Loop Quantum Gravity and Spin Foams

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Hal M. Haggard,
Bard College

Loop gravity has its roots in casting General Relativity as a gauge theory

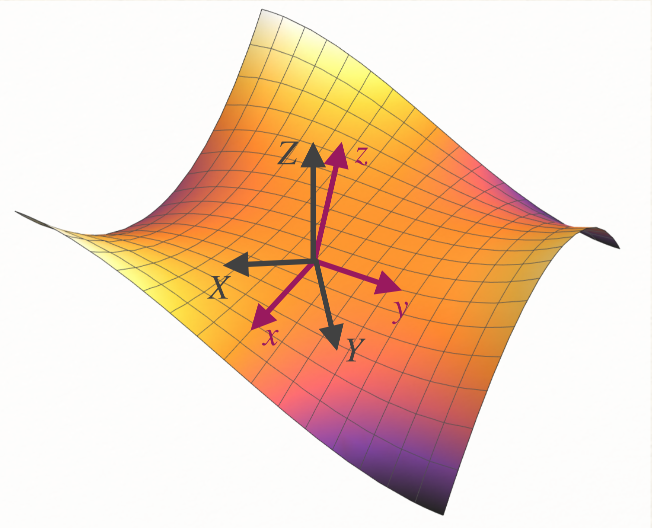
In particular,

Loop Quantum Gravity refers to the canonical theory where spacetime is split into space and time

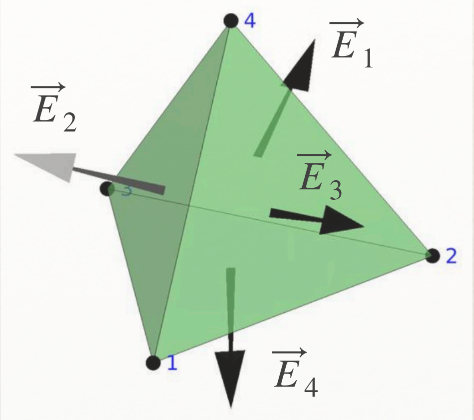
and

a Spin foam is a discrete geometry path integral approach based on similar algebraic structures

Outline

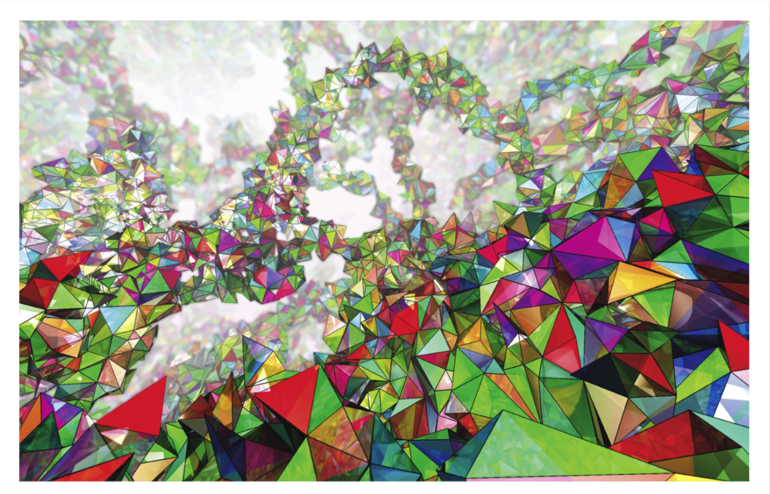


I. Loop Quantum Gravity:
GR as a canonical gauge theory



II. Connecting Approaches: Classical and Quantum Tetrahedra

III. Spin Foams: Discrete Geometry Path Integrals



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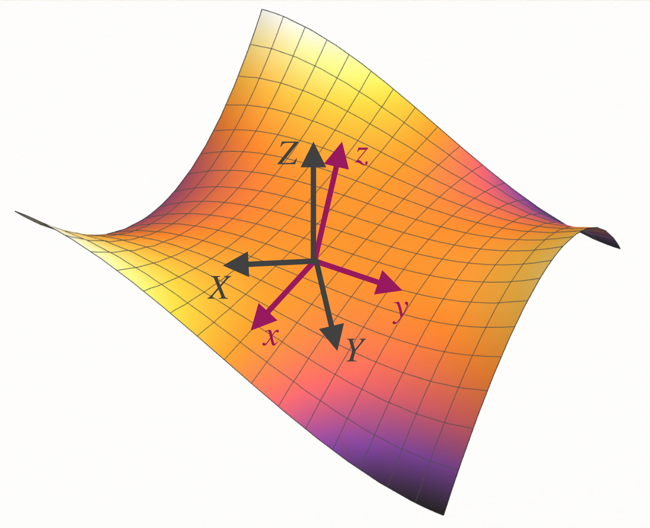

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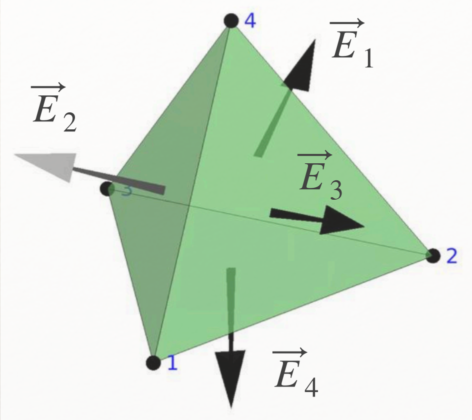
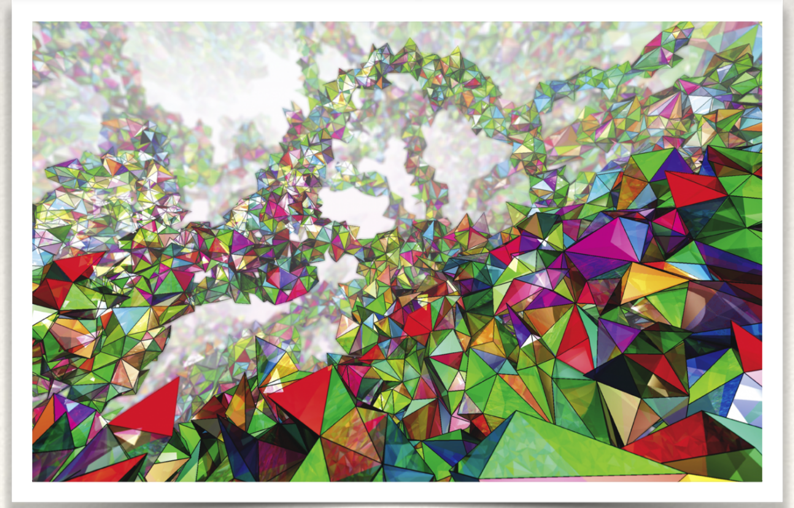
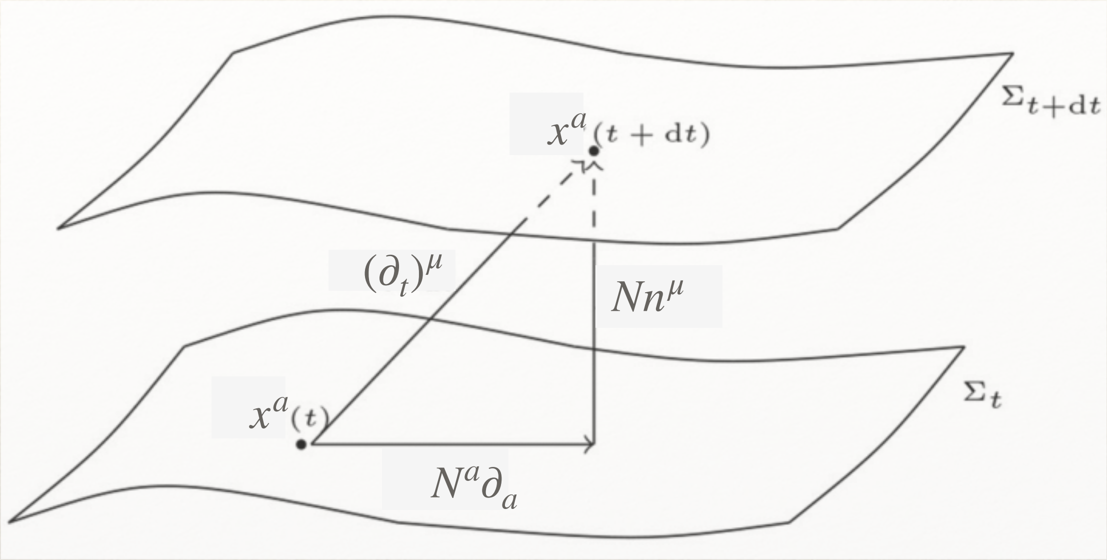


III. Spin Foams: Discrete Geometry Path Integrals



GR is deeply a theory of a connection, but…

The symplectic structure of the theory, in metric variables, doesn't cast the connection as one of the canonical variables.



In ADM:

canonical variables are $(q_{ab}, \tilde{\pi}^{ab})$

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \vec{N}^2 & q_{ab}N^a \\ q_{ab}N^b & q_{ab} \end{pmatrix},$$

$$K_{ab} = \frac{1}{2N}(\dot{q}_{ab} - 2D_{(a}N_{b)}), \;\; \& \;\; \tilde{\pi}^{ab} = \partial L/\partial \dot{q}_{ab} = \sqrt{q}(K^{ab} - q^{ab}K).$$

…so, the key to formulating GR as a gauge theory is to change variables

The Ashtekar electric field takes over for the spatial metric: it is a densitized triad field

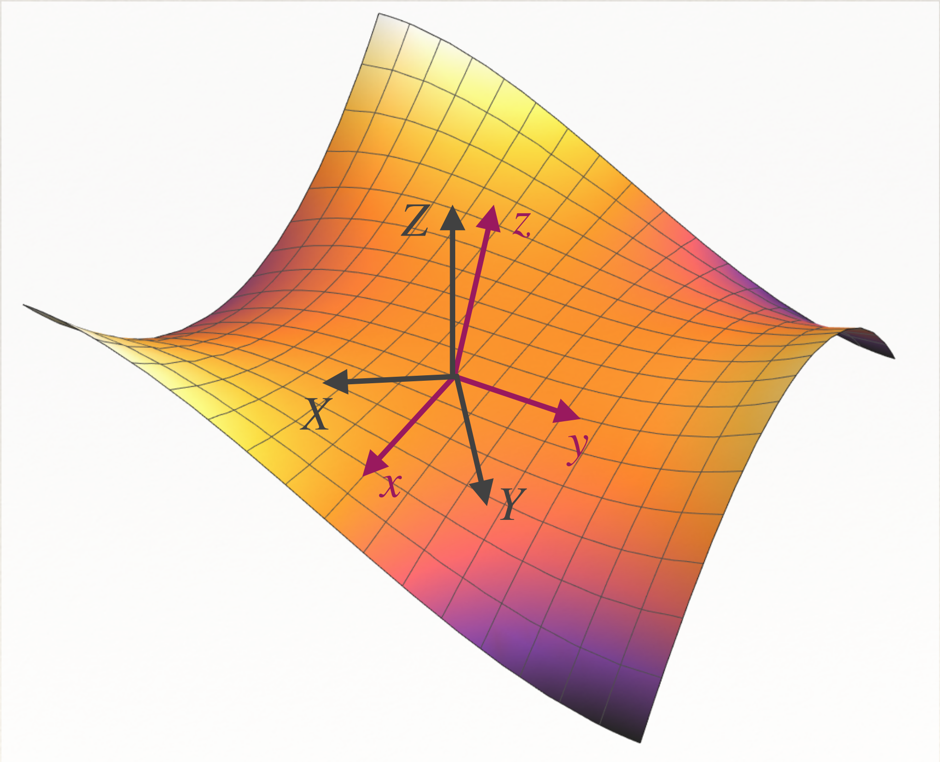
$$\tilde{E}^a_i = \sqrt{\det q}\, E^a_i,$$

that provides a 'square root' of the inverse metric

$$\tilde{E}^a_i \tilde{E}^{ib} = \det q\, q^{ab}.$$

A particularly nice organization of this variable is the 2-form

$$E^i(x) = \tilde{E}^{ia}(x)\epsilon_{abc} dx^b \wedge dx^c.$$



Spin connection split

Next we make a space-time split of the spin connection:

$$\omega^{0i} \to \text{boosts}$$

$$\omega^{ij} \to \text{spatial rotations,}$$

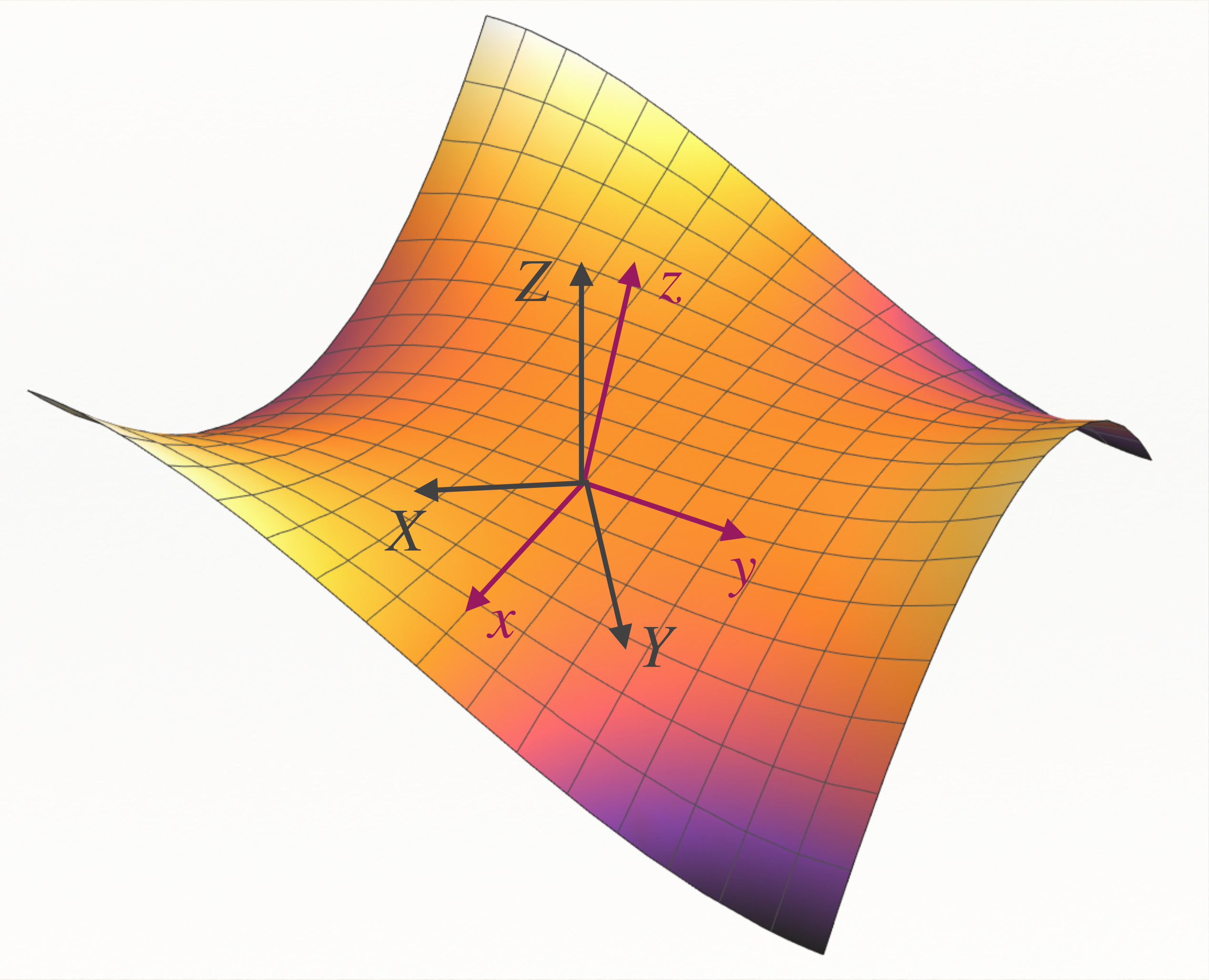
and, define

$$\Gamma_a{}^i := \frac{1}{2}\epsilon^i{}_{jk}\omega_a{}^{jk}.$$

Just as the (compatible, torsionless) spacetime spin connection is determined by the tetrad, here we have

$$\Gamma_a{}^i = \Gamma_a{}^i(E),$$

is determined by the triad.



The Ashtekar connection

Defining, $K_a^i := K_{ab}E^{bi}$, the $p\dot{q}$ term of the ADM Lagrangian becomes

$$\tilde{\pi}^{ab}\dot{q}_{ab} = \sqrt{q}(K^{ab} - q^{ab}K)2\dot{E}_{i(a}E^i_{b)} = 2\tilde{E}^a_i\dot{K}^i_a + \partial_t(\,*\,).$$

Thus, $\tilde{E}$ and K are conjugate variables, and schematically

$$\{q, \tilde{\pi}\} = 1, \quad \{q, q\} = 0, \quad \{\tilde{\pi}, \tilde{\pi}\} = 0$$
$$\{K, \tilde{E}\} = 1, \quad \{K, K\} = 0, \quad \{\tilde{E}, \tilde{E}\} = 0.$$

Connections have the freedom that you can add any vector, so

Ashtekar connection: $A^i_a := \Gamma^i_a + \mathrm{i}\,K^i_a$, with $\mathrm{i} := \sqrt{-1}$.

Thus, retaining conjugacy, and making A a connection,

$$\{A^i_a(x), \tilde{E}^b_j(y)\} = \mathrm{i}\,8\pi G\,\delta^i_j\delta^b_a\delta^{(3)}(x, y).$$

Gravity as an SU(2) gauge theory

The action is now

$$S[A^i_a, E^b_j] = \frac{1}{2\kappa}\int dt\, d^3x \left[\tilde{E}^a_i \dot{A}^i_a - N\mathscr{H} - A^i_0 \mathscr{G}_i - N^a \mathscr{V}_a\right]$$

with

Gauss constraint	$\mathscr{G}_i := D_a \tilde{E}^a_i \simeq 0$
Spatial diffeos	$\mathscr{V}_a := \tilde{E}^b_i F_{ab}{}^i \simeq 0$
Scalar constraint	$\mathscr{H} := \frac{1}{2}\epsilon^{ij}{}_k \tilde{E}^a_i \tilde{E}^b_j F_{ab}{}^k \simeq 0$

and the field strength

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + \epsilon^i{}_{jk} A^i_a A^j_b\,.$$

Full invariance is semidirect product of diffeos and SU(2) gauge.

