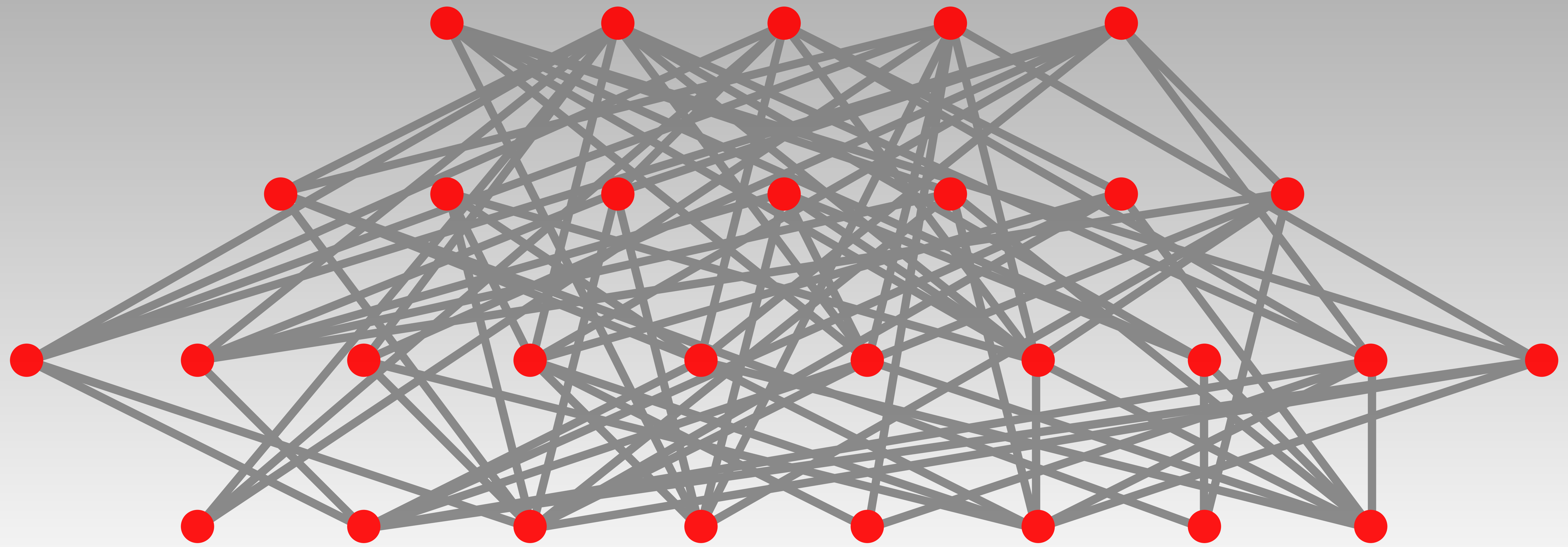
Calibrating the continuum approximation of discrete quantum gravity



Sumati Surya
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Quantum Gravity 2023

Outline

- The Continuum Approximation of quantum gravity: With or Without Discreteness
- The Causal Set Way : Quantising the causal structure
    - The Continuum Approximation: random discretisation
    - Does the Continuum emerge from Dynamics?
    - Uniqueness?
- Calibrating the continuum approximation: new techniques

The Continuum Approximation: With or Without Discreteness

- Different Physics at different scales:

GR: $V \gg V_p$, Planck scale: $V \sim V_p$, Trans-Planckian scale: $V \ll V_p$ (?)

- GR emergent from Planck scale : what Planck scale physics should one forget to get GR?
  - Quantum Gravity DOF $\mathbb{QG}$: Discrete or Continuum
  - $\mathbb{QG} \sim (M, g)$ at $V >> V_p$
- How do we recognise the continuum approximation when we see it?

Continuum or Discrete DOF

- If $\mathbb{QG}$ is a quantum spacetime geometry (or any other classically equivalent set of continuum quantities), with some continuity requirements: $C^2, C^1, C^0 \ldots$ OR is discrete (graph, triangulation, network, causal set, ..)

- What distinguishing features does $\mathbb{QG}$ have at the Planck scale V_p? (Is there physics below V_p for continuum theories?)

- Say your theory spits out a (coherent state of) continuum geometry $(M, g) \in \mathscr{M}_r$ but $r < 2$ or some pre-geometric $\mathbf{Q}$ (example: piece-wise continuous, orbifolds, extra dimensions, etc.)

