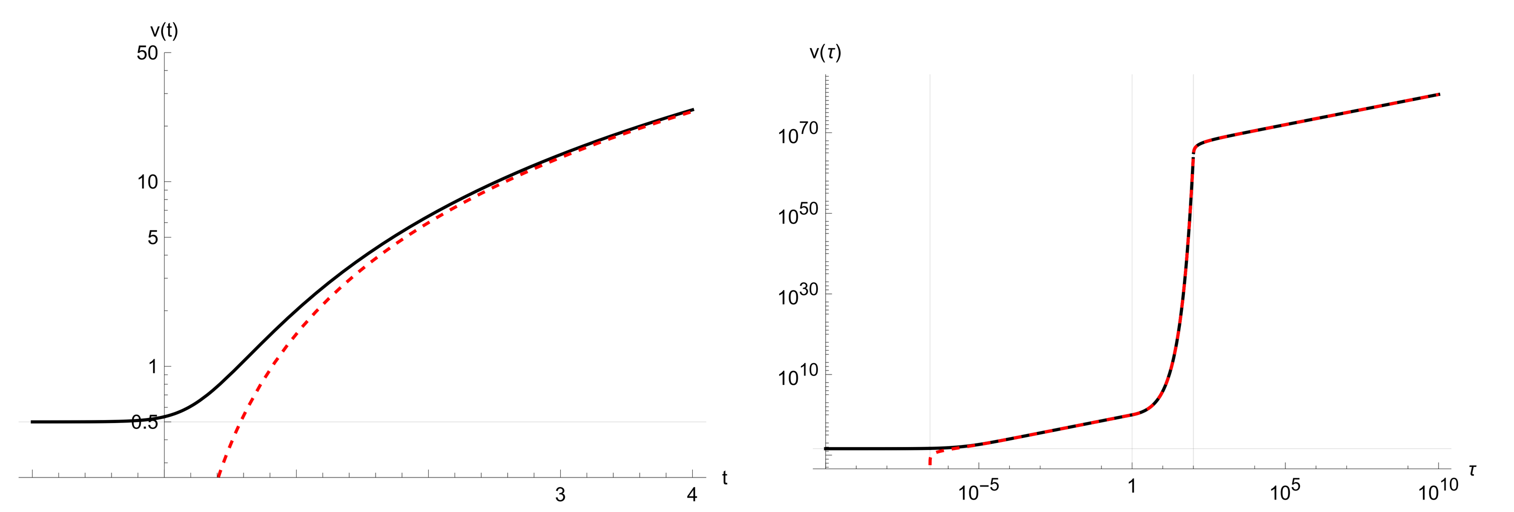

Figure 2: Volume $v(t)$ for a generic energy density in the modified algebra picture (left); full numerically-integrated evolution of $v(t)$ with the three phases separated by vertical lines, normalized at the start of inflation (right).

Standard Power Spectrum

Here we compute the Power Spectrum of primordial scalar perturbations.

- Introduce conformal time $d\eta = dt/a$ and the Mukhanov-Sasaki (MS) variable ξ [4]:
- Perform a Fourier decomposition and quantize each mode separately:

$$\left[\hat{\xi}_k, \hat{\pi}_k\right] = i, \qquad \pi_k = d\xi_k/d\eta, \qquad \hat{\xi}_k\psi(\xi_k) = \xi_k\psi(\xi_k), \qquad \hat{\pi}_k\psi(\xi_k) = i\frac{\mathrm{d}}{\mathrm{d}\xi_k}\psi(\xi_k).$$