Crux challenges

You will have noticed the $\mathrm{i} = \sqrt{-1}$ appearing in A. This makes the original Ashtekar connection a complex variable. There is a good reason for this choice…

…further analysis reveals that $K_a^i = \omega_a{}^{0i}$, the boost part. And the Lorentz group has a very nice decomposition over $\mathbb{C}$:

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{su}(2,\mathbb{C}).$$

The original Ashtekar connection is the self-dual factor.

To make sense of the quantum theory, one needs to be able to extract the 'real & imaginary parts' of operators and this has been a sticking point…

[However, see Alexander, Herczeg, & Freidel CQG 40, 145010]

The Ashtekar-Barbero connection

Instead the most common practice is to work with a real connection variable

Ashtekar-Barbero connection: $A_a^i := \Gamma_a^i + \gamma K_a^i$, with $\gamma \in \mathbb{R}$.

The 'Barbero-Immirzi' parameter γ is a new free parameter of the theory. We will see its physical meaning briefly.

Again: $\{A_a^i(x), \tilde{E}_j^b(y)\} = \gamma \delta_j^i \delta_a^b \delta^{(3)}(x,y)$, $\{A, A\} = 0$, & $\{\tilde{E}, \tilde{E}\} = 0$.

But, there is a tension between: [Alexandrov CQG 17, 4255]

	Ashtekar	Asht-Barb	Alexandrov
(i) Real variable	✗	✓	✓
(ii) Poisson commuting A	✓	✓	✗
(iii) Spacetime covariance	✓	✗	✓

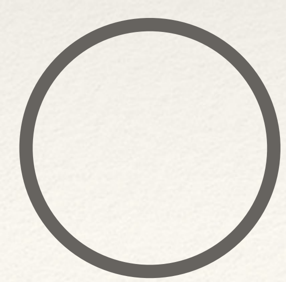
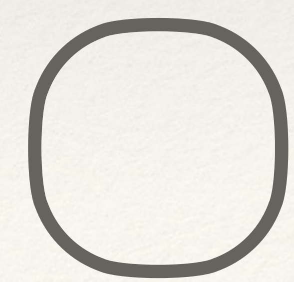
What are the observables in a gauge theory?

Abelian	Non-Abelian
1. $F = (E, B)$ local	F is not gauge invariant
2. $A^T_\mu = \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) A^\nu$	Still possible
3. $\oint A \to \oint A + \cancelto{0}{\oint d\lambda}$	$h(x,y) = \mathscr{P} e^{\int_x^y A}$ $\to g(x)h(x,y)g^{-1}(y)$

Both lead to Wilson loops

$$W(\gamma) = \mathrm{tr}\left[g(x)h_\gamma(x,x)g^{-1}(x)\right] = \mathrm{tr}\left[h_\gamma(x,x)\right].$$

Why aren't the $W(\gamma)$ observables used more often?

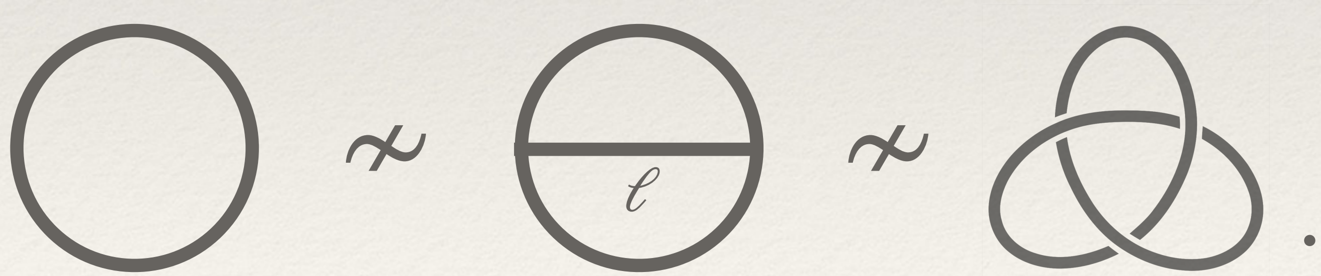
The trouble is that they distinguish  &  .

Gravity as a $\mathrm{Diff}(\mathscr{M}) \ltimes \mathrm{SU}(2)$ gauge theory

GR is a gauge theory, but an unusual one, with a gigantic gauge group: in addition to local changes of frame we have the entirety of the diffeos to consider.

LQG leverages the diffeos in a wonderful fashion, taking diffeomorphic loops in Σ to be equivalent

Thus, we need only consider inequivalent classes of loops—spin networks provide a basis for these inequivalent loops



The holonomies h_ℓ will provide half of the quantum variables.

LQG is similar to lattice gauge theory (LGT)

Working with holonomies h_ℓ there is a natural Hilbert space and inner product:

$$\mathscr{H} = L^2[G, \mu_H],$$

the space of square-integrable functions of the group elements with respect to the Haar measure μ_H.

We can extend this to a Hilbert space over a graph Γ with L links ℓ and N nodes n using the tensor product:

$$\mathscr{H}^L_\Gamma = L^2[G^L, \mu_H].$$

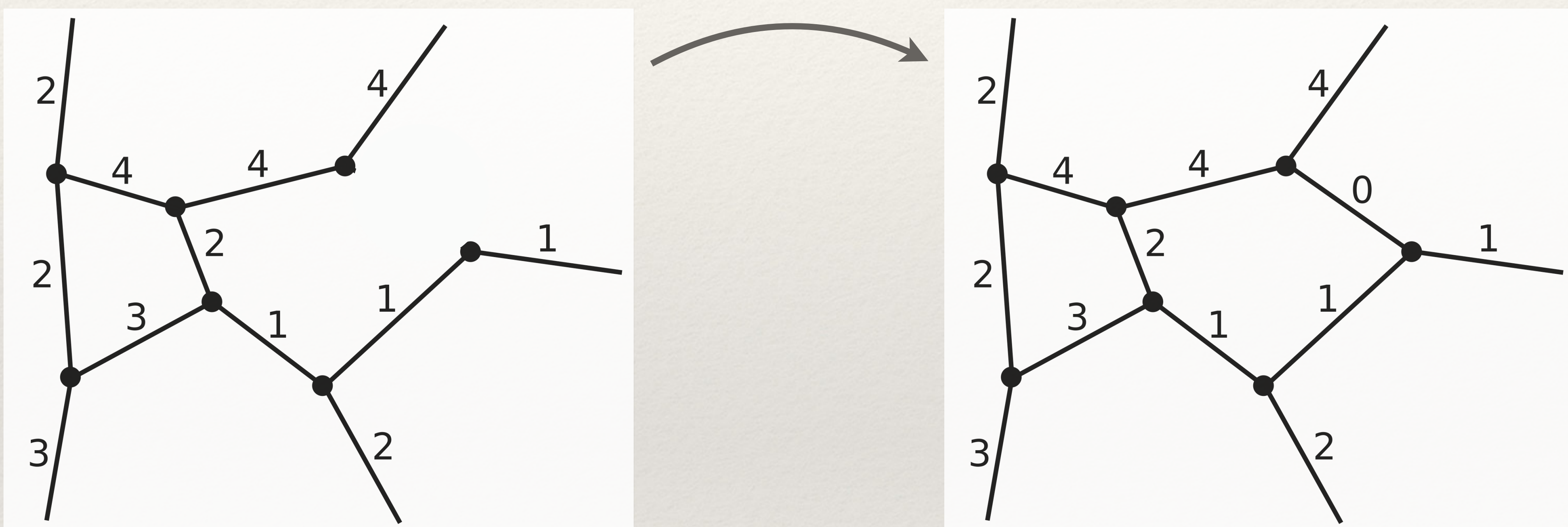
But, this space is not yet gauge invariant, so we finally divide by the gauge invariance at the nodes:

$$\mathscr{H}_\Gamma = L^2[G^L/G^N, \mu_H].$$

[Bianchi GRG 46, 1668]

Similarities to lattice gauge theory

"Cylindrical consistency" allows us to extend this inner product to get a notion of inner product on two different graphs Γ and Γ'. Idea:



Then, $$\langle \Psi_\Gamma, \Psi_{\Gamma'} \rangle = \langle \Psi_\Gamma, \Psi_{\Gamma'} \rangle_{\Gamma''}.$$

This allows a rich connection to continuum field theory in limit of finer and finer graph.

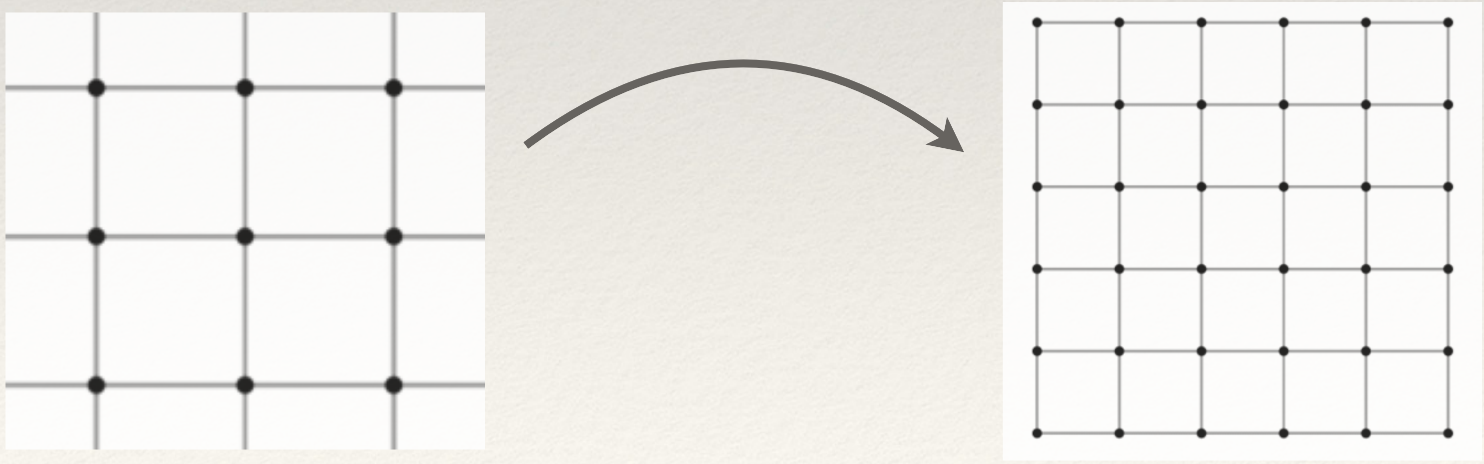
[Ashtekar & Lewandowski JMP 36, 2170]

An important difference: in loop gravity there is no fixed lattice spacing

The lattice spacing *l* of LGT represents the metrical spacing of the points of the lattice.

There can be no such fixed background structure in a fully dynamical treatment of quantum gravity.

Ideally you refine the theory by increasing the number of nodes *N* of the graph, not changing the spacing *l*:



[Dittrich Adv. Sci. Lett., Bahr, Dittrich, Steinhaus PRD 83]

Result: a well-defined construction of a kinematical Hilbert space

We have the Hilbert space:

$$\mathscr{H}_\Gamma = L^2[G^L/G^N, \mu_H],$$

which consists of cylindrical functions:

$$f(A) = \Psi_\Gamma(\{h_\ell\}).$$

The holonomies probe the space time curvature around closed loops of the graph Γ

But, …

…what's become of the triad degrees of freedom?

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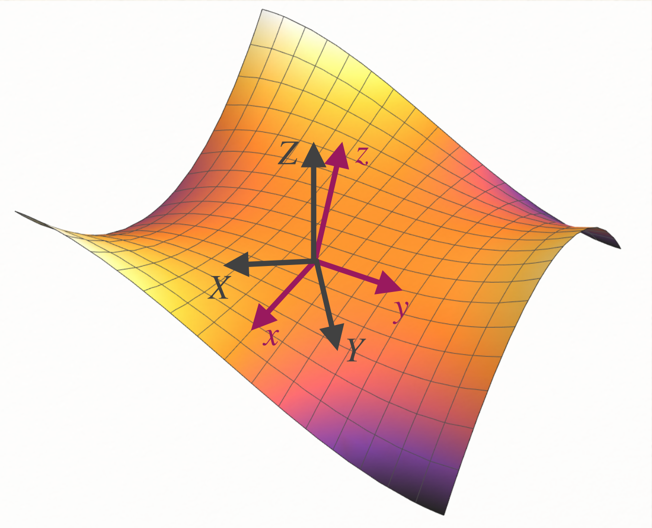

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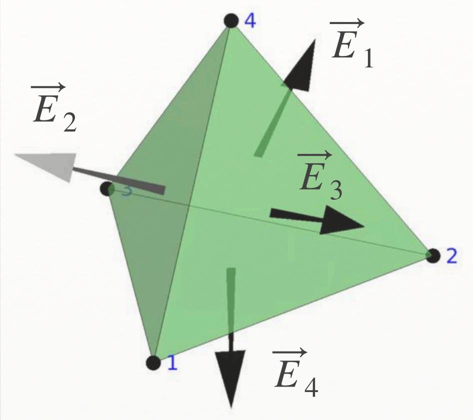

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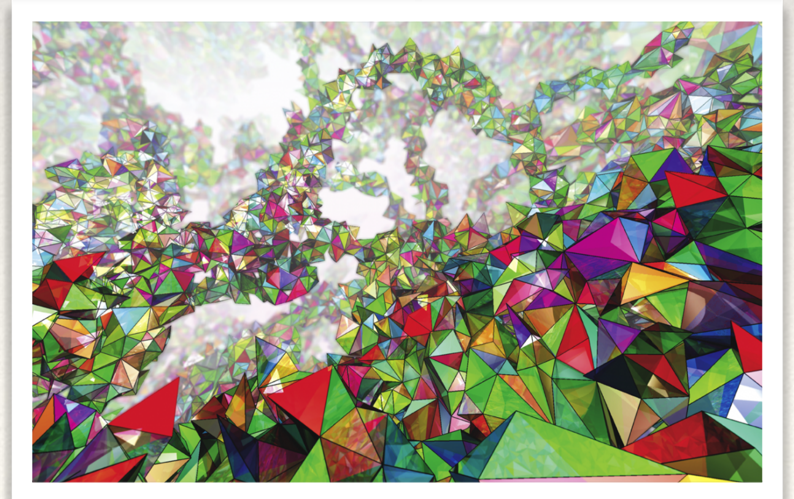


The Gauss constraint

So far, I've put little emphasis on the Gauss constraint

$$\mathscr{G}_i := D_a \tilde{E}^a_i \simeq 0.$$

This is because you know how to solve it: over a closed 2D surface $\mathcal{S}$, the 2-form electric flux must satisfy

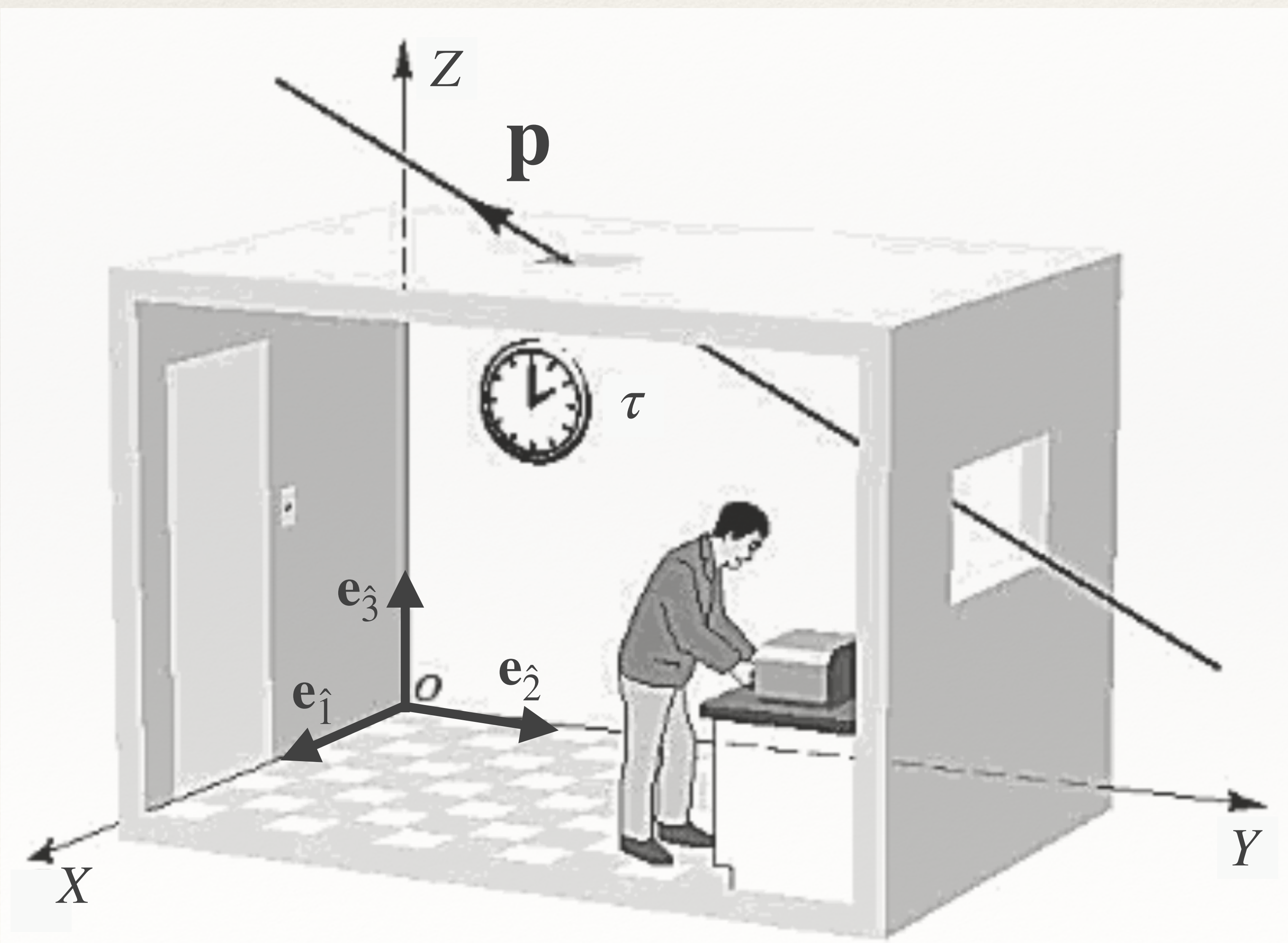
$$\overrightarrow{E}_{\mathcal{S}} = \oint_{\mathcal{S}} \tilde{E}^{ia}(x)\epsilon_{abc}\tau_i dx^b \wedge dx^c = 0.$$