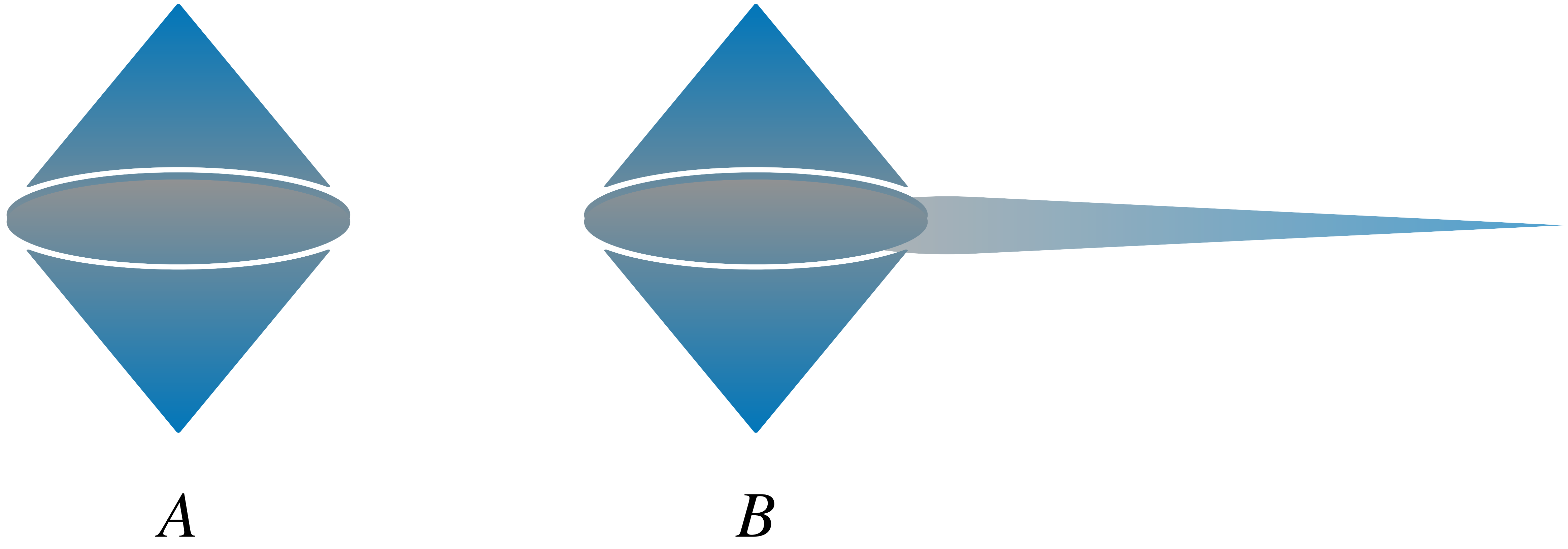
- Does (M, g) or $\mathbf{Q}$ ***approximate*** $(\mathbf{M}, \mathbf{g}) \in \mathscr{M}_2$ (or desired differentiability) at a scale $V \gg V_p$?

Uniqueness

- Is this approximation unique?

$$\mathbb{QG} \sim_V (\mathbf{M}, \mathbf{g}) \text{ AND } \mathbb{QG} \sim_V (\mathbf{M}', \mathbf{g}') \text{ are } (\mathbf{M}, \mathbf{g}) \text{ and } (\mathbf{M}', \mathbf{g}') \text{ "close" at } V \gg V_p?$$

- What does it mean for two spacetimes to be close at a given scale?



— Bombelli, 2000, Bombelli and Noldus, 2004
—Burtscher and Allen, 2021,
--Kunzinger and Steinbauer, 2021, ...

The Causal Set Way: Quantising the Causal Structure

Lorentzian Spacetime (Causal, Distinguishing) = $(M, \prec\ ) + \epsilon$

-Hawking, Hawking-King-MacCarthy, Malament, Kronheimer-Penrose

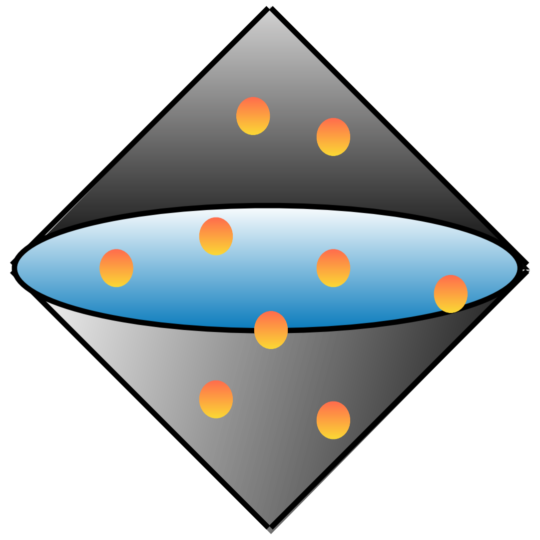
$(M, \prec\ )$ is a poset :(i) Acyclic: $x \prec y, y \prec x \Rightarrow x = y$ (ii) Transitive: $x \prec y, y \prec z \Rightarrow x \prec z$

The Causal Set Hypothesis:

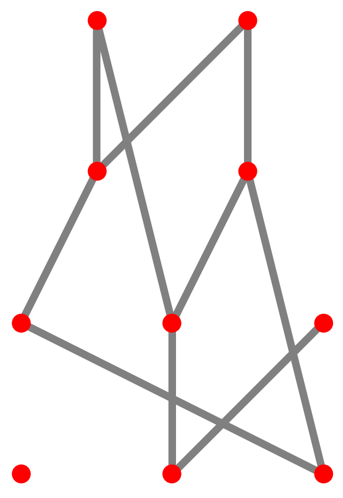
-Myrheim, 1978
-Bombelli, Lee, Meyer and Sorkin,1987

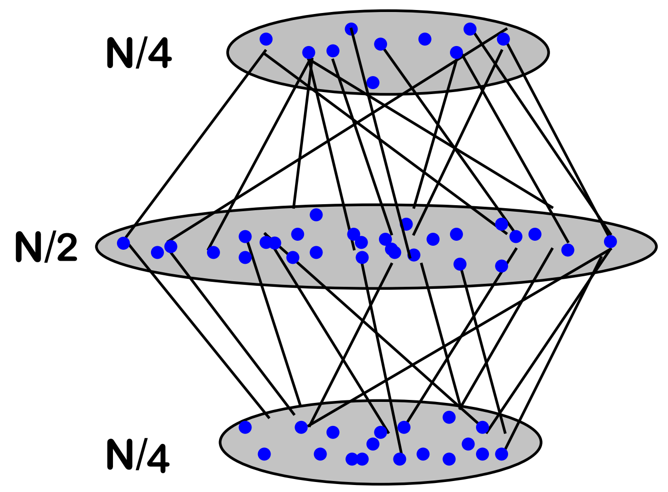
...Robb, Zeeman, Penrose, Kronheimer, Finkelstein, Myrheim, Hemion, 't Hooft ...