- Result: each mode is a quantum harmonic oscillator with time-dependent frequency $\omega_k(\eta)$.
- Solve the time-dependent harmonic oscillator (TDHO) with the method of Invariants [5]:

$$\psi_n(\eta, \xi_k) = \frac{h_n(\frac{\xi_k}{f})}{\sqrt{2^n\, n!}}\,\frac{e^{-\frac{\xi_k^2}{2f^2}}}{(\pi f^2)^{\frac{1}{4}}}\, e^{i\frac{f'}{2f}\xi_k^2}\, e^{i\alpha_n}, \qquad \alpha_n(\eta) = -(n+\frac{1}{2})\int f^{-2} d\eta, \qquad f'' + \omega_k^2 f - \frac{1}{f^3} = 0.$$

- Compute the expectation value $\left\langle \hat{\xi}_k^2 \right\rangle$ on the vacuum state i.e. the TDHO ground state.

$$\mathcal{P}^{\text{std}}(k) = \frac{k^3}{4\pi^2}\frac{\langle 0|\hat{\xi}_k^2|0\rangle}{a^2\epsilon}\bigg|_{-k\eta\ll 1} = \frac{k^3}{4\pi^2}\frac{f^2(\eta)}{2a^2\epsilon}\bigg|_{-k\eta\ll 1} = \frac{H_s^2}{8\pi^2\epsilon}(1 + k^2\eta^2)\bigg|_{-k\eta\ll 1} = \frac{H_s^2}{8\pi^2\epsilon}.$$

The result is the usual Harrison-Zel'dovich scale-invariant spectrum.

Modified Power Spectrum

Here we quantize the Fourier mode of the MS variable according to the modified algebra prescription.

- We must work in the momentum polarization because of modified actions:

$$\left[\hat{\xi}_k, \hat{\pi}_k\right] = i(1 - \mu^2\hat{\pi}_k^2), \qquad \hat{\xi}_k\psi(\pi_k) = -i\frac{\mathrm{d}\psi}{\mathrm{d}\pi_k}, \qquad \hat{\pi}_k\psi(\pi_k) = \frac{\tanh(\mu\pi_k)}{\mu}\psi(\pi_k).$$

- Result: TDHO with a modified kinetic term, its Schrödinger equation cannot be solved.
- We perform a perturbative analysis through a series expansion in powers of μ^2:
- Result: two PDEs, first is the same as the standard case and solved with method of Invariants:

$$\psi_n^0(\eta, \pi_k) = (-i)^n \frac{h_n\left(\frac{\pi_k f}{|R|}\right)}{\sqrt{2^n\, n!}}\sqrt{\frac{(R^*)^n f}{R^{n+1}\sqrt{\pi}}}\; e^{-\frac{\pi_k^2 f^2}{2R}} e^{i\alpha_n}, \qquad R = 1 - iff'.$$

- Second is solved by using $\psi_n(\eta, \pi_k)$ as a complete basis and finding the coefficients:

$$\psi^0 = \sum_n c_n(\eta)\,\psi_n(\eta, \pi_k), \qquad \psi^1 = \sum_n d_n(\eta)\,\psi_n(\eta, \pi_k), \qquad i\sum_n \frac{\mathrm{d}\,d_n}{\mathrm{d}\eta}\,\psi_n^0(\eta, \pi_k) = -\frac{\pi_k^4}{3}\sum_n c_n(\eta)\,\psi_n^0(\eta, \pi_k).$$

- The ground state results to be $\psi_0^{\text{tot}}(\eta, \pi_k) = \psi_0 + \mu^2(d_0\psi_0 + d_2\psi_2 + d_4\psi_4)$.
- Now we calculate the expectation value $\left\langle \hat{\xi}_k^2 \right\rangle$ and the modified Power Spectrum:

$$\frac{\left\langle \hat{\xi}_k^2 \right\rangle}{|N|^2} = \frac{f^2}{2}\left(1 + \frac{2\sqrt{2}\,\mu^2\,\mathrm{Re}(d_2 e^{-2i\varphi})}{1 + 2\mu^2\,\mathrm{Re}(d_0)}\right), \qquad \mathcal{P}^{\text{mod}}(k) = \frac{H_s^2}{8\pi^2\epsilon}\left(1 - \frac{4\mu^2}{7k^5\eta^6}\right)\bigg|_{-k\eta\ll 1}.$$