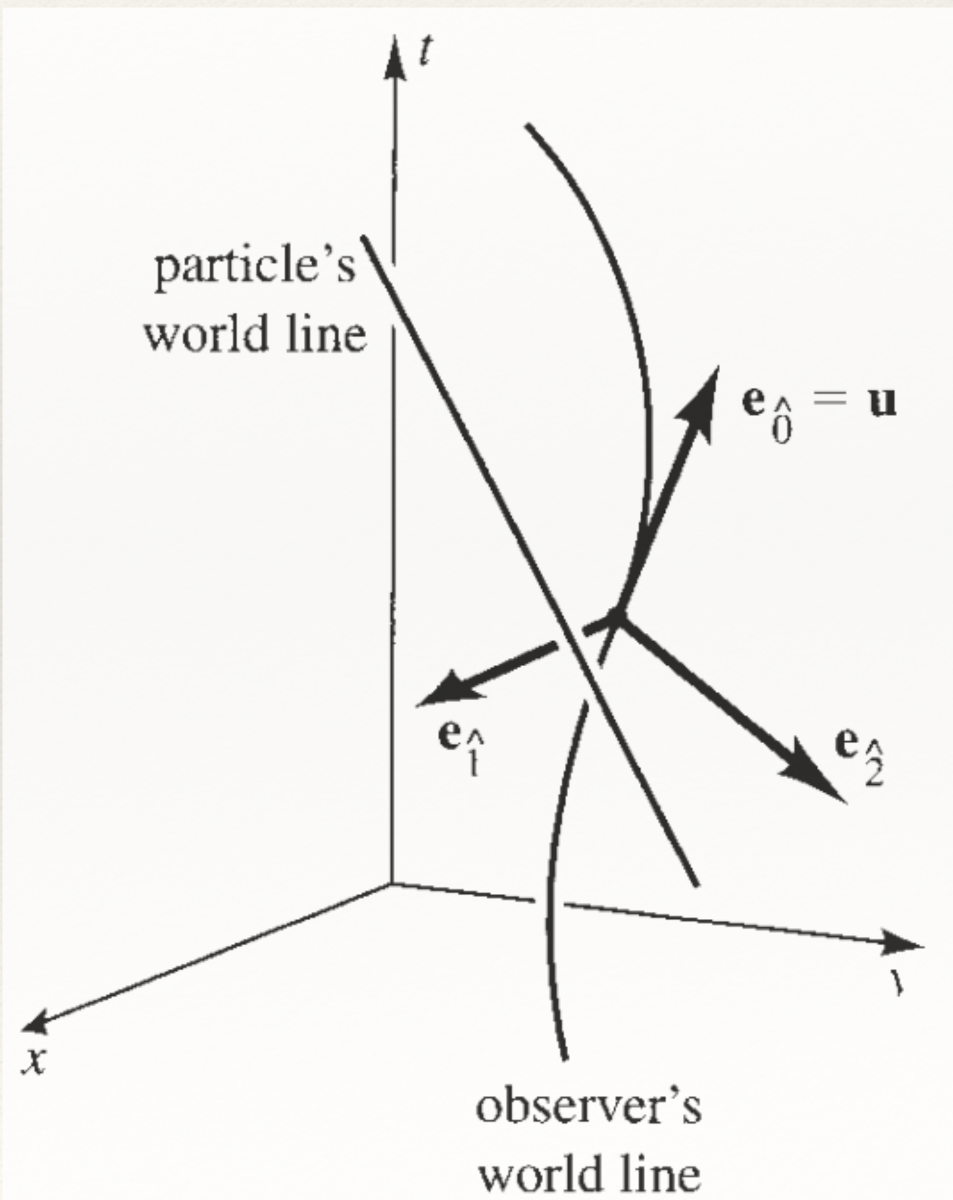
However, even here there are fascinating aspects of this gauge theory.

Metrically, this integral is picking out an oriented area of the 2-surface…but, oriented how?

Local orientations

The internal index (the vector orientation) is indicating how area elements would be measured in a local inertial frame; much like the energy of a particle is $E = -\mathbf{p}\cdot\mathbf{u}_{\text{obs}} = -\mathbf{p}\cdot\mathbf{e}_{\hat{0}}$.



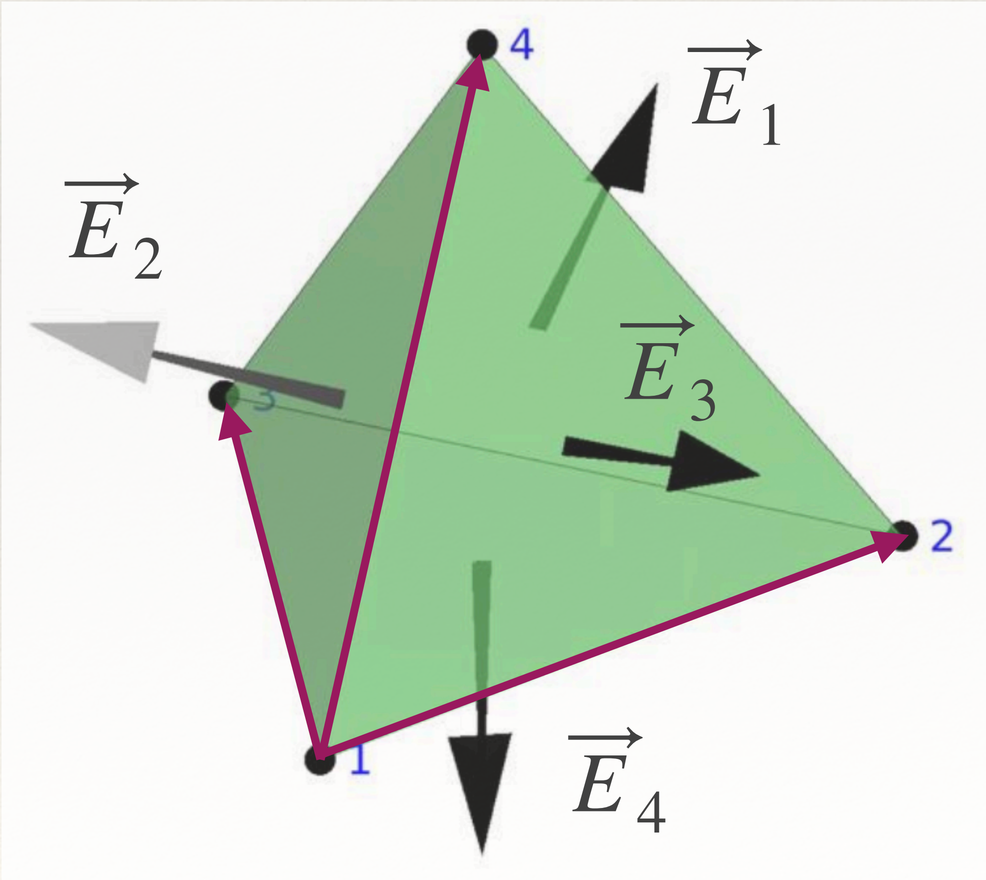


A particularly nice case

Remarkably, even the simplest possible case of a constant field over a polyhedral region is rich:

$$\vec{E}_{\mathcal{S}} = \oint_{\mathcal{S}} d\vec{E} = 0 \quad \Longrightarrow \quad \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 = 0.$$

This identity was used by Hermann Minkowski to give a complete characterization of convex polyhedra at the close of the 19th century.



[Minkowski, Nach. vd Ges. 1897]

As $\vec{E}_2 = \frac{1}{2}\vec{l}_{14} \times \vec{l}_{13}$, we can write

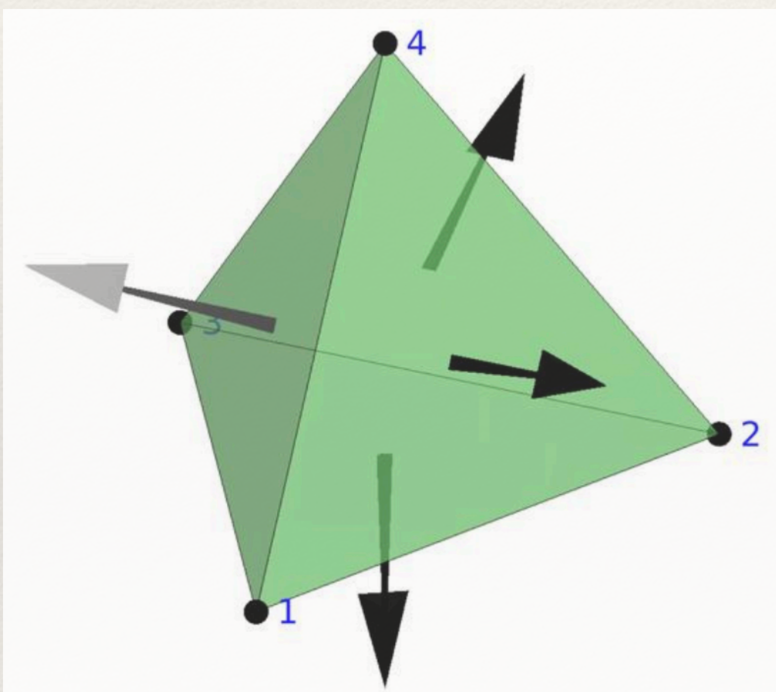
$$V = \frac{1}{6}\vec{l}_{12} \cdot (\vec{l}_{13} \times \vec{l}_{14}),$$

or equally well,

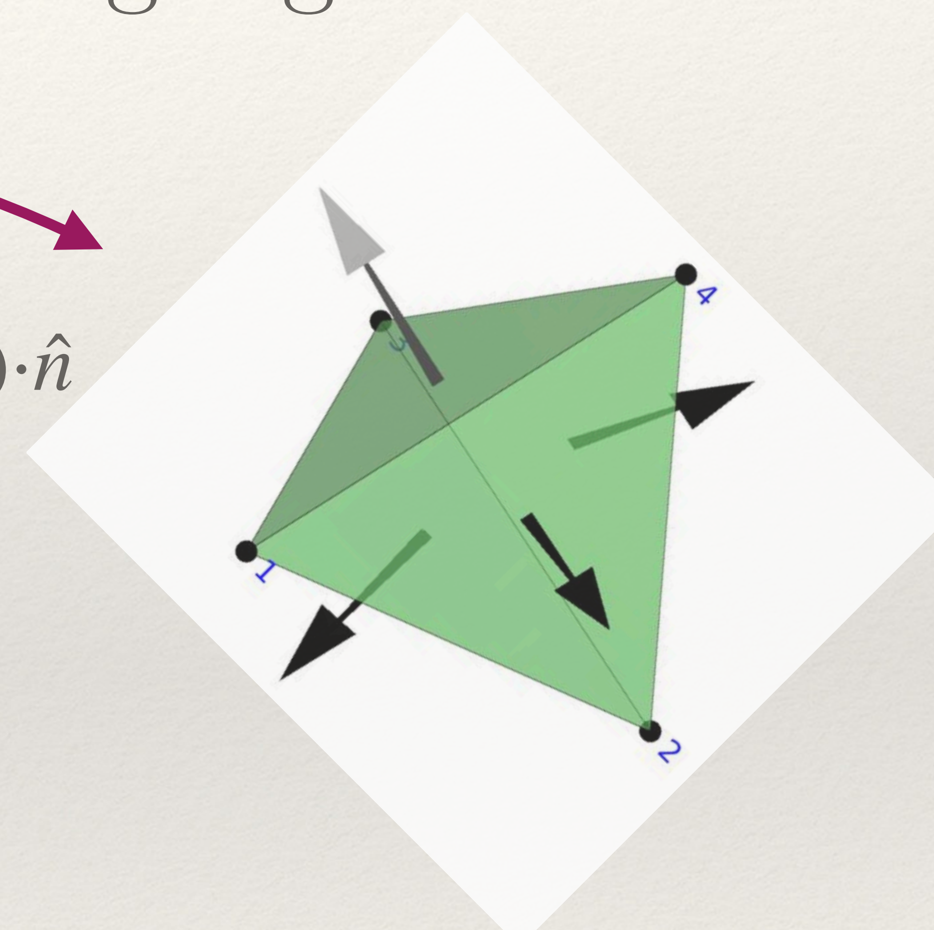
$$V^2 = \frac{2}{9}\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3).$$

Gauge invariance and shape

The internal index is also $\mathfrak{su}(2)$-valued, hence the $\overrightarrow{E}$'s & closure have a 2nd role: the vector $\overrightarrow{E}_{\mathcal{S}} = \overrightarrow{E}_1 + \overrightarrow{E}_2 + \overrightarrow{E}_3 + \overrightarrow{E}_4$ generates gauge rotations

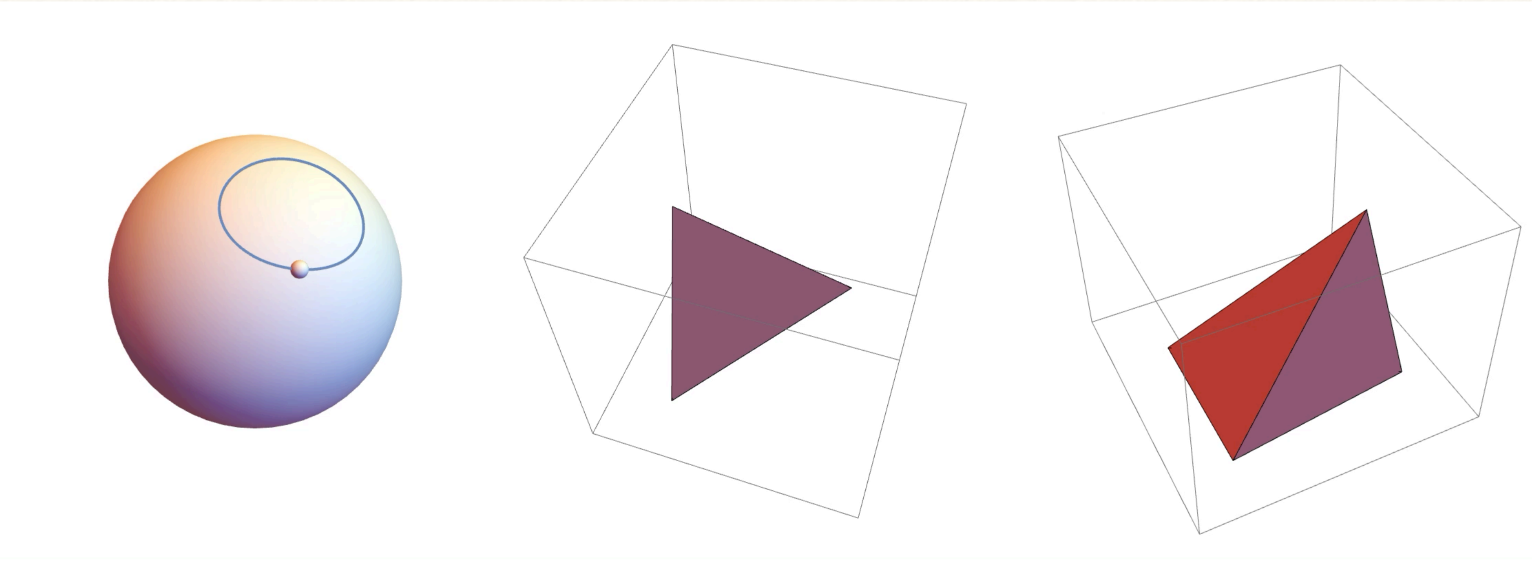


$$R(\theta, \hat{n}) = e^{\theta(\overrightarrow{E}_{\mathcal{S}})\cdot\hat{n}}$$



….and $\overrightarrow{E}_{\mathcal{S}} = 0$ means that these rotations change the tetrahedron's orientation, but don't change its shape (metric geometry)!