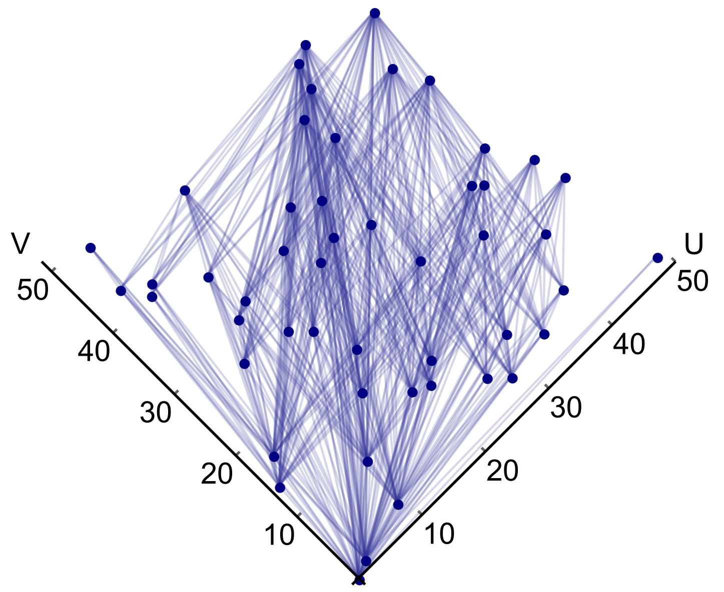
1. Locally finite posets or Causal Sets are the fine grained structure of spacetime



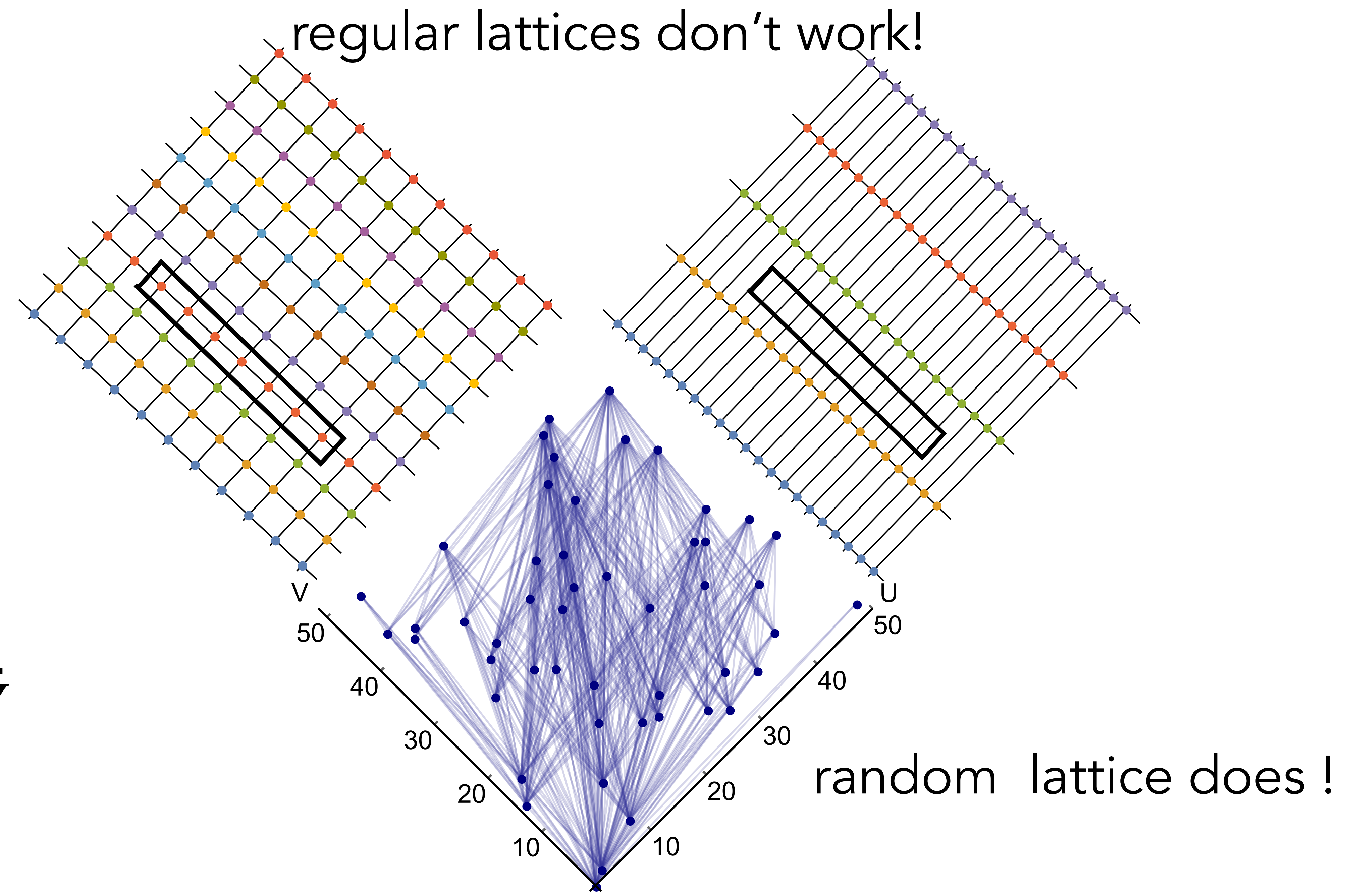
2. Continuum Approximation: $Order + Number \sim Spacetime$ (counting replaces volume)







