Lorentzian functional renormalization group: Hadamard property and state (in)-dependence.

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Quantum Gravity 2023

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J. Math. Phys **61** 103511 (2020)
Nucl. Phys. B **980** 115814 (2022)
arXiv: 2305.11388 (2023)

and To Appear...

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Introduction

The Functional Renormalization Group (FRG) is a leading non-perturbative tool for the study of Quantum Gravity, c.f. the Asymptotic Safety program.

However, the formalism relies heavily on Euclidean signature, and the adaptation to physically relevant Lorentzian manifolds is technically and conceptually challenging.

Different approaches towards Lorentzian signature pursued: [Litim et al. (2021), RB-MN (2022), d'Angelo et al. (2022), (2023)]

The formulation of the FRG flow equations for QFTs on Lorentzian signature backgrounds (and QG via background field method) has a number of qualitatively new features, in particular the dependence of the flow on an underlying "vacuum-like" state.

We illustrate this using scalar QFTs here.

UV flow by and large the same as that in Euclidean signature obtained via heat kernel technology.

IR flow depends on the "choice of state". Highly non-local – not accessible even through the non-local heat kernel.

State choice and the FRG

Take a globally hyperbolic Lorentzian manifold $\mathcal{M}$ as basic,

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,.$$

FRG flow equation for a scalar field on $\mathcal{M}$,

$$\boxed{k\partial_k \Gamma_k[\varphi] = \frac{i\hbar}{2}\mathrm{Tr}\Big\{ k\partial_k \mathcal{R}_k \cdot G_k[\varphi] \Big\}\,, \quad G_k[\varphi] := \big(-\Gamma^{(2)}_{k,\varphi\varphi} + \mathcal{R}_k \big)^{-1}\,,}$$

is obtained by introducing a mode modulator term $\Delta S_k[\chi] = \frac{\epsilon_g}{2}\chi \cdot \mathcal{R}_k(g) \cdot \chi$ to the bare scalar action $S[\chi]$.

To illustrate the issue of "state-choice/dependendence", consider the Local Potential Approximation (LPA)

$$\Gamma_k[\varphi] = -\int d^D y \sqrt{-g}\Big\{ -\frac{1}{2}\varphi \nabla^\mu \nabla_\mu \varphi + U_k(\varphi) \Big\}\,.$$

In Lorentzian signature, the flow is driven by the inverse of $-\Gamma^{(2)}_{k,\varphi\varphi} + \mathcal{R}_k$.

Green's functions

Omitting the modulator, consider the inverse of $-\Gamma^{(2)}_{k,\varphi\varphi} = \mathcal{D}\delta$, where

$$\mathcal{D} = -\frac{1}{N\sqrt{\gamma}} e_0\left(\frac{\sqrt{\gamma}}{N} e_0(\cdot)\right) + \frac{1}{N\sqrt{\gamma}}\partial_i\left(N\sqrt{\gamma}\gamma^{ij}\partial_j(\cdot)\right) + \partial^2_\varphi U_k(\varphi)\,, \quad e_0 := \mathcal{L}_{\partial_t - \vec{N}}$$

is a wave-type operator. Such an inverse is highly non-unique.

General results from perturbative QFT on curved backgrounds: the inverse $G[\varphi]$ of $\mathcal{D}$ should be of Hadamard form

$$G^{\text{Had}}(y,y') \asymp \frac{U(y,y')}{\sigma_\epsilon(y,y')^{\frac{d-1}{2}}} + V(y,y')\ln \mu^2\sigma_\epsilon(y,y')^2 + \mathcal{W}(y,y')\,,$$

$$\sigma_\epsilon(t,x;t',x') = \sigma(t,x;t',x') + i\epsilon(t-t') + O(\epsilon^2)$$

- U, V are fully determined by $\mathcal{D}$, have expansion in σ_ϵ with coefficients U_n and V_n related to the heat kernel coefficients A_n.
- $\mathcal{W}(y,y')$ is a smooth, state-dependent piece. Unconstrained outside of maximally symmetric or static geometries.

Goal: **Lift Hadamard property** to $(-\Gamma^{(2)}_{k,\varphi\varphi} + \mathcal{R}_k)^{-1}$.

State-independent UV

State-dependent IR

Full FRG: lift in perturbation theory

Starting from the flow equation in Lorentzian signature,

$$k\partial_k \Gamma_k[\varphi] = \frac{i\hbar}{2}\mathrm{Tr}\Big\{ k\partial_k \mathcal{R}_k \cdot G_k[\varphi] \Big\}\,, \qquad \Big[-\frac{\delta^2 \Gamma_k}{\delta\varphi\delta\varphi} + \mathcal{R}_k \Big] \cdot G_k[\varphi] = \mathbb{1}\,,$$

inserting the ansatz

$$\Gamma_k[\varphi] = S[\varphi] + \sum_{n\geq 1} \hbar^n \Gamma_{k,n}[\varphi]\,, \qquad G_k[\varphi] = G_{k,0}[\varphi] + \sum_{n\geq 1} \hbar^n G_{k,n}[\varphi]\,,$$