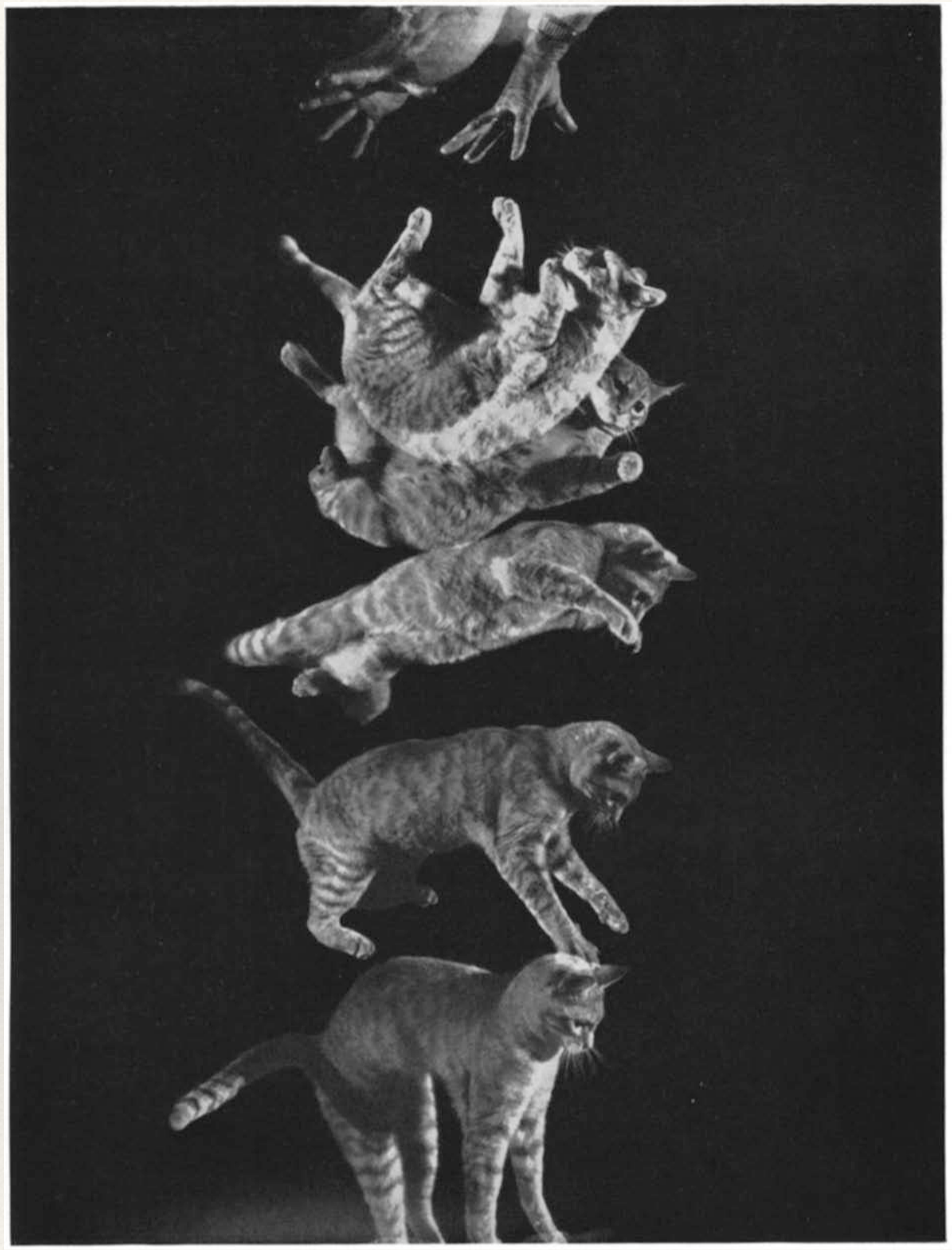
Stunningly, this is the same gauge theory that explains how a falling cat lands on its feet



Fixing the facet areas $\{E_1, \ldots, E_4\}$ and the volume

$$V^2 = \frac{2}{9}\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3),$$

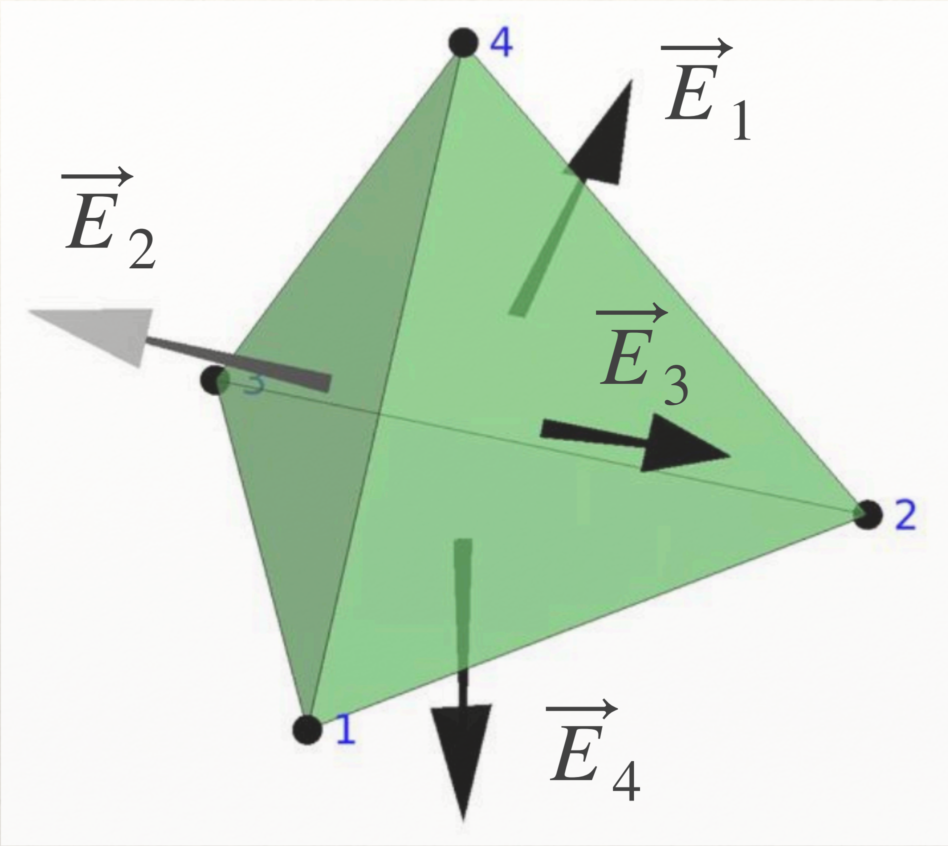
The tet still has room to change shape.



[Littlejohn & Reinsch, Rev. Mod. Phys. 69, 1997]

Quantization of Geometry: Area

As we have seen, each of the fluxes $\vec{E}_\ell$ can be thought of as an angular momentum vector:



Let $\mathscr{H}_{j_\ell}$ be the carrier space of an SU(2) irrep with basis $|j_\ell m_\ell\rangle$, then

$$|\hat{\mathbf{E}}_\ell|\,|j_\ell m_\ell\rangle = \gamma a_P \sqrt{j_\ell(j_\ell+1)}\,|j_\ell m_\ell\rangle$$

where $a_P := 8\pi\hbar G/c^3$ & the Barbero-Immirzi γ sets the spectral spacing, or 'area gap'.

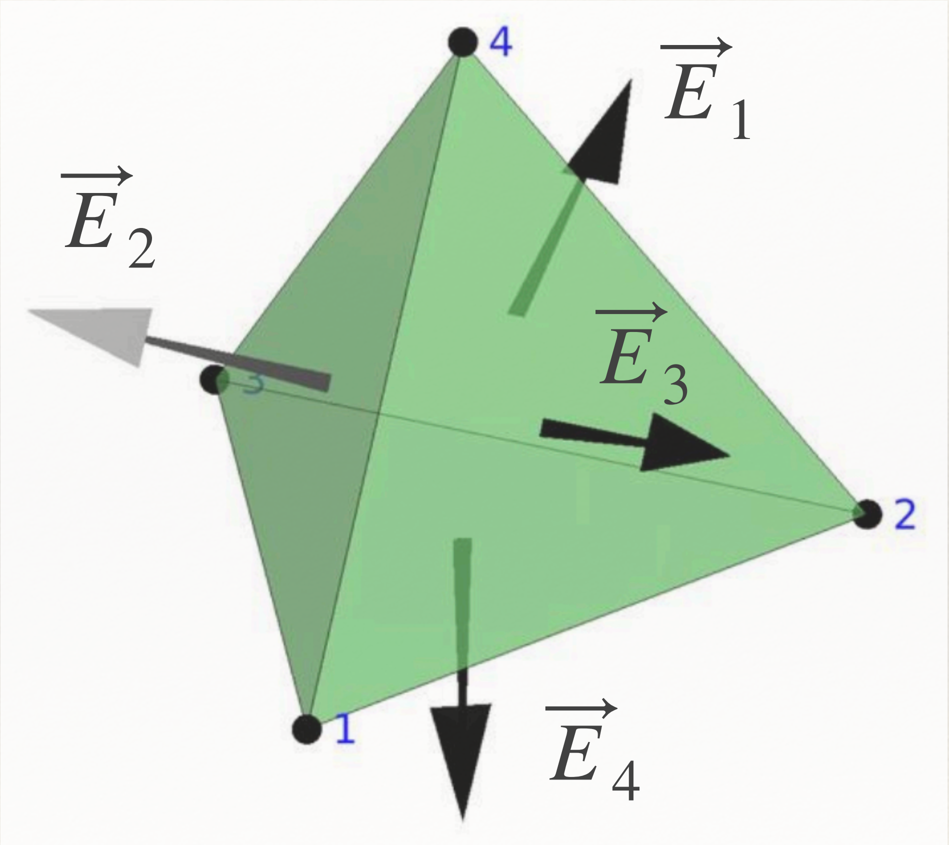
[Rovelli and Smolin, Nuc.Phys. B442, 593; Ashtekar & Lewandowski, CQG 14, A55, Friedel, Geiller, Wieland, Handbook QG 2023]

Quantization of Geometry: Tetrahedra

The magnetic quantum number m_ℓ belies orientation dependence. This makes sense for each of the facets, but it must go away for the tet as a whole.

To achieve this at the quantum mechanical level, we must search for rotationally invariant states of the product of the irreps:

$$|\,i\rangle \in \mathrm{Inv}(\mathscr{H}_{j_1} \otimes \mathscr{H}_{j_2} \otimes \mathscr{H}_{j_3} \otimes \mathscr{H}_{j_4}).$$



We call such an invariant state an "intertwiner" and

$$|\,i\rangle = |\,i\; j_1\, j_2\, j_3\, j_4\rangle := \sum_{m's} i^{m_1\cdots m_4}\,|\,j_1\, m_1\rangle\,|\,j_2\, m_2\rangle\,|\,j_3\, m_3\rangle\,|\,j_4\, m_4\rangle.$$

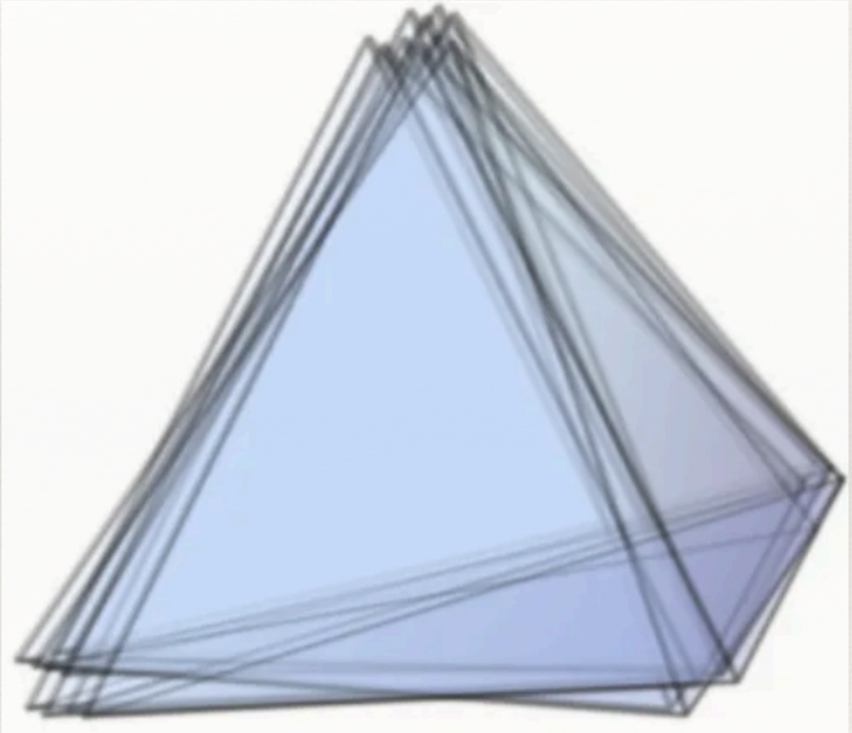
Quantization of Geometry: Volume

The classical geometry of V suggests one way to construct an intertwiner,

$$\hat{V} = \frac{\sqrt{2}}{3}\sqrt{|\hat{\mathbf{E}}_1 \cdot (\hat{\mathbf{E}}_2 \times \hat{\mathbf{E}}_3)|}\,.$$

This is a rotational invariant and its eigenvalues, v say, provides a very physical set of basis states:

$$|i\rangle = |v\, j_1\, j_2\, j_3\, j_4\rangle.$$



We see again that a quantum tet is specified by only 5 parameters and hence is quantum mechanically fuzzy—don't over read the polyhedral description.

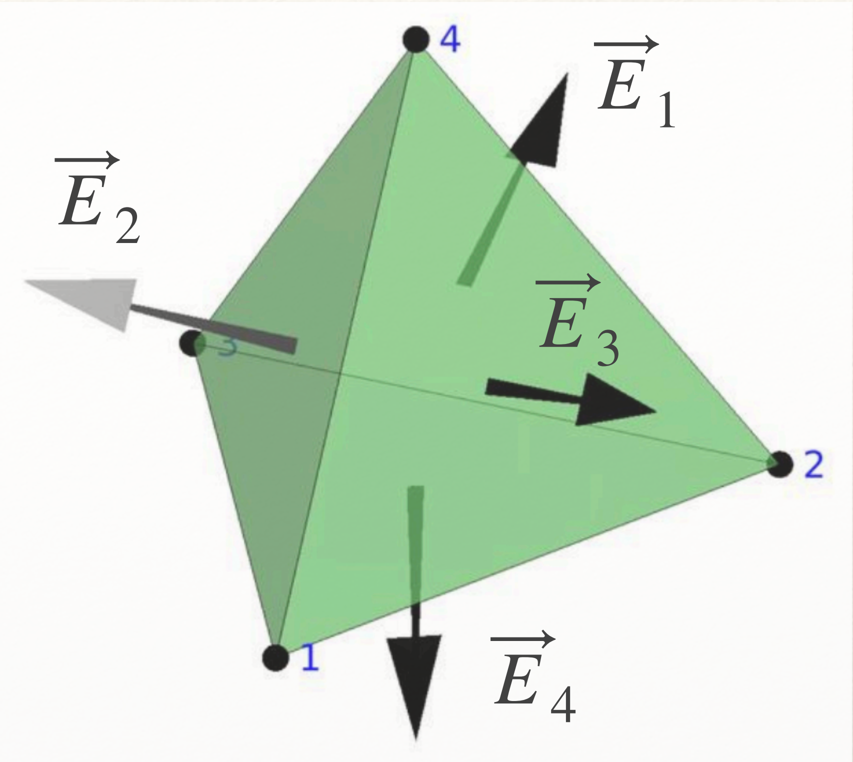
[Levy-Leblond & Levy-Nahas, J.Math.Phys 6]

Warning: A change in notation

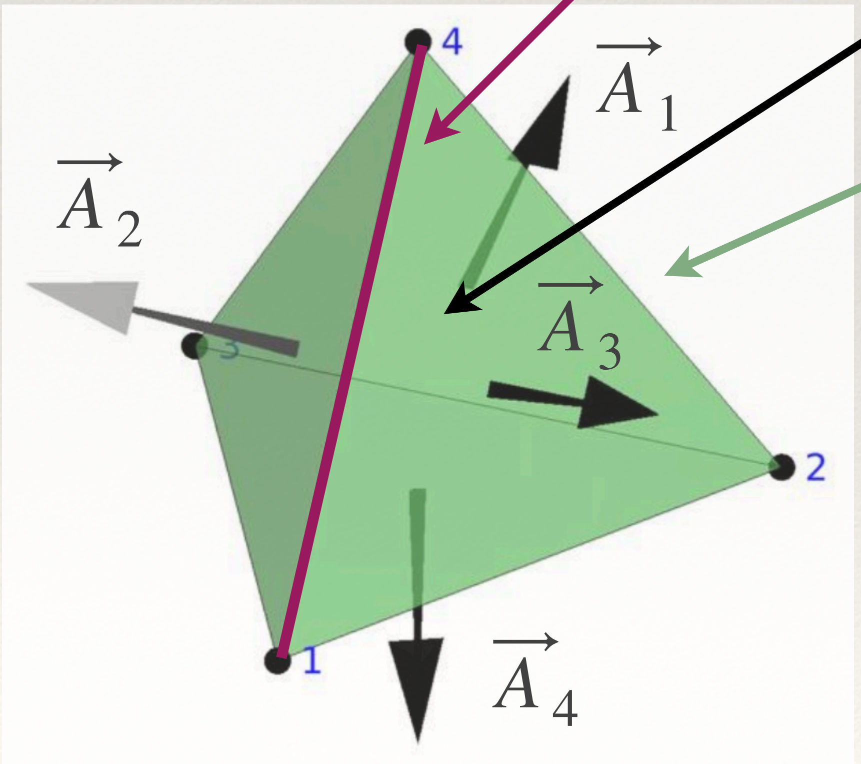
We have been discussing the tetrahedron in terms of electric fluxes:

$$\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 = 0.$$

Moving forward we will have less need to refer to the Ashtekar connection A_a^i…



more notation: edge e, triangle t, tet τ



…and so I will switch to a geometrical notation with:

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0.$$

This will make it more intuitive to refer to areas and 'area vectors'.

The area vectors provide a unified framework for Euclidean and Lorentzian discrete geometries.

Taking $\vec{A}_t \in \mathfrak{so}(3)$ (Euclidean) or $\vec{A}_t \in \mathfrak{so}(2,1)$ (Lorentzian), the closure $\sum_t \vec{A}_t = 0$, expresses invariance in either case.

Consider again

$$9V^2 = 2\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3) = 2\det(\vec{A}_1\ \vec{A}_2\ \vec{A}_3).$$

Squaring yields

$$81V^4 = 4\det(\vec{A}_1\ \vec{A}_2\ \vec{A}_3)\det(\vec{A}_1\ \vec{A}_2\ \vec{A}_3)^t$$

$$= 4\begin{vmatrix} A_1^2 & \vec{A}_1\cdot\vec{A}_2 & \vec{A}_1\cdot\vec{A}_3 \\ \vec{A}_2\cdot\vec{A}_1 & A_2^2 & \vec{A}_2\cdot\vec{A}_3 \\ \vec{A}_3\cdot\vec{A}_1 & \vec{A}_3\cdot\vec{A}_2 & A_3^2 \end{vmatrix} := 4G,$$

Area metric

with $V^4 > 0$ (Euclidean tet) and $V^4 < 0$ (Lorentzian tet).