The Continuum Approximation

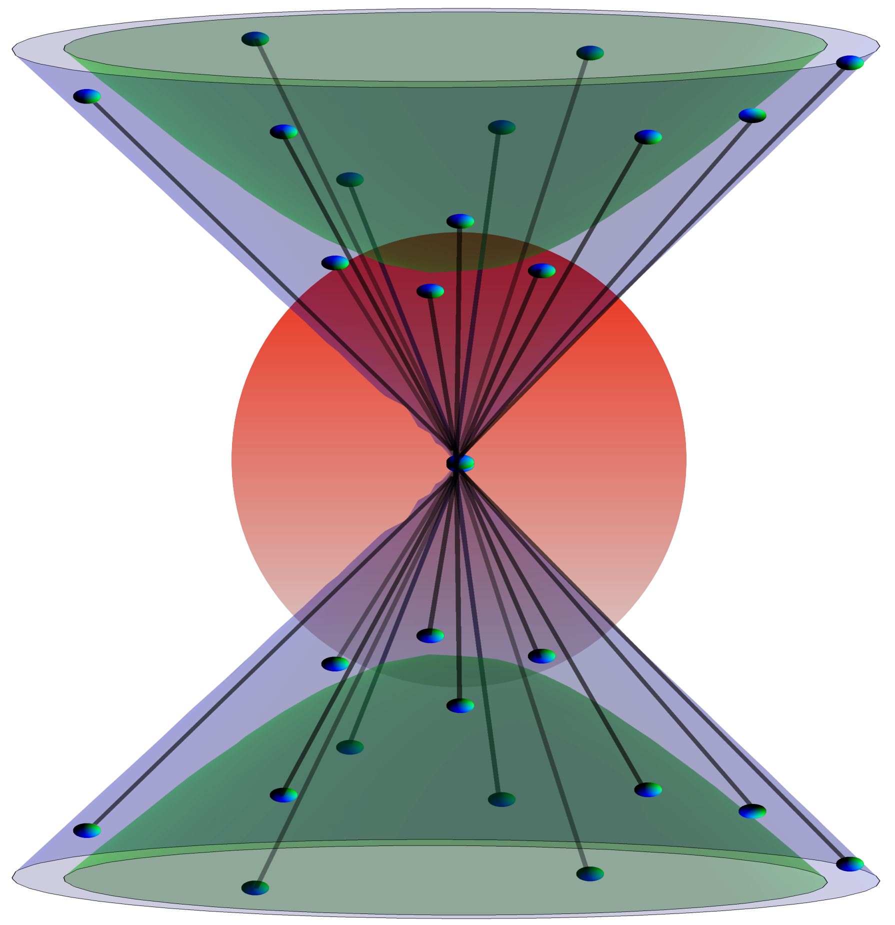


- $n \sim \rho V$ correspondence has to be diffeo invariant
- Random discretisation via a Poisson sprinkling process:
    - $P_V(n) = \frac{(\rho V)^n}{n!} e^{-\rho V}$, $\langle n \rangle = \rho V$, $\Delta n = \sqrt{\rho V}$
- For every causal spacetime (M, g) there is a kinematic ensemble $\{C\}_\rho$ (first quantisation)
- $C \sim_\rho (M, g)$ is a faithful embedding at density ρ if :
    - $C \hookrightarrow (M, g)$ is order preserving
    - n_V : number of points in spacetime volume V is a random variable, $P_V(n) = \frac{(\rho V)^n}{n!} e^{-\rho V}$

ρ sets a scale

Causal Set Non-locality

- The nearest neighbours lie all along the light cone
- A continuum-like causal set is a graph without a fixed valency



Preserving Local Lorentz invariance

Theorem : — *Bombelli, Henson and Sorkin,2006*

There is no measurable map $D : \Omega \to H$ *which is equivariant, i.e.,* $D \circ \Lambda = \Lambda \circ D$.

- *Measure triple* (Ω, Σ, μ)
- *unit hyperbola* $H \subset \mathbb{M}^d$ *(fdtl directions)*

Proof: If such a map existed, then $\mu_D = \mu \circ D^{-1}$ *is a Lorentz invariant probability measure on* H *which is not possible since is* H *non-compact.*

Calibrating the Continuum Approximation

- Is this approximation unique?

FUNDAMENTAL CONJECTURE:

If $C \sim_\rho (\mathbf{M}, \mathbf{g})$ AND $C \sim_\rho (\mathbf{M}', \mathbf{g}')$ then $(\mathbf{M}, \mathbf{g})$ and $(\mathbf{M}', \mathbf{g}')$ are "close" at some $\rho^{-1} \gg V_p$.