gives recursive system of flow equations

$$[-S^{(2)}[\varphi] + \mathcal{R}_k] \cdot G_{k,0}[\varphi] = \mathbb{1} \qquad \boxed{-S^{(2)}(\varphi) = \text{Wave type operator} + \text{potential}}$$

$$[-S^{(2)}[\varphi] + \mathcal{R}_k] \cdot G_{k,n}[\varphi] = \sum_{l=1}^{n} \Gamma^{(2)}_{k,l}(\varphi) \cdot G_{k,n-l}[\varphi]\,, \quad n \geq 1\,,$$

$$k\partial_k \Gamma_{k,n}[\varphi] = \frac{i}{2}\mathrm{Tr}\Big\{ k\partial_k \mathcal{R}_k \cdot G_{k,n-1}[\varphi] \Big\}\,, \quad n \geq 1\,.$$

In principle iteratively defines

$$G_{k,0} \;\to\; \Gamma_{k,1} \;\to\; G_{k,1} \;\to\; \Gamma_{k,2} \;\to\; \ldots$$

Lifted Hadamard property via modulated inverse Hessian

Appearance of "state dependence" in the FRG is clearly visible for:

1. Perturbative iteration of full FRG.
2. Non-perturbative LPA.

Lift of the Hadamard property to the (non-unique) inverse of $\mathcal{D} + \mathcal{R}_k$, a modulated wave-type operator, is technically feasible.

Coarse graining in Lorentzian signature?

- In Euclidean signature, the modulator $\mathcal{R}_k(\nabla_E^2)$ coarse-grains relative to $-\nabla_E^2 \geq 0$. Properties responsible for finiteness of RHS of flow eq.
- On foliated backgrounds $-\nabla_L^2 = \nabla_t^2 - \nabla_s^2$ with $-\nabla_s^2 \geq 0$.

Proposal: Modulate modes according to $-\nabla_s^2$, $\mathcal{R}_k(\nabla_E^2) \rightsquigarrow \mathcal{R}_k(\nabla_s^2)$. Renders RHS of the flow equation manifestly finite.

Other approaches preserve manifest covariance, but RHS is not finite.
[Litim et al. (2021), d'Angelo et al. (2022), (2023)]

State-independent UV aspects: Spatial FRG on FRW spacetimes

Study for LPA on FRW backgrounds $ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j$,

$$k\partial_k V_k + (d+1)V_k - \frac{d-1}{2}\varphi_0 V_k' = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} k\partial_k R_k(t,p)\mathcal{G}_k[\varphi_0](t,p)\,,$$

with $V_k(\varphi_0)$ the dimensionless potential and

$$-i\mathcal{G}_k[\varphi_0](t,p) := \hat{G}_k[\varphi_0](t,t,p)$$

is the temporal coincidence limit of the spatially Fourier transformed Green function.

$\mathcal{G}_k$ satisfies the Gelfand-Dickey equation,

$$2\mathcal{G}_k(a^d\partial_t)^2\mathcal{G}_k - (a^d\partial_t\mathcal{G}_k)^2 + 4a^{2d}\left[p^2/a^2 + k^2\left(r\left(\tfrac{p^2}{k^2a^2}\right) + V_k''\right)\right]\mathcal{G}_k^2 = +1\,.$$

Enables lift of Hadamard property to FRG

Theorem (RB-MN 2020, 2023)

Hadamard property is characterized by generalized resolvent expansion with coefficients determined by algebraic recursion.

This generalizes to the R_k modulated version: for large k and bounded $\wp := p/k$ one has

$$\mathcal{G}_k(t,p) \asymp \frac{1}{2a^d[\wp^2/a^2 + r(\wp^2/a^2) + V_k'']^{1/2} k} \Big\{ 1 + \sum_{n\geq 1} (-)^n \bar{G}_n(t) k^{-2n} \Big\} .$$

- Coefficients $\bar{G}_n(t)$ are again determined by an algebraic recursion.
- Large k behavior determined by coefficients $\bar{G}_n(t)$, independent of state.
- There is a correspondence

  $\bar{G}_n$ coeff. $\longleftrightarrow$ Spatially off diagonal Hadamard coeff. $\longleftrightarrow$ Spatially off diagonal Heat kernel coeff.

  so (up to justifiable Wick rotation in coefficients) the flow of V_k for large k is by-and-large the same as those obtained with the Euclidean formalism.

For details, see:

RB-MN, J. Math. Phys **61** 103511 (2020) , RB-MN, Nucl. Phys. B **980** 115814 (2022)

State dependent IR aspects: Spatial FRG on FRW and States of Low Energy

The small k regime of the flow is of considerable interest as the link to observables is through $\Gamma = \lim_{k\to 0} \Gamma_k$.

In the Euclidean formulation, the small k form of the flow equation

$$k\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\big\{ k\partial_k \mathcal{R}_k \, G_k[\varphi] \big\} \, ,$$

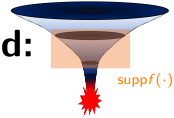
is accessible for (a) maximally symmetric backgrounds; or (b) via (non-local) heat kernel.