[Asante, Dittrich, Padua-Arguelles, arXiv:2112.15387, arXiv:2302.11586]

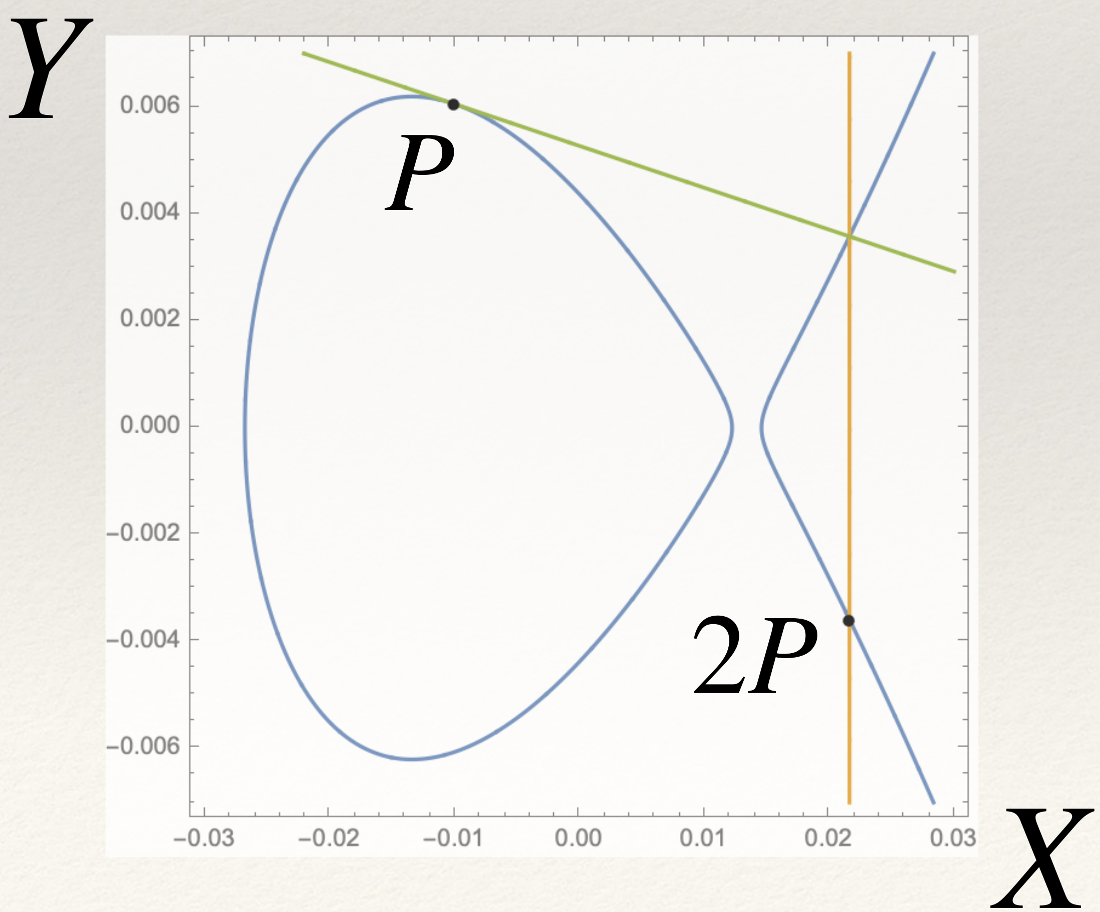
With this insight you can show that to every tetrahedron there corresponds an elliptic curve

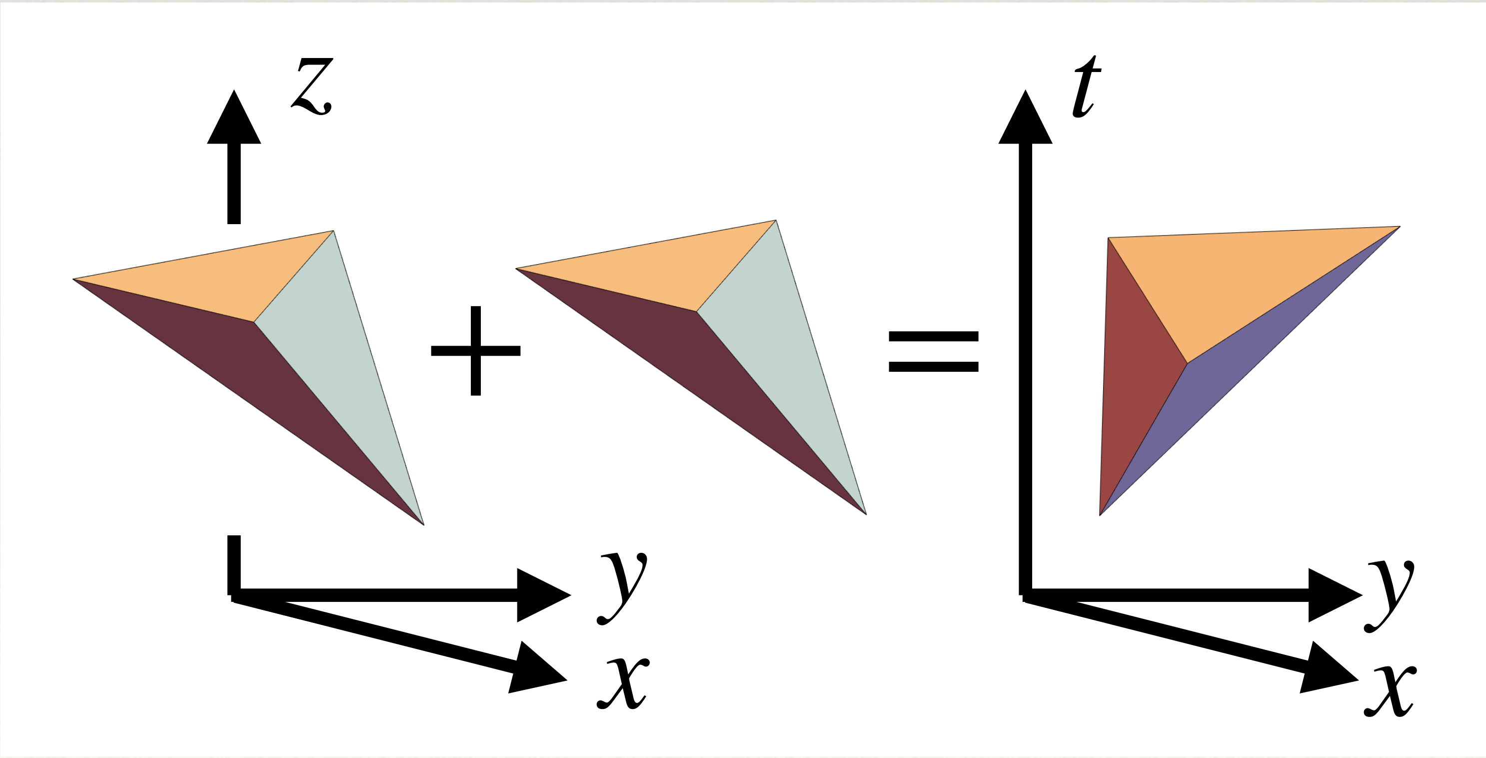
Let $x = (\overrightarrow{A}_1 + \overrightarrow{A}_2)^2$, $y = (\overrightarrow{A}_2 + \overrightarrow{A}_3)^2$, and $z = (\overrightarrow{A}_1 + \overrightarrow{A}_3)^2$, then

Elliptic curve

$$\begin{cases} x + y + z = A_1^2 + A_2^2 + A_1^3 + A_4^2 \quad \text{and} \\ 81V^4 = xyz - (A_1^2A_2^2 + A_3^2A_4^2)\,x - (A_2^2A_3^2 + A_1^2A_4^2)\,y - (A_1^2A_3^2 + A_2^2A_4^2)\,z + 2\sigma_3 \end{cases}$$

An elliptic curve is a plane cubic algebraic curve with a pt P.





[Antu, Doran & HMH, in progress]

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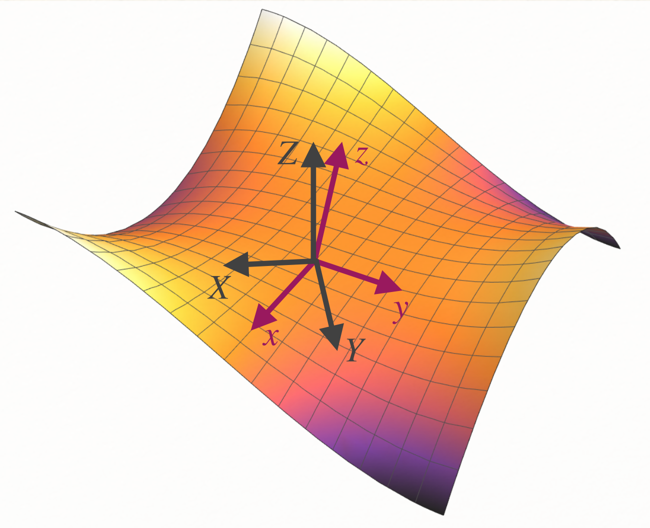
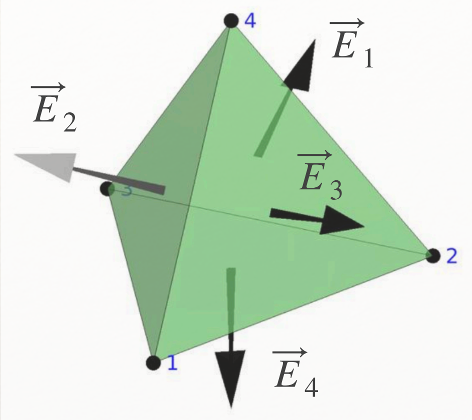
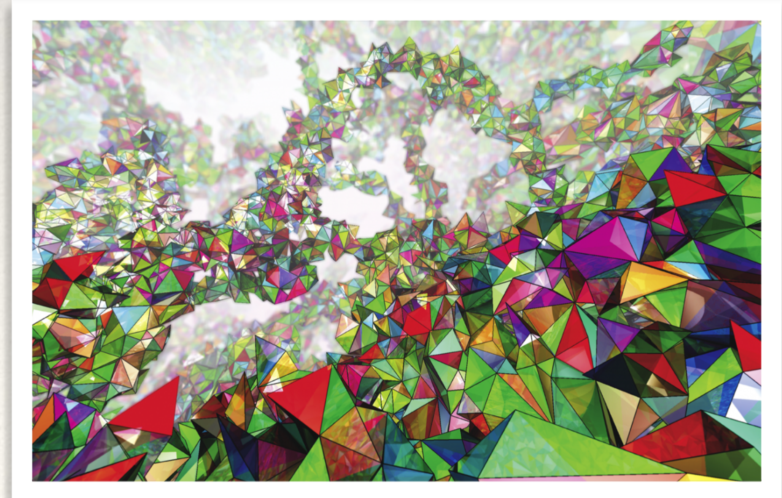
I. Loop Quantum Gravity:
GR as a canonical gauge theory



II. Connecting Approaches: Classical and Quantum Tetrahedra



III. Spin Foams: Discrete Geometry Path Integrals



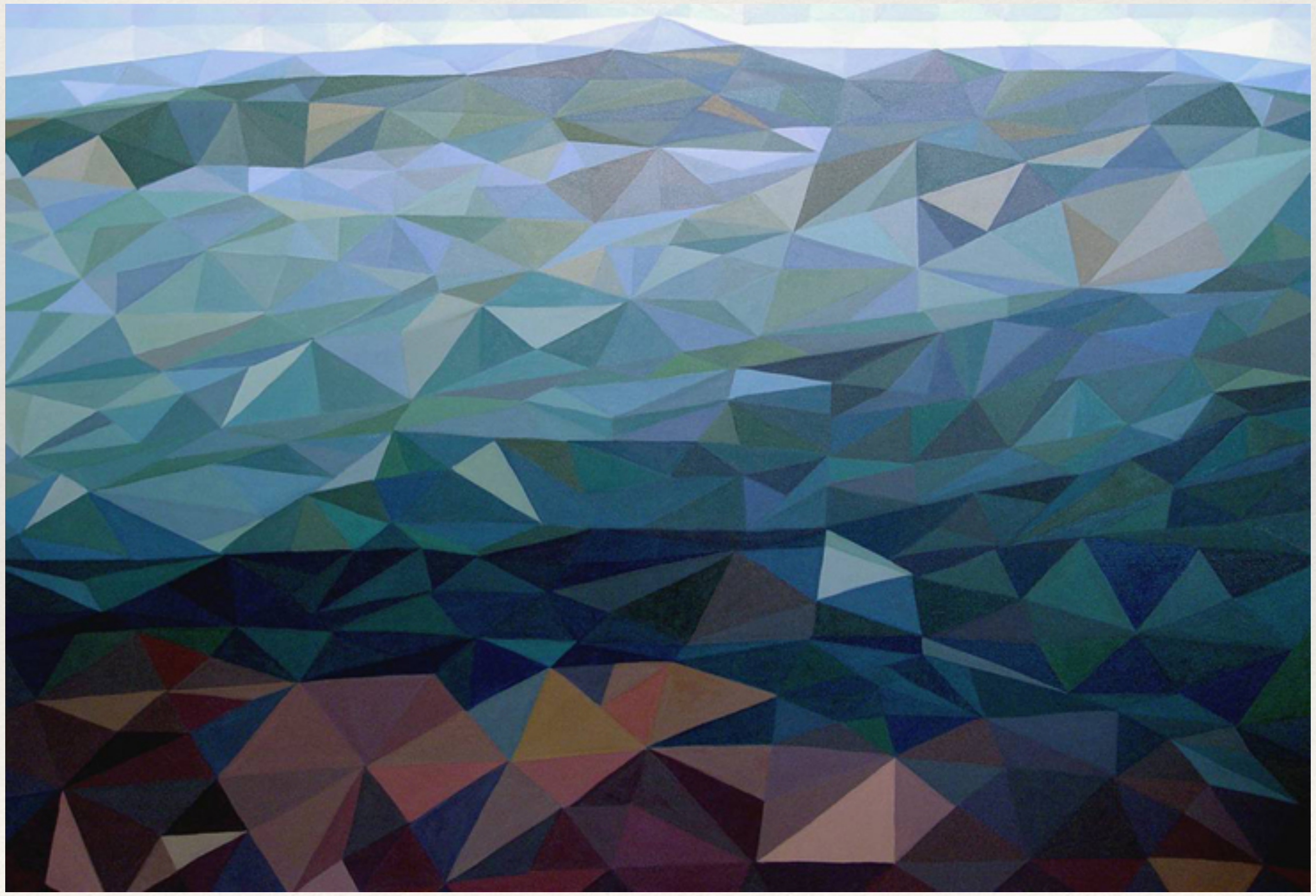
In Regge Calculus we describe spacetime by a triangulation of flat pieces glued together to give curvature.

Regge Calculus

This cuts the degrees of freedom of the gravitational field down to a finite number and greatly eases their study.

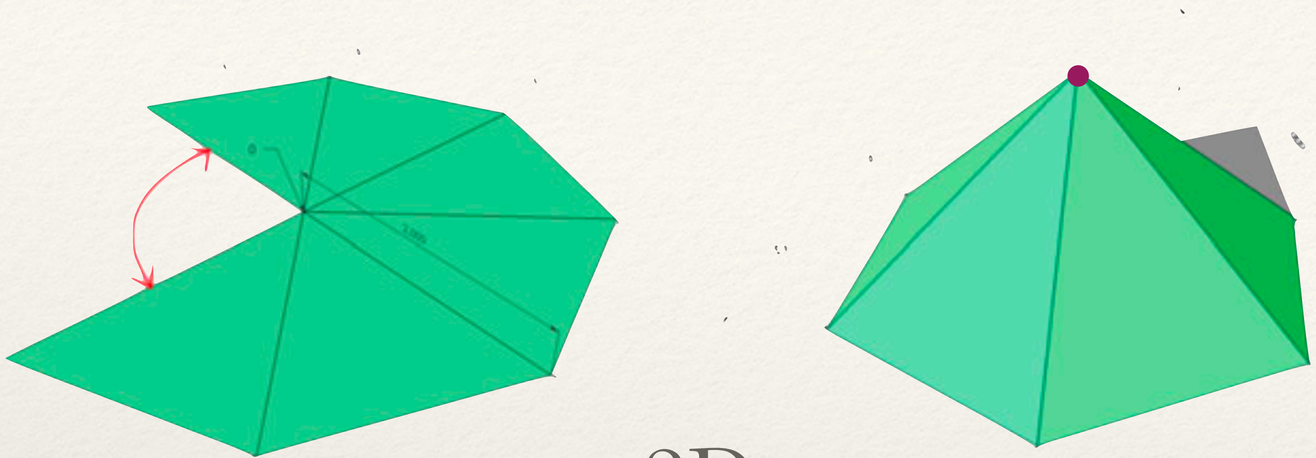
It is also immensely useful for doing numerics.

Mt. Rainier



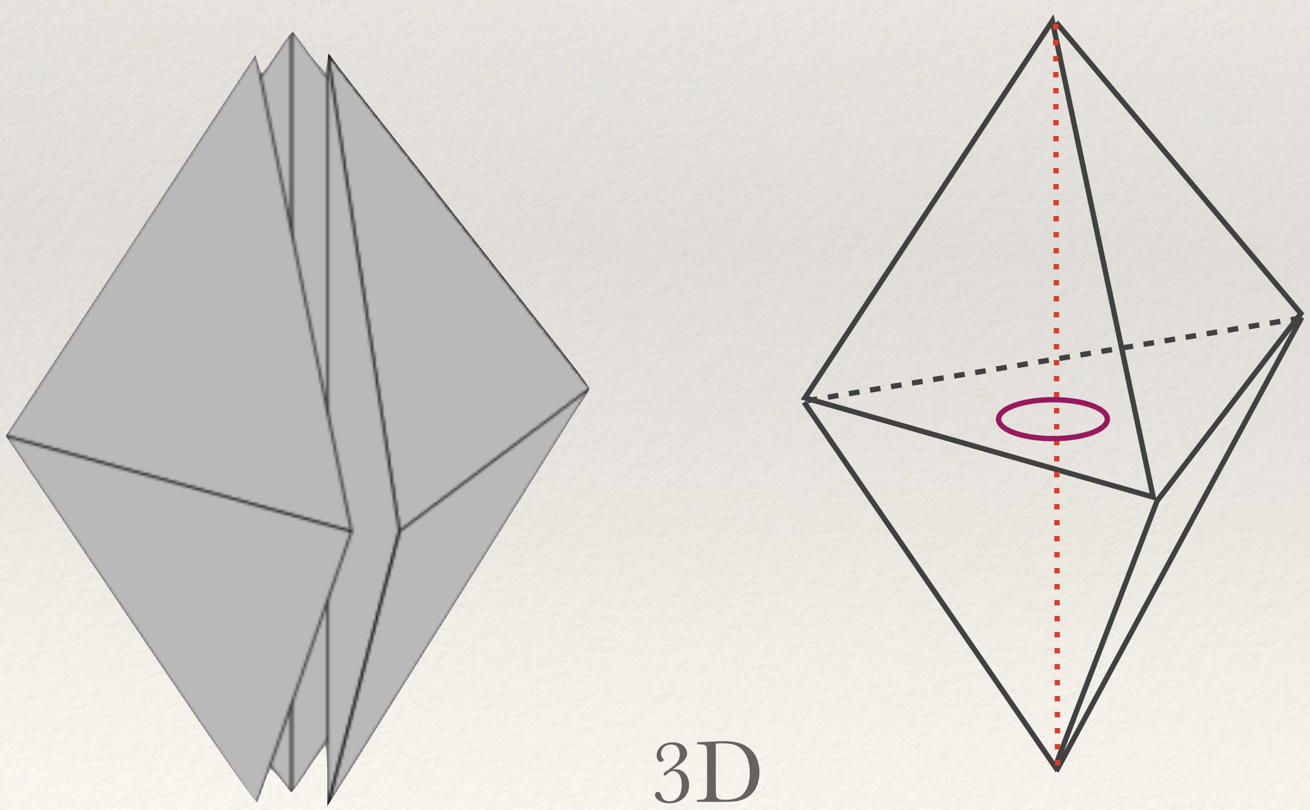
Scott Bailey, *47°19′38″ N, 120°27′36″W*

A dimensional ladder helps to illustrate some salient aspects of Regge Calculus



2D

In 2D it is clear how curvature becomes concentrated on the $(d-2)$-dimensional 'bones'.