- When $-k\eta \to 0$ our correction diverges; thus we compute the spectrum at the end of inflation η_f.

$$\mathcal{P}^{\text{mod}}(k) \approx \mathcal{P}^{\text{std}}\left(1 - 10^{-65}\left(\frac{m_P c^2}{E_\mu}\right)\beta^5\right), \qquad \beta = \frac{\overline{k}}{k}, \qquad \overline{k} = 0.002Mpc^{-1}.$$

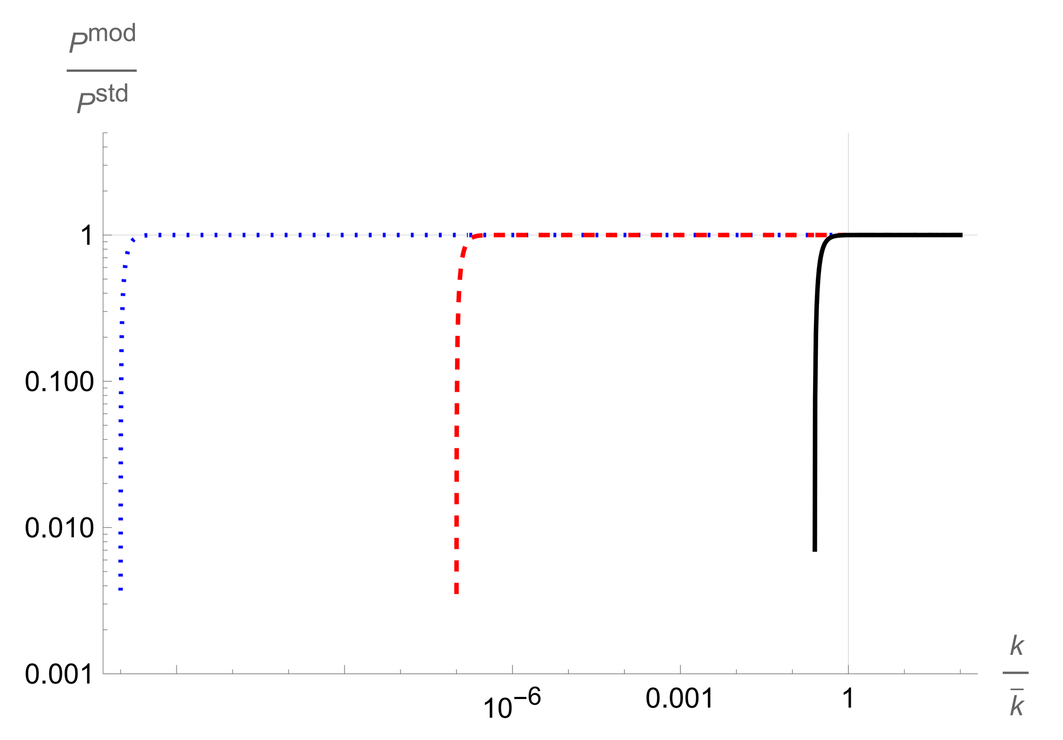

Figure 3: The modified Power Spectrum rescaled to the standard one for $r = m_P c^2/E_\mu = 10^{62}$ (black continuous line), $r = 10^{30}$ (red dashed line) and $r = 1$ (blue dotted line). The pivotal scale $k = \overline{k}$ is indicated by a faded grey line.

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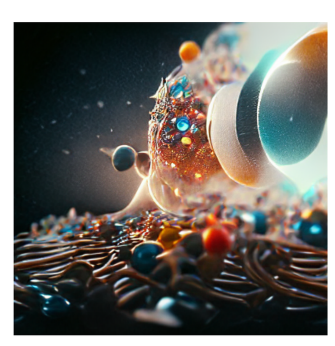

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