- Order Invariants $\mathcal{O}$ as Geometric Invariants $\mathbb{G}$

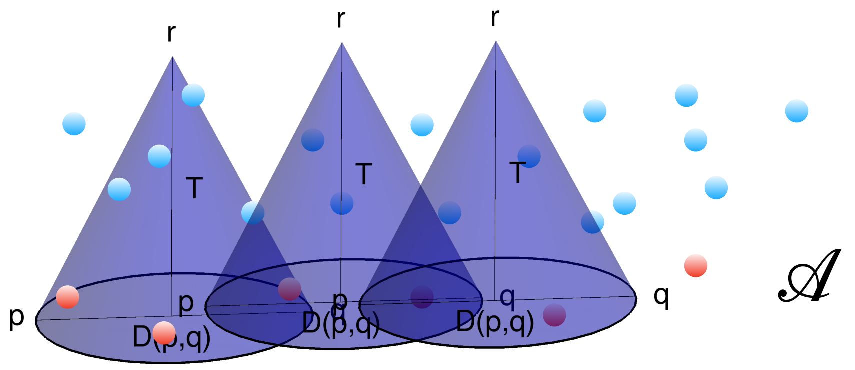
- **WEAK FORM OF FUNDAMENTAL CONJECTURE:**

If $C \sim_{(\rho,\mathcal{O})} (\mathbf{M}, \mathbf{g})$ AND $C \sim_{(\rho,\mathcal{O})} (\mathbf{M}', \mathbf{g}')$ then $(\mathbf{M}, \mathbf{g}) \sim_{(\rho,\mathbb{G})} (\mathbf{M}', \mathbf{g}')$

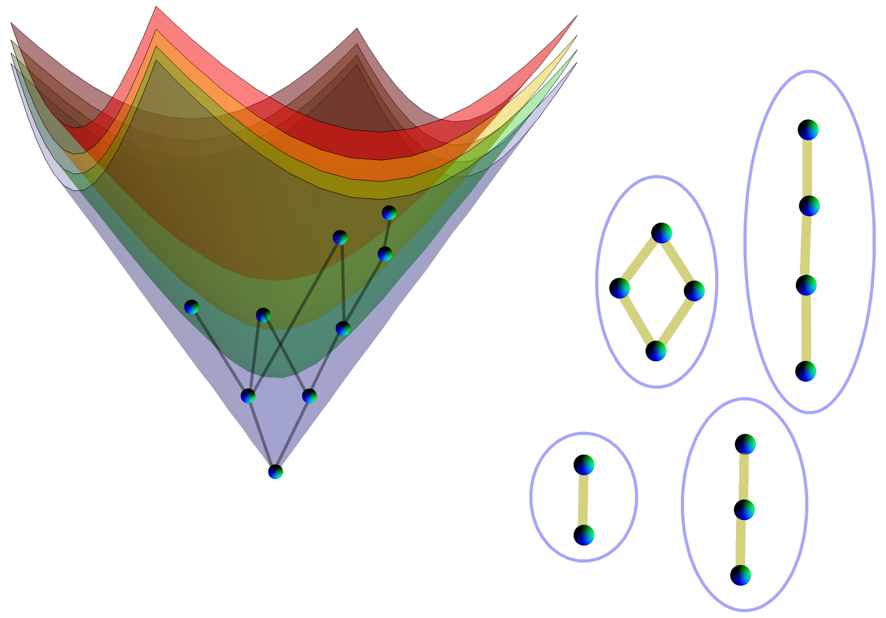
- Which order invariants $\mathcal{O}$ correspond to Geometric Observables $\mathbb{G}$?

Geometric Reconstruction: geometry from counting

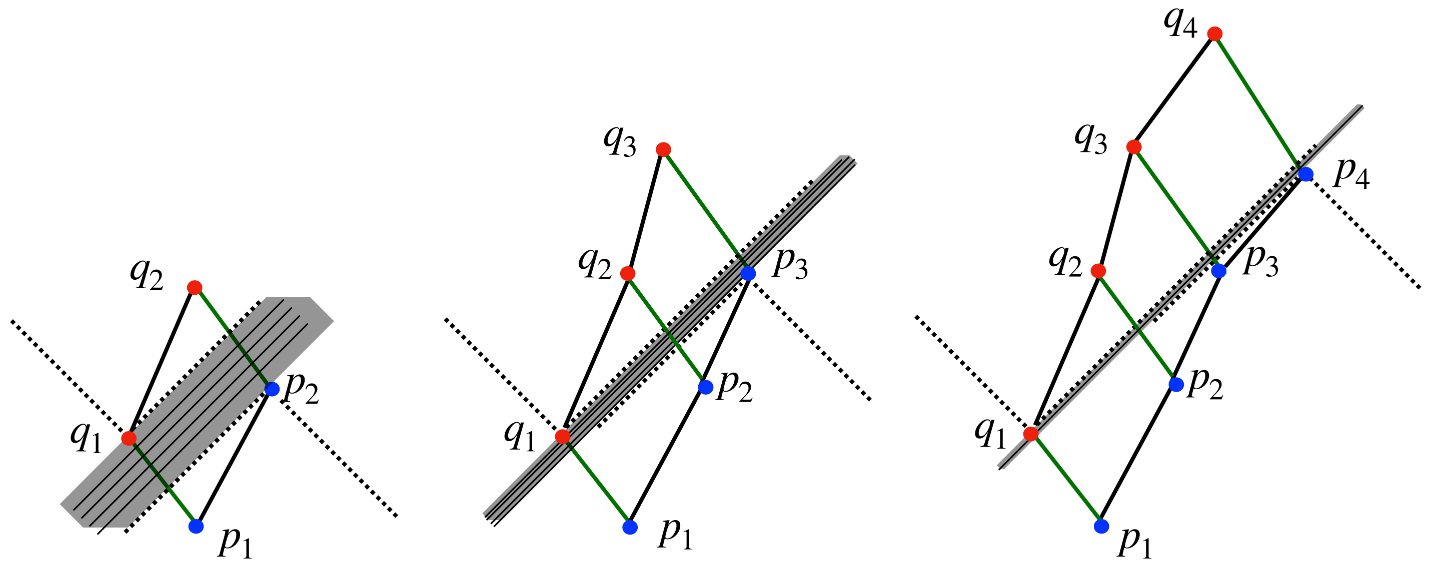
- Dimension Estimators — *Myrheim, Myer, Glaser & Surya, ..*
- Timelike Distance — *Brightwell & Gregory*
- Spatial Homology —*Major, Rideout & Surya*