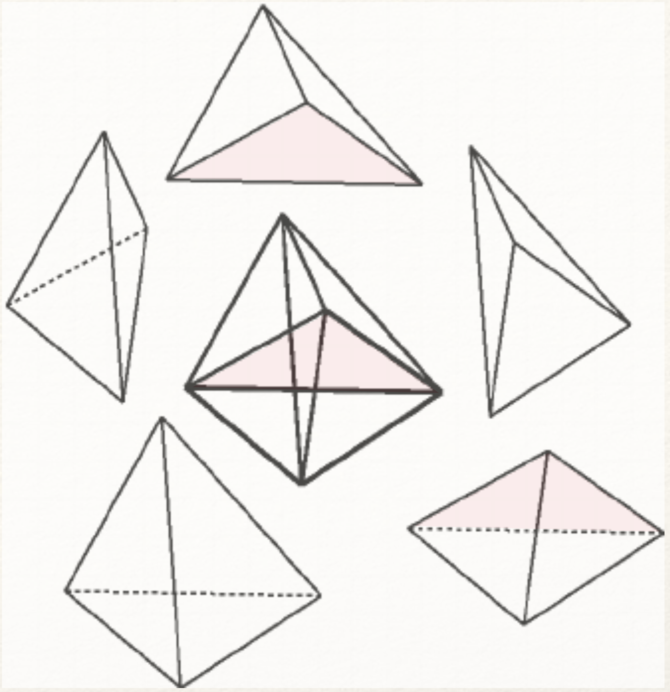
3D

In 3D we see an intriguing alignment between the *metrical* and *symplectic* aspects: the bones are 1D edges, whose lengths give the metric;

meanwhile the conjugate curvature angle is compact and leads to quantization of lengths

In 4D the bones are 2D triangles t.

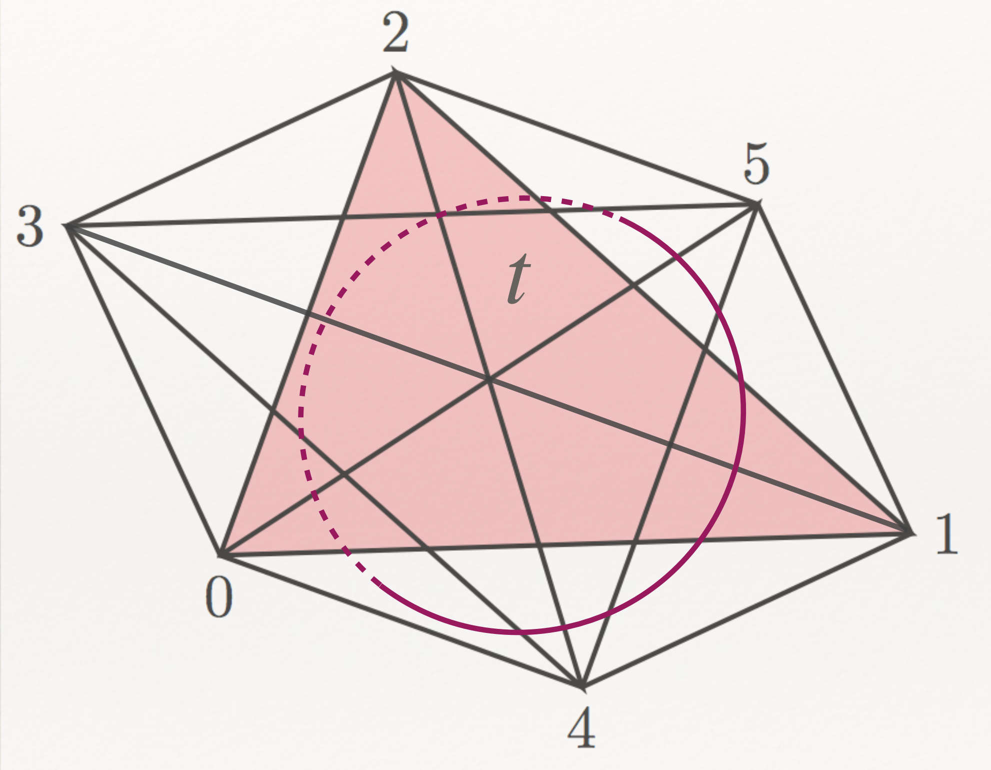
5 tetrahedra glue into a 4D simplex



One is forced to choose between: the apparent metrical length variables l, with a complicated conjugate variable

or

The area of the triangle t, which is conjugate to the curvature angle around the bone. This curvature angle is compact, indicating the areas will be quantized.



The 2nd choice is harmonious with loop quantum gravity (LQG), & the focus of the discrete geometry path integrals of spin foams.

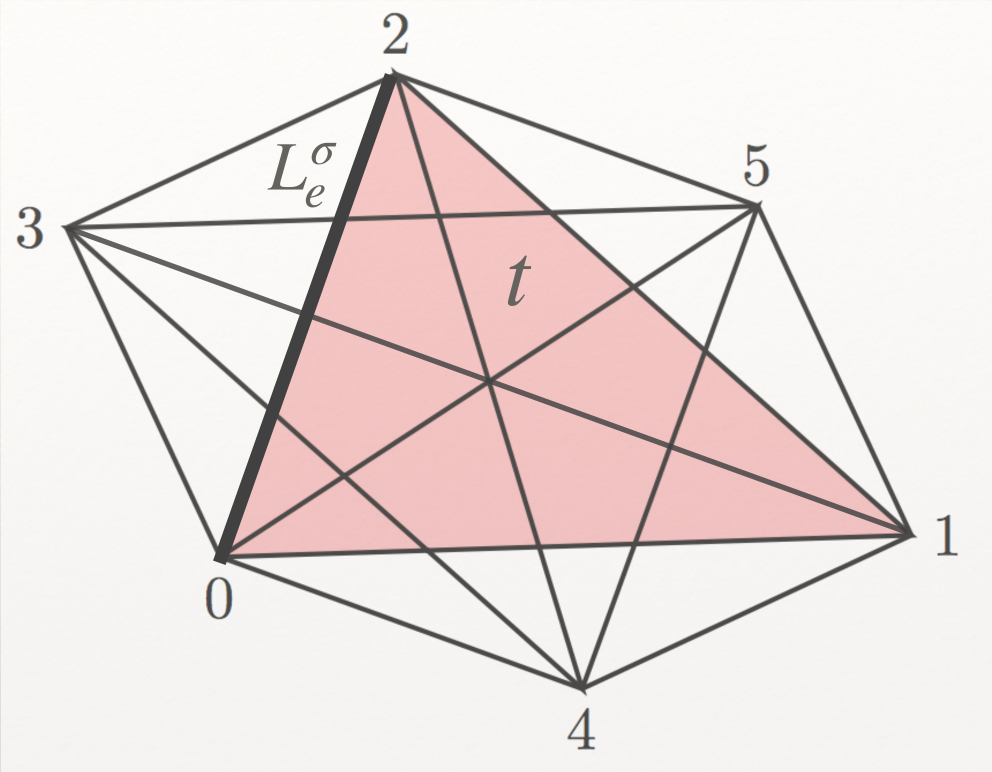
In standard Regge Calculus we treat the lengths of edges as variables…

Area Regge Calculus

…in Area Regge Calculus it's the areas of triangles. This provides a closer connection to area geometry, its quantization, and loop quantum gravity.

In standard Regge Calculus we treat the lengths of edges as vars, while in Area Regge Calculus it's the areas of triangles

A 4-simplex has ten edges and ten faces. Locally the functions $A_t(l)$ can be inverted to give edge lengths $L_e^\sigma(a)$.



Considering areas a as variables we can define Area Regge Calculus (ARC) via the action

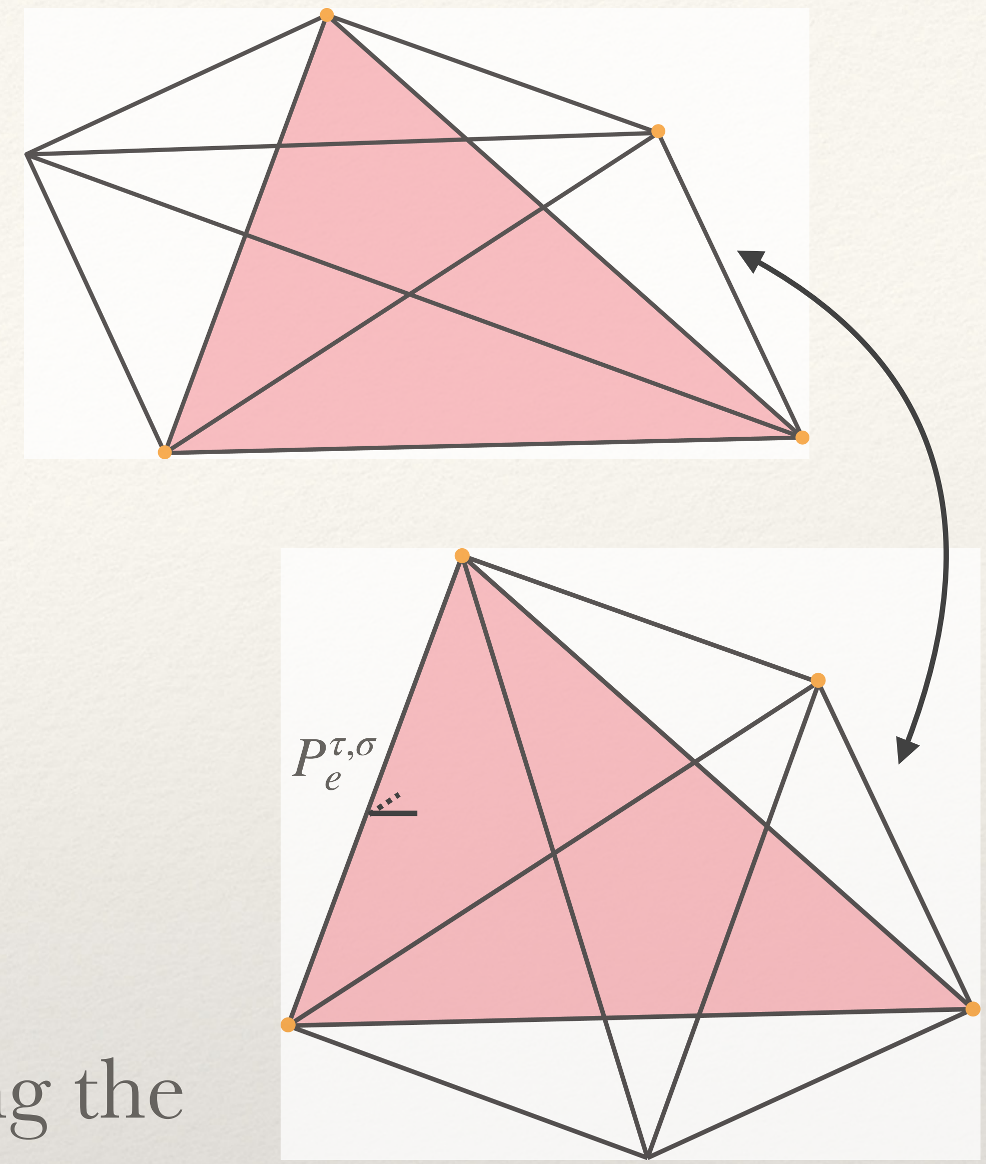
$$S_{\text{ARC}} = \sum_t a_t\, \epsilon_t(a)\ , \quad \text{with } \epsilon_t = 2\pi - \sum_{\sigma \supset t} \theta_t^\sigma$$

The dihedral and deficit angles are obtained using $\theta_t^\sigma(a) = \theta_t^\sigma(L^\sigma(a))$. Strikingly, variation of this action gives eqs. of motion

$$\delta S_{\text{ARC}} = \epsilon_t(a) + \sum_t a_t \delta\epsilon_t = \epsilon_t(a) = 0, \quad \text{which impose flatness on } \Delta.$$

0 (due to the Schläfli identity)

Adding Constraints to the Theory



We can understand this difference in eqs. of motion between ARC and LRC as due to a differing # of degrees of freedom.

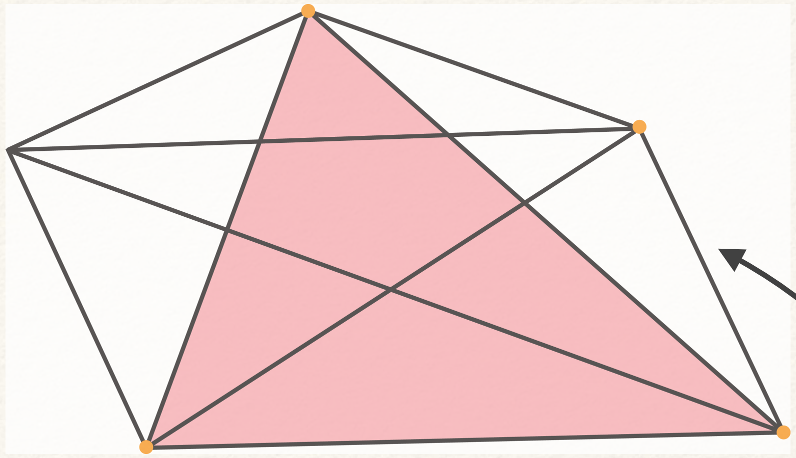
Gluing along the tetrahedron with orange vertices, 6 edge lengths are matched, but only 4 areas.

This mismatch can be resolved by introducing the dot products $p^\tau_{tt'} := p^\tau_e = \mathrm{sgn}(\mathrm{V}^2_\tau)\,\hat{n}_t \cdot \hat{n}_{t'}$, & $p^\tau_{tt} = A^2_t$:

$P^{\tau,\sigma}_e(a) = P^\tau_e(L^\sigma(a))$ is the dihedral angle around edge e in tet τ.

Two neighboring simplices $\{\sigma, \sigma'\}$, glued along τ, will have the same lengths in τ if the constraints

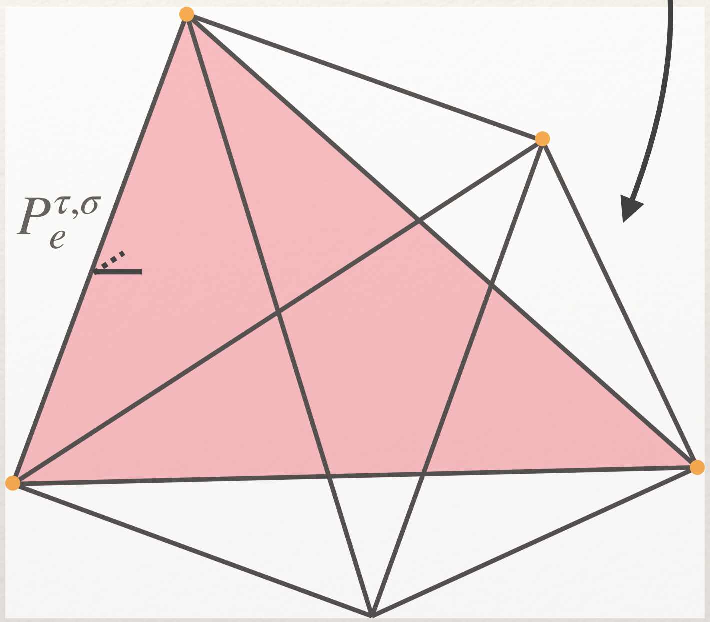
$P^{\tau,\sigma}_{e_i}(a) - P^{\tau,\sigma'}_{e_i}(a) = 0$, $i = 1{,}2$ are imposed on non-opposite edges e_i.



We can localize constraints to a single 4-simplex by introducing the two additional variables $p^\tau_{e_i}$ per τ to our theory and imposing

$$\mathscr{C}_i \equiv p^\tau_{e_i} - P^{\tau,\sigma}_{e_i}(a) = 0, \quad i = 1,2.$$

The advantage of these localized constraints is that they preserve additive factorization of the Regge action and allow us to write the path integral in a product factorized form.