In the Lorentzian setting:

- Need to show existence of $G_k[\varphi]$ with lifted "Hadamard" property.
- Conceptually, expect "state-dependent" parts of $G_k[\varphi]$ to impact deep IR flow.
- Technically – how to extract φ dependence of RHS for small k?

Need explicit "Hadamard state" to study. Focus on spatial FRG with "States of Low Energy" (SLE).

SLE induced flow in the deep infrared: beyond the heat kernel



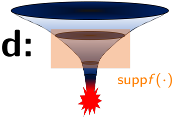
States of Low Energy (SLE) are a class of exact Hadamard states on FRW spacetimes, defined through the minimization of a temporally averaged energy functional (see [RB-MN, arXiv: 2305.11388 (2023)] for generalization to Bianchi I).

Construction readily generalizes to FRG effective potential flow eq. to give "lifted-Hadamard" $\mathcal{G}_k^{SLE}$ at *all* scales

$$k\partial_k V_k + (d+1)V_k - \frac{d-1}{2}\varphi_0 V_k' = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d} k\partial_k R_k(t,p)\mathcal{G}_k^{SLE}[\varphi_0](t,p)\,,$$

Can systematically access small RG scale flow analytically – $\mathcal{G}_k^{SLE}$ admits a convergent IR expansion!

SLE induced flow in the deep infrared: beyond the heat kernel



$$k\partial_k V_k + (d+1)V_k - \frac{d-1}{2}\varphi_0 V_k'$$

$$= \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty d\wp\, \wp^{d-1} \left(\frac{\int dt\, f(t)^2 a(t)^{-d}}{\int dt\, f(t)^2 a(t)^d [V_k'' + \wp^2/a^2 + r(\wp^2/a^2)]} \right)^{1/2} + O(k^2)$$

Small k form of flow equation:

- Valid for all FRW spacetimes, specializes correctly to Minkowski.
- Has well-defined infrared fixed point equation.
- Non-autonomous $O(k^2)$ terms arise from temporal non-locality in the SLE construction.
- Flow depends explicitly on the temporal averaging function $f(t)^2$, not recoverable from non-local HK.
- Cross-over to non-local HK regime to be studied.

Towards general backgrounds: Wick rotated Heat kernel

Euclidean FRG: heat kernel methods central.

On Lorentzian globally hyperbolic manifolds, a variant of the Wick rotation arises via lapse complexification

$$ds_\theta^2 = -e^{-i2\theta}N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \quad \theta \in [0,\pi]\,,$$

All coordinates and transition functions stay real!

This complexified line element has an associated complexified Laplacian ∇_θ^2, and QFT Hessian is $\mathcal{D}_\theta = -\nabla_\theta^2 + e^{-i\theta}U''(\varphi)$ with $U''(\varphi) \geq 0$.

Theorem (RB-MN 2023, to appear)

On generic globally hyperbolic manifolds, the complexified Hessian $-i\mathcal{D}_\theta$ generates for $\theta \in (0,\pi)$ a strongly continuous contractive semigroup $e^{-is\mathcal{D}_\theta}$ with a smooth kernel $K_s^\theta(t,x;t',x')$.

This rigorously defined Wick rotated HK's diagonal has a small s expansion

$$K_s^\theta(t,x;t,x) = \frac{(-ie^{i\theta})^{\frac{d+1}{2}}}{(4\pi s)^{\frac{d+1}{2}}}\left\{1 + (ie^{-i\theta}s)A_1^\theta(t,x) + (ie^{-i\theta}s)^2 A_2^\theta(t,x) + \dots\right\}$$

obtainable from the Euclidean version by $s \mapsto ie^{-i\theta}s$, and $A_n \mapsto A_n^\theta$.

The relation to a lapse Wick rotated Green's function is given by

$$G_z^\theta(t,x;t',x') = \int_0^\infty ds\, e^{-sz} K_s^\theta(t,x;t',x')\,, \quad z \in \rho(-i\mathcal{D}_\theta).$$

Use this in RHS of lapse Wick rotated FRG.

UV aspects: Small s asymptotic expansion of K_s^θ provides state independent UV aspects of the FRG flow with rigorous interpolation between Lorentzian and Euclidean signatures.

IR aspects: Expect that the state dependent IR aspects relate to coordinated $s \to \infty$ and $\theta \to 0^+$ (Lorentzian) limit. This highlights the need to have K_s^θ defined beyond a formal expansion.

Conclusions and Outlook

Lorentzian signature FRG is inherently state dependent.

- Only UV aspects of flow remain independent of details of state.
- Technical control over state dependence so far only in perturbation theory and local potential approximation.
- The state/Green's functions need to be known at all energy scales. So far only available for spatially homogenous backgrounds via States of Low Energy construction.
- Route to generic backgrounds via lapse Wick rotated heat kernel to be further investigated.

Thank you!