- Spatial and Spacelike Distance — *Rideout & Wallden, Eichhorn, Mizera & Surya, Eichhorn, Surya & Versteegen*
- D'Alembertian — *Sorkin, Henson, Benincasa & Dowker, Dowker & Glaser*



- Benincasa-Dowker-Glaser Action — *Benincasa & Dowker, Dowker & Glaser*
- GHY terms in the Action — *Buck, Dowker, Jubb & Surya*
- Locality and Interval Abundance — *Glaser & Surya*
- Horizon Molecules — *Barton, Counsell, Dowker, Gould & Jubb, Machet and Wang*
- Scalar Field Greens functions —*Johnston, Dowker, Surya & Nomaan X*
- Scalar Field SJ vacuum — *Johnston, Sorkin, Yazdi, Nomaan X, Surya*
- Entanglement Entropy —*Dou & Sorkin, Sorkin & Yazdi, Yazdi, Nomaan X, Surya*
- **Null Geodesics from Ladder molecules -- *with A. Bhattacharya and A. Mathur, 2022***



Example: Dimension Estimator

- If $C \sim_\rho (\mathbb{M}^d, \eta)$
- Myrheim-Myer dimension estimator $\langle r \rangle = \dfrac{2\langle R \rangle}{\langle n \rangle^2} = \dfrac{\Gamma(d+1)\Gamma(d/2)}{4\Gamma(3d/2)}$
- If $C \sim_\rho (\mathbf{M}', \mathbf{g}')$, then $(\mathbf{M}', \mathbf{g}') \sim_{(\rho,d)} (\mathbb{M}^d, \eta)$
- Example $\mathbf{M}' = \mathbb{R}^d \times S^1$, $\mathbf{g} = \eta \oplus l$: For $\rho^{-1} >> V_c \times |S^1|$, $C \sim_{\rho,d} (\mathbf{M}', \mathbf{g}')$
- Therefore if C has Myrheim-Myer dimension d, then it cannot approximate a spacetime of dimension $d' \neq d$.

But is the continuum emergent from dynamics?