Importantly, dot products at a pair of non-opposite edges (e_1, e_2) do not commute. Instead

$$\{p^\tau_{e_1}, p^\tau_{e_2}\} = \pm\,\gamma\frac{9}{2}\mathrm{Vol}^2_\tau, \quad \text{with } (e_1, e_2) \text{ non-opposite.}$$

We can localize constraints to a single 4-simplex by introducing the two additional variables $p^{\tau}_{e_i}$ per τ to our theory and imposing

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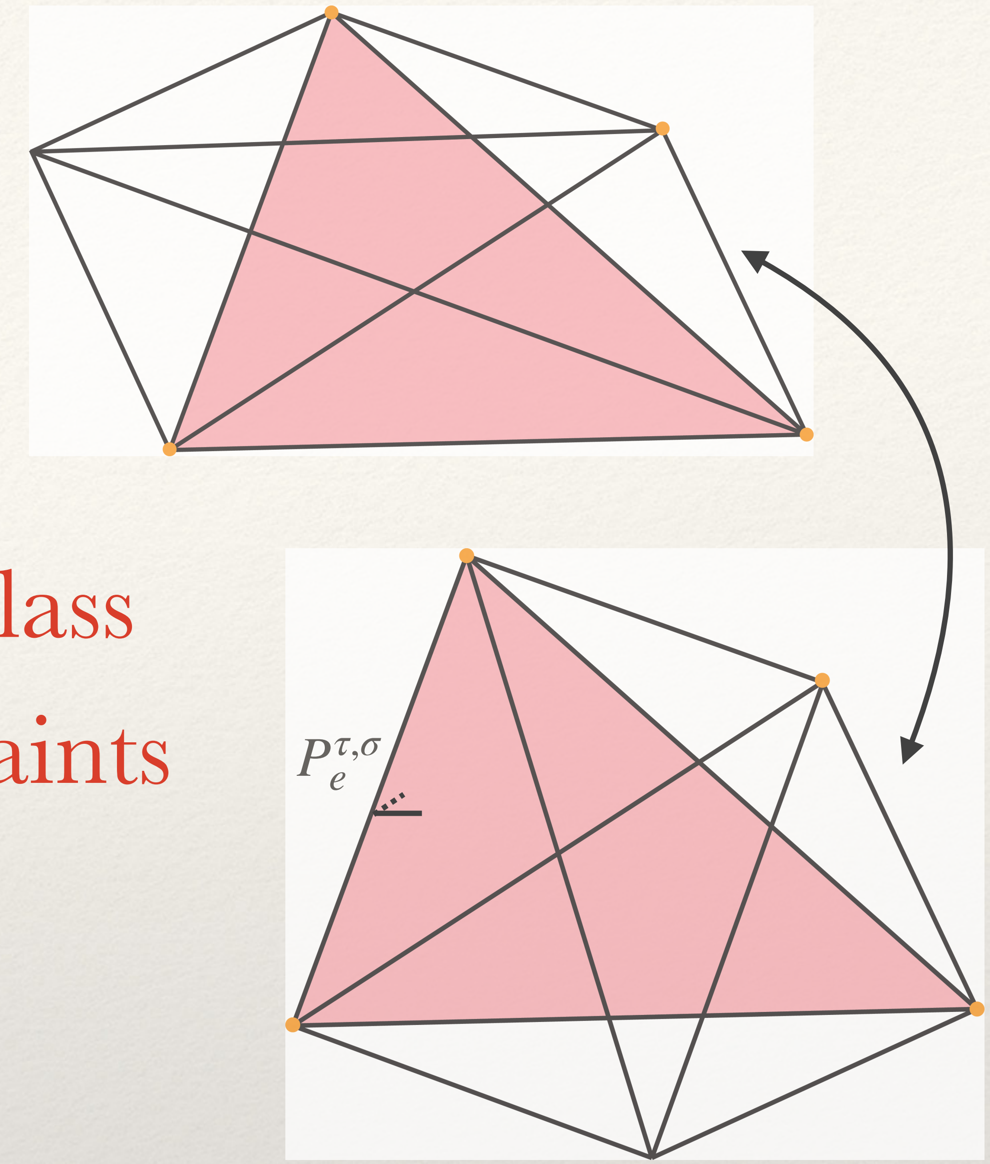


2nd class constraints

The advantage of these localized constraints is that they preserve additive factorization of the Regge action and allow us to write the path integral in a product factorized form.

Importantly, dot products at a pair of non-opposite edges (e_1, e_2) do not commute. Instead

$$\{p^{\tau}_{e_1}, p^{\tau}_{e_2}\} = \pm\,\gamma\frac{9}{2}\mathrm{Vol}^2_{\tau}, \quad \text{with } (e_1, e_2) \text{ non-opposite.}$$

The Area Regge action, $S_{\text{Regge}} = \sum_t a_t \epsilon_t$, factorizes additively. Boundaries of the triangulation Δ are readily included.

From the definition of the deficit angle

$$\epsilon_t = 2\pi - \sum_{\sigma \supset t} \theta_t^\sigma,$$

we see that the area Regge action factorizes

$$S_{\text{ARC}} = \sum_t a_t \epsilon_t = \sum_t n_t \pi a_t - \sum_\sigma \sum_{t \supset \sigma} a_t \theta_t^\sigma(a) \equiv \sum_t S_t^a(a) + \sum_\sigma S_\sigma^a(a).$$

The last equality defines the triangle and simplex actions

$$S_t^a = n_t \pi a_t \quad \text{and} \quad S_\sigma^a(a) = -\sum_{t \supset \sigma} a_t \theta_t^a(L^\sigma(a)).$$

Here the index $n_t \in \{1,2\}$ allows for triangulations with boundary: it is 1 for triangles on the boundary and 2 for triangles in the bulk.

To this action we add a set of functions g that impose the constraints; these act to glue simplices through tetrahedra τ

$$S_{\text{Tot}} = \sum_t S_t^a(a) + \sum_\sigma S_\sigma^a(a) + \sum_{\tau \subset \text{blk}} g_\tau^{\sigma,\sigma'}(a).$$

This constraint discussion was classical, finally we come to our quantum input: the discrete area spectrum found above

$$a(j) = \gamma a_P \sqrt{j(j+1)} \sim \gamma a_P (j + 1/2), \quad \text{with} \;\; a_P = 8\pi\hbar G/c^3.$$

(Again j is an half-integer spin label and γ is the area gap.) But, this leads to an important tension…

If we impose the constraints too strongly, there will be no tetrahedra with (half-integer) areas that satisfy them.

We are forced to navigate between Scylla—reducing too much the density of states—and Charybdis—imposing dynamics that does not match GR ⇝ weak imposition of constraints.

Defining an Effective Spin Foam model

In this context we can define the spin foam

$$\mathcal{Z} = \sum_{\{j_t\}} \mu(j) \prod_t \mathscr{A}_t(j) \prod_\sigma \mathscr{A}_\sigma(j) \prod_{\tau \in \mathrm{blk}} G_\tau^{\sigma,\sigma'}(j),$$

with

$$\mathscr{A}_t = e^{\mathrm{i}\gamma n_t \pi (j_t + \frac{1}{2})} \qquad \text{and} \qquad \mathscr{A}_\sigma = e^{-\mathrm{i}\gamma \sum_{t \supset \sigma} (j_t + \frac{1}{2}) \theta_t^\sigma(j)}.$$

In practice, we take $\mu(j) = 1$ for spins satisfying the constraints.

The factors $G_\tau^{\sigma,\sigma'}$ implement the constraints: imposing these sharply, with $G_\tau^{\sigma,\sigma'} = 1$ if satisfied and 0 else, leads to diophantine eqs. for the constraints that will only be satisfied for rare and special labels $\{j_t\}$;

this is the key fact that $\rightsquigarrow$ weak imposition of the constraints

Coherent state peaked on P's

We implement the constraints with

$$G_\tau^{\sigma,\sigma'}(j) = \langle \mathscr{K}_\tau(\,\cdot\,; P_{e_i}^{\tau,\sigma}(j)) \,|\, \mathscr{K}_\tau(\,\cdot\,; P_{e_i}^{\tau,\sigma'}(j)) \rangle.$$

Inputs and Approximations for the Numerics

The spin foam

$$\mathscr{Z} = \sum_{\{j_t\}} \mu(j) \prod_t \mathscr{A}_t(j) \prod_\sigma \mathscr{A}_\sigma(j) \prod_{\tau \in \mathrm{blk}} G_\tau^{\sigma,\sigma'}(j),$$

with $\mu(j) = 1$,

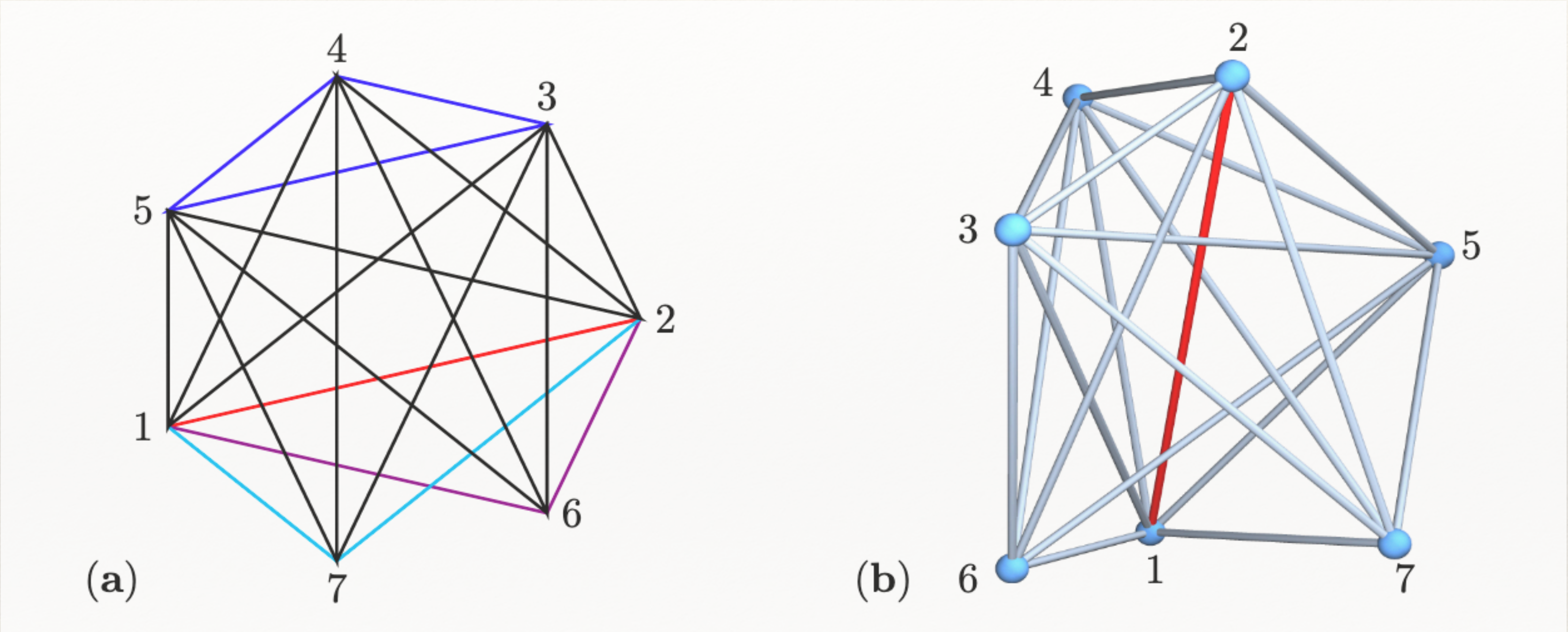
$$\mathscr{A}_t = e^{\mathrm{i}\gamma n_t \pi (j_t + \frac{1}{2})} \qquad \text{and} \qquad \mathscr{A}_\sigma = e^{-\mathrm{i}\gamma \sum_{t \supset \sigma} (j_t + \frac{1}{2}) \theta_t^\sigma(j)}.$$

To keep the numerics tractable researchers:

- consider symmetry reduced triangulations
- approximate the coherent inner products by real gaussians with widths determined by the $\{\mathscr{C}_i, \mathscr{C}_j\} = \pm\gamma(9/2)\mathrm{Vol}_\tau^2$ non-commutation
- and consider scaling with both j and γ.

Numerical Results

I will present results for the following triangulation



[Asante, Borissova, Dittrich, Gielen, HMH, Kogios, Padua-Arguelles, Schander, Simao, Steinhaus, …: arXiv:2004.07013, arXiv:2011.14468, arXiv:2104.00485, arXiv:2105.10808, arXiv:2109.00875, arXiv:2112.15387 , arXiv:2203.02409, arXiv:2207.03307, arXiv:2303.07367, arXiv:2306.06012, arXiv:2206.13540,...]

Symmetry reduced numerical triangulation:

Δ consists of 6 simplices around one edge

We apply a certain symmetry reduction, so that there are only 3 bndry and 3 bulk areas (4 bndry lengths and 1 bulk length).

There are 3 simplices of type 1 and three simplices of type 2. In each type, all simplices have the same geometry.

The path integral involves 1 bulk variable in LRC and 3 area variables in (constrained) ARC. However, making use of the fall off of the G functions, we can significantly reduce the summation range and gain time savings in the numerics.

Symmetry reduced numerical triangulation:

Δ consists of 6 simplices around one edge

For completeness, here is the definition of this Δ:

vertices:	m,n=0,1; i,j=2,3,4	k=5,5′
simplices:	(0,1,2,3,5)	(01,2,3,5′)
	(0,1,2,4,5)	(0,1,2,4,5′)
	(0,1,3,4,5)	(0,1,3,4,5′)
lengths:	$l_{01} = t$ blk	$l_{01} = t$ blk
	$l_{mi} = l_{ik} \equiv x$	$l_{mi} = l_{ik} \equiv x$
	$l_{ij} \equiv y$	$l_{ij} \equiv y$
	$l_{m5} \equiv z$	$l_{m5'} \equiv z'$
areas:	$A(x, x, y)$	$A(x, x, y)$
	$A(x, x, t)$ blk	$A(x, x, t)$ blk
	$A(x, x, z)$	$A(x, x, z')$
	$A(z, z, t)$ blk	$A(z', z', t)$ blk

This model illustrates that spin foams can avoid the flatness problem in a range of spin j and Barbero-Immirzi parameter γ

E.g. at $\gamma = 0.1$, for Δ_3 we have

$$\epsilon(A(x,x,t)) = 3.19 - 0.20\mathrm{i}, \quad \epsilon(A(z,z,t)) = -1.32 + 0.18\mathrm{i}, \text{ and}$$

$$\epsilon(A(z',z',t)) = -0.59 + 0.07\mathrm{i}$$

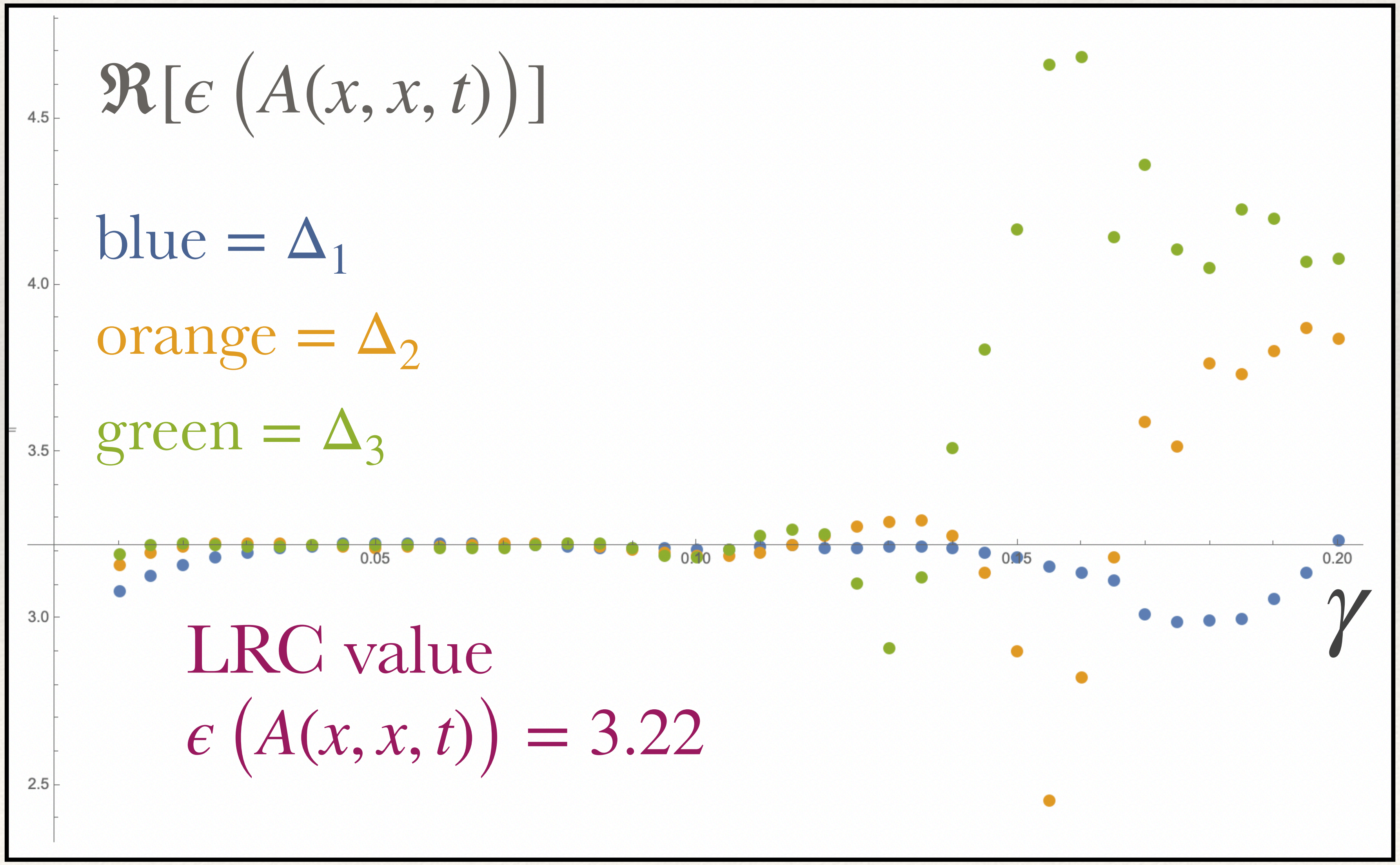
Compare the LRC values:

$\epsilon(A(x,x,t)) = 3.22$

$\epsilon(A(z,z,t)) = -1.36$

and

$\epsilon(A(z',z',t)) = -0.607$



Effective Spin Foam models

The structure of an effective spin foam, on a fixed triangulation Δ, can be decomposed into three parts:

$$\mathcal{Z}_{\text{ESF}} = \sum_{\{a_t\}} \mu(a) \exp\left(\frac{\mathrm{i}}{\hbar} S_{\text{ARC}}(a)\right) \prod_{\tau} G_{\tau}^{\sigma,\sigma'}(a)$$

$\mu(a)$ is the measure term, possibly fixed by course grain or diffeos

$S_{\text{ARC}}(a)$ is the area Regge action

$G_{\tau}^{\sigma,\sigma'}(a)$ implements the constraints on areas weakly via a Gaussian

Spin foam models are beginning to access the dynamical regime of Quantum Gravity. In particular, ESFs are being used to study many features of quantum gravity, such as sum over orientations, causal structures, topology change, etc.

The End

I am hugely grateful to the Quantum Information Structure of Spacetime (QISS) Project and to the Perimeter Institute for Theoretical Physics for their support of my work.

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Connections, connections, connections

With the introduction of a Lorentz bundle, we have a new kind of vector over every point of $\mathscr{M}$ (internal vectors)

How should we parallel transport these?

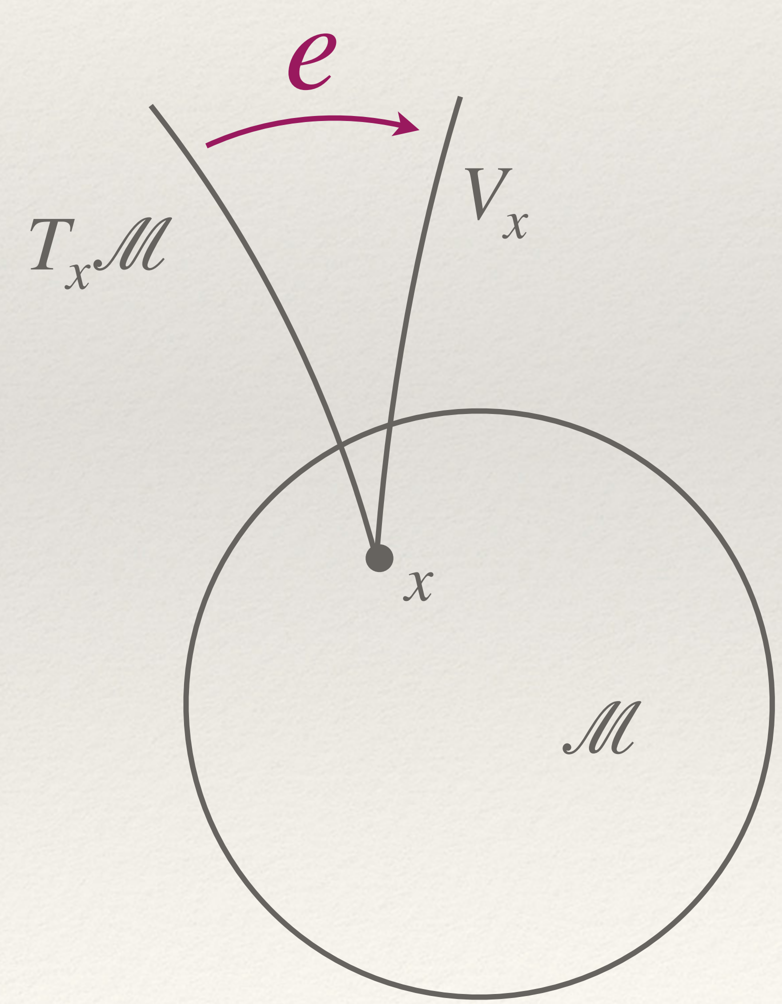
Ans: the spin connection. The idea is

$$\mathscr{D}_\mu v^I = e^I_\nu \nabla_\mu v^\nu$$
$$= \partial_\mu v^I + \omega_\mu{}^I{}_J v^J,$$

with ω_μ^{IJ} the spin connection.

We have $\omega_\mu{}^{IJ} = e^I_\nu \nabla^{LC}_\mu e^{\nu J}$ when

$$\mathscr{D}_\mu \eta^{IJ} = 0 \iff \omega_\mu{}^{(IJ)} = 0$$
$$d_\omega e^I := de^I + \omega^{IJ} \wedge e_J = 0$$

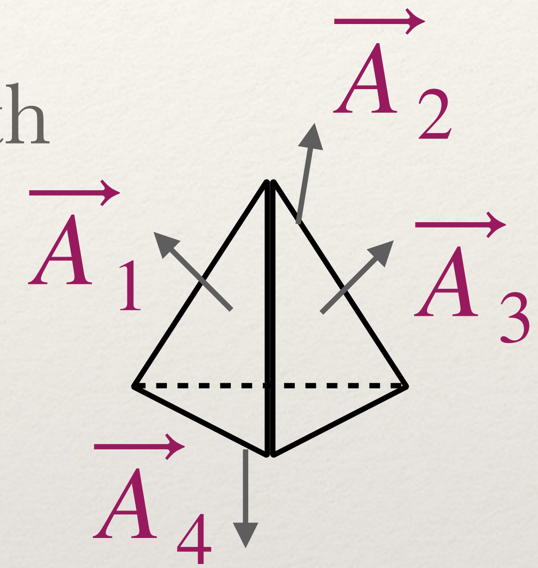


Intriguingly, the quantum area geometry of tetrahedra is non-commutative. We can see this by looking into the variables p_e.

The areas A_t and 2 inner products p_{e_1}, p_{e_2} completely describe a tetrahedron τ: $p^\tau_{tt'} := p^\tau_e = \mathrm{sgn}(V^2_\tau)\, \hat{n}_t \cdot \hat{n}_{t'}$, & $p^\tau_{tt} = A^2_t$.

The area vectors satisfy $\{A^i_t, A^j_t\} = \gamma c^{ij}_k A^k_t = \gamma \epsilon^{ijm} \kappa_{mk} A^k_t$ with

$$\kappa_{ij} = \begin{cases} \delta_{ij} & \text{if Euclidean} \\ \eta_{ij} & \text{if Lorentzian} \end{cases}.$$



For a triple of triangles (t, t', t'') with angle parameters $p^\tau_{tt'}$ and $p^\tau_{t't''}$:

$$\{p^\tau_{tt'}, p^\tau_{t't''}\} = \kappa_{ii'} \kappa_{jj'} A^i_{t'} A^j_{t''} \{A^{i'}_t, A^{j'}_t\} = \gamma \epsilon^{i'j'k'} \kappa_{ii'} \kappa_{jj'} \kappa_{kk'} A^i_{t'} A^j_{t''} A^k_t$$

$$= \gamma \vec{A}_t \cdot (\vec{A}_{t'} \times \vec{A}_{t''}) = \pm \gamma \frac{9}{2} \mathrm{Vol}^2_\tau$$

For fixed areas these degrees of freedom *do not commute*. Quantum mechanically they encode the shape of a fuzzy quantum tetrahedron.

Weak Constraints

It will be useful to zoom out and consider a simple toy model with weak constraints. This will give a sense of their general behavior:

Consider the oscillatory integral

$$\iint e^{i\lambda S(x,y)} e^{-\mu \mathscr{C}(x,y)^2} dxdy.$$

We impose a constraint $\mathscr{C}$ in both a strong & a weak manner and compute expectation values for $\mathscr{O} = e^{-x^2}$ using

$$\langle \mathscr{O} \rangle_\mu = \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(i\lambda(x^2+y^2)\right) \exp\left(-\mu(y-x+2)^2\right) \exp\left(-x^2\right) dydx}{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(i\lambda(x^2+y^2)\right) \exp\left(-\mu(y-x+2)^2\right) dydx}$$

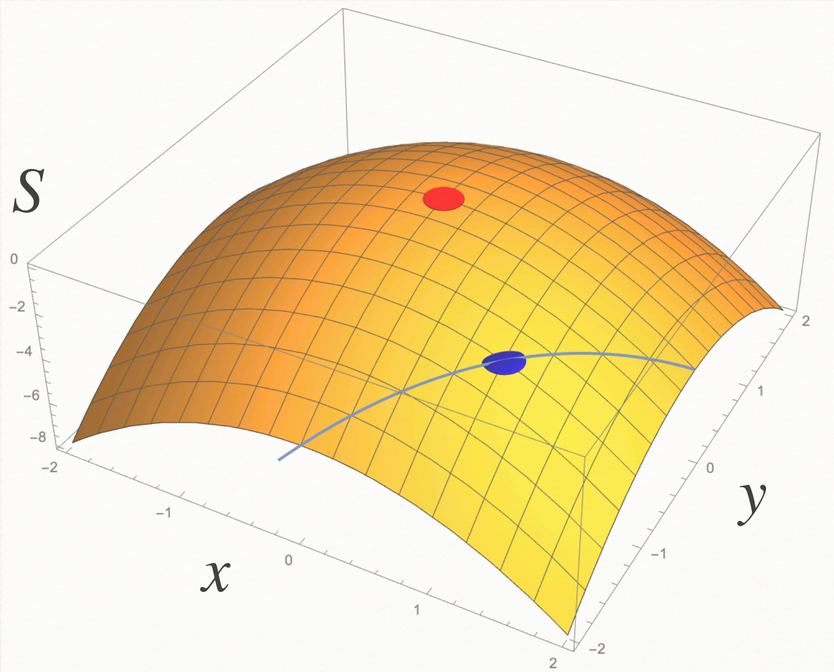
with $S = x^2 + y^2$ and $\mathscr{C} = y - (x - 2)$.

Weak Constraints

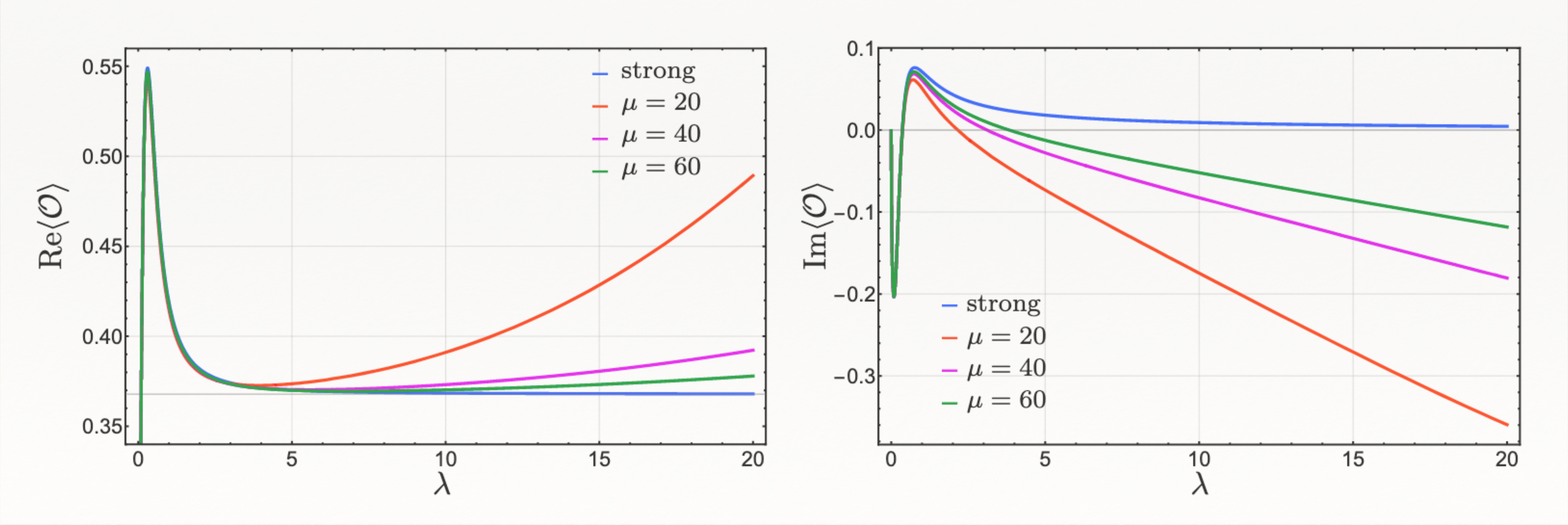
Take the oscillatory integral

$$\iint e^{i\lambda S(x,y)} e^{-\mu \mathscr{C}(x,y)^2} dxdy,$$

with $S = x^2 + y^2$ and $\mathscr{C} = y - (x-2)$.

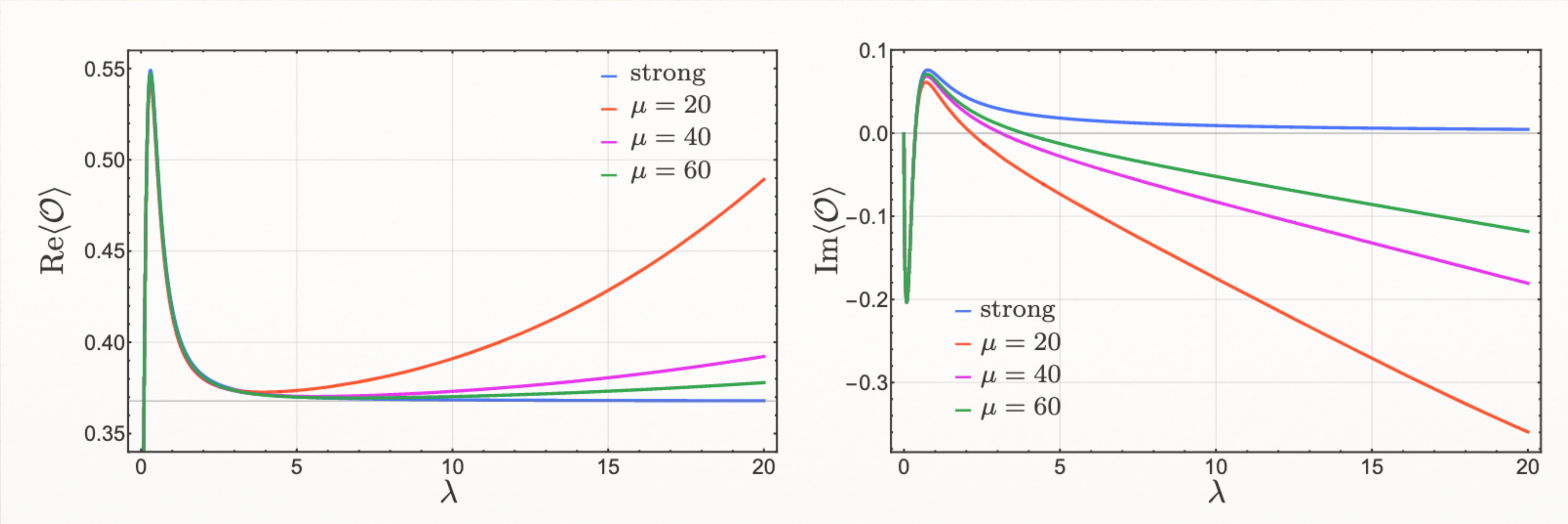


The constrained action $S = x^2 + (x-2)^2$ has a critical point at $(x, y) = (1, -1)$, and hence the classical expectation value is $\langle \mathcal{O} \rangle_{Cl} = e^{-1} \approx 0.368$. Compare

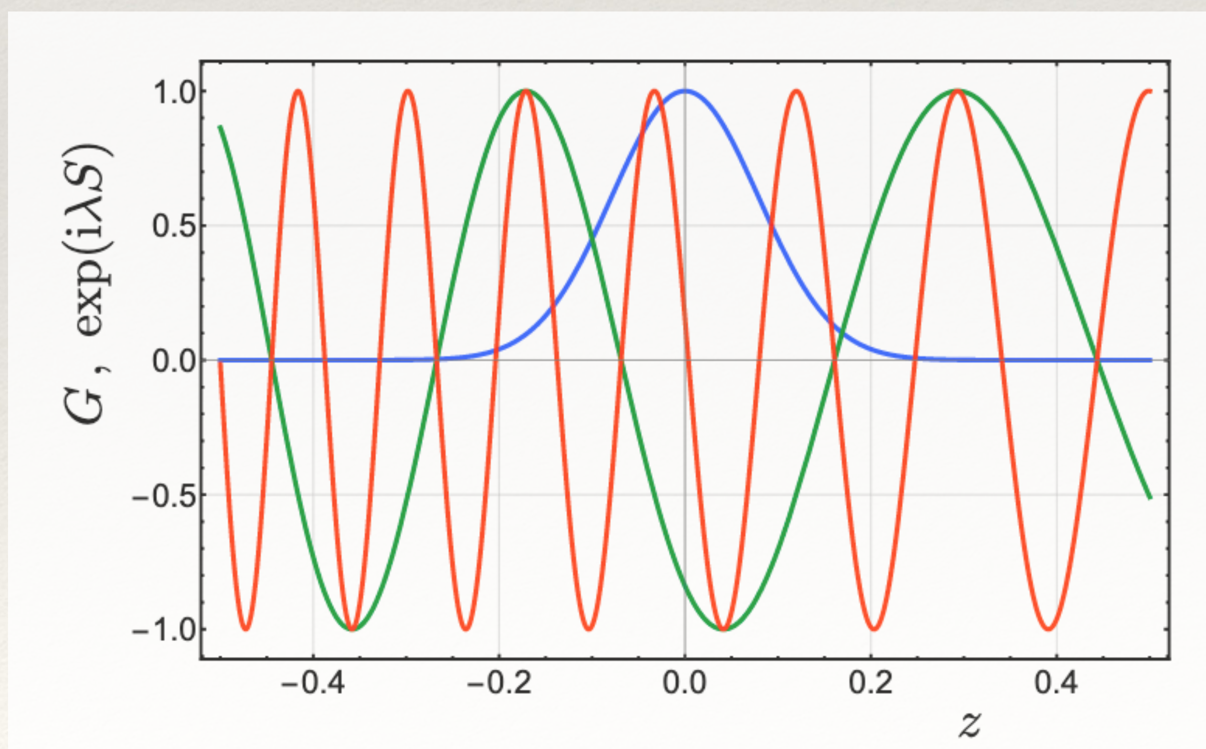


Weak Constraints

Why does the expectation value escape at large λ?



There is an interplay between the integrand's oscillations and the gaussian constraints:



G

$\lambda = 5$

$\lambda = 15$

Semiclassical Regime

The # of oscillations of the phase factor occurring over the width of the Gaussian (near const. crit. pt.) should be less than a number of order 1.

We turn this into a 1D problem by considering the direction of the steepest change of the constraint $\vec{c} = \vec{\nabla} C / |\vec{\nabla} C|$ and require

gaussian width in $\vec{c}$ direction

$$\lambda \times \left(\left| \vec{\nabla} S \cdot \vec{c} \right| \right) \Big|_{\text{const. crit. pt.}} \times \sigma(\vec{c}) \lesssim \mathcal{O}(1).$$

Plugging in these factors for the Effective Spin Foam models gives

$$\frac{\sqrt{\gamma a_t}}{\ell_P} \epsilon_t = \gamma \sqrt{j} \epsilon_t \lesssim \mathcal{O}(1).$$

This formula is the key to understanding the 'flatness problem': the semiclassical regime is *not* just $j \gg 1$!