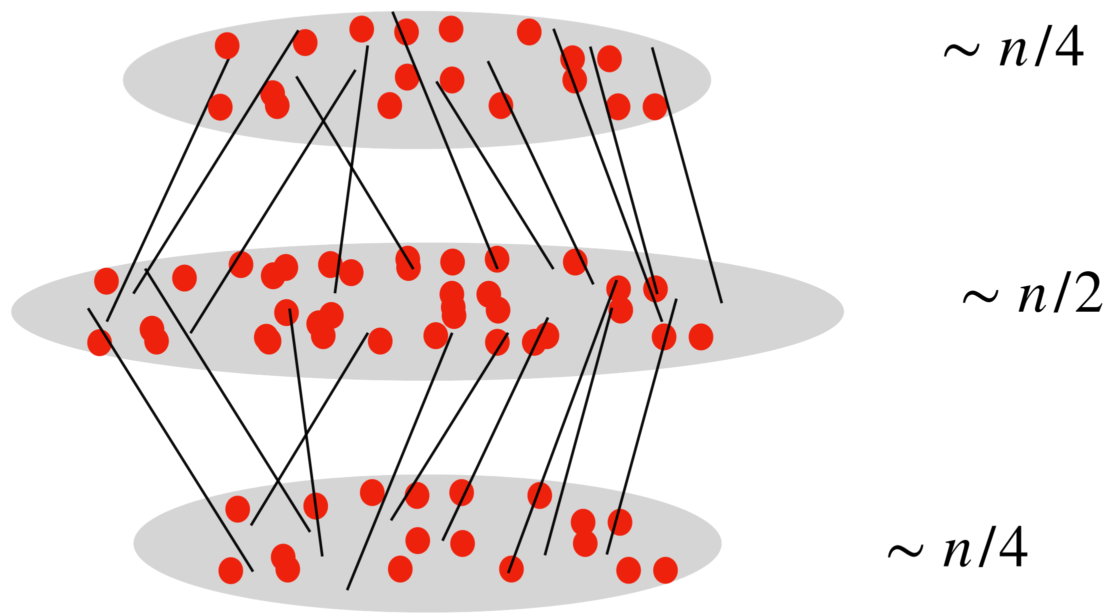
Lorentzian Path Sum over Ω_n

$$Z = \sum_{c \in \Omega_n} \exp(i S_{BDG}(c))$$

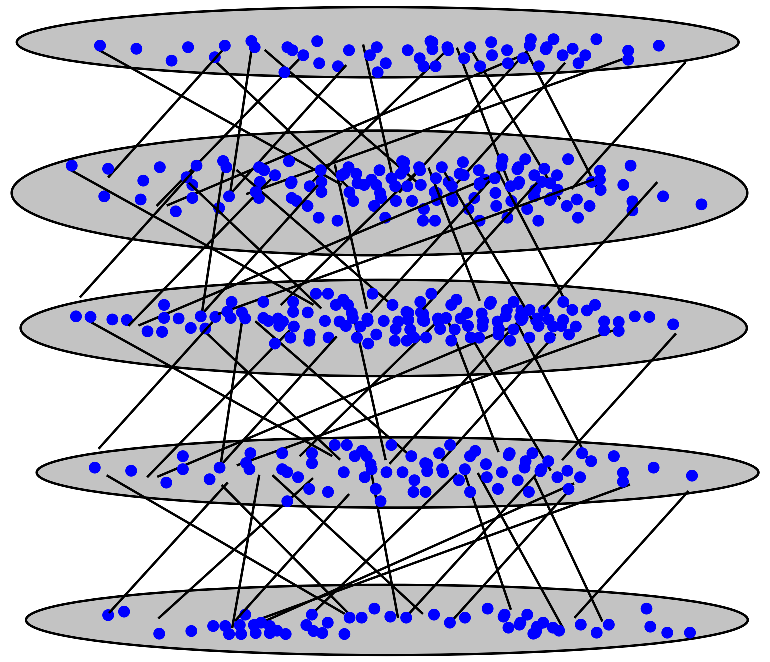
- Ω_n : sample space of all n-element causal sets

- $|\Omega_n| \sim 2^{\frac{n^2}{4}+\frac{3n}{2}+o(n)}$

- Typical causal sets are Kleitmann-Rothschild: $|\Omega_{KR}| \sim 2^{\frac{n^2}{4}+\frac{3n}{2}+o(n)}$

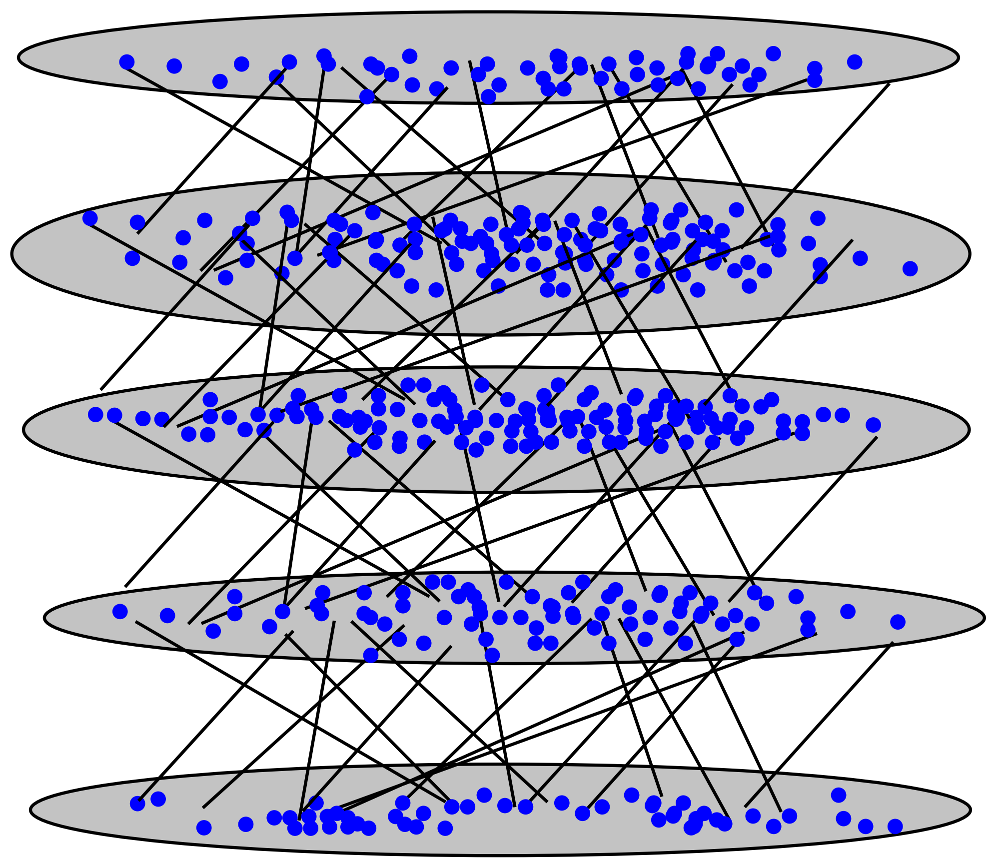


- Kleitman and Rothschild, Trans AMS, 1975
- J. Henson, D. Rideout, R. Sorkin and S.Surya, JEM, 2015

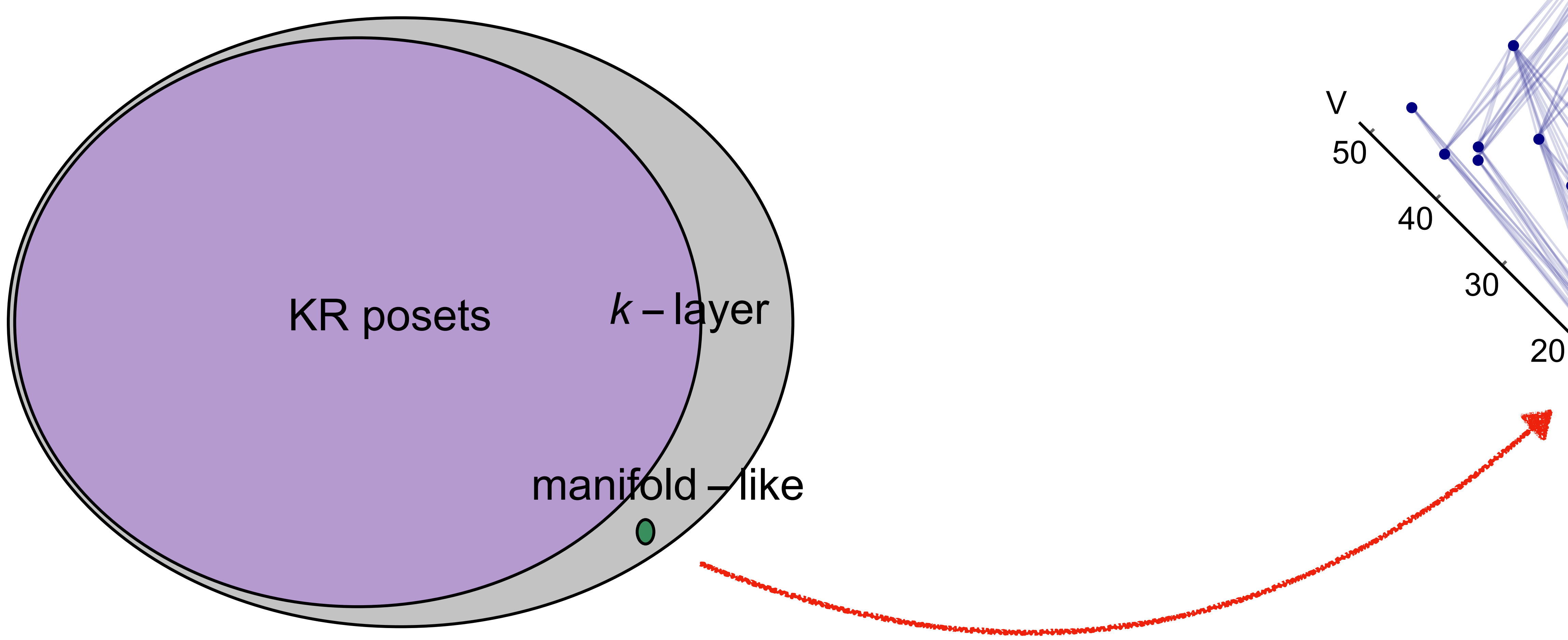


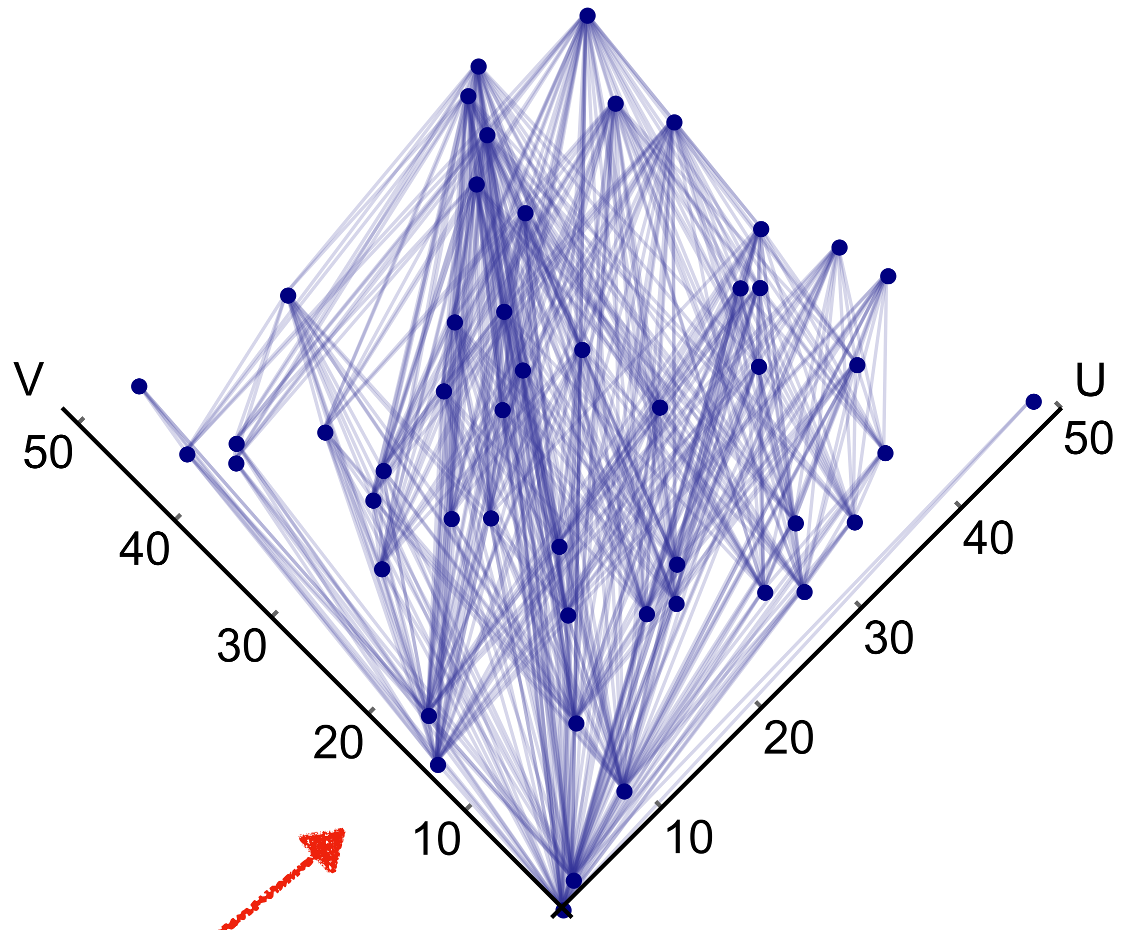
- Other layered Posets are subdominant: $\sim 2^{c(d)n^2+o(n^2)}, \quad c(d) \leq 1/4$

-D. Dhar, JMP, 1978
- Promel, Steger, Taraz 2001



Layered Posets are not manifold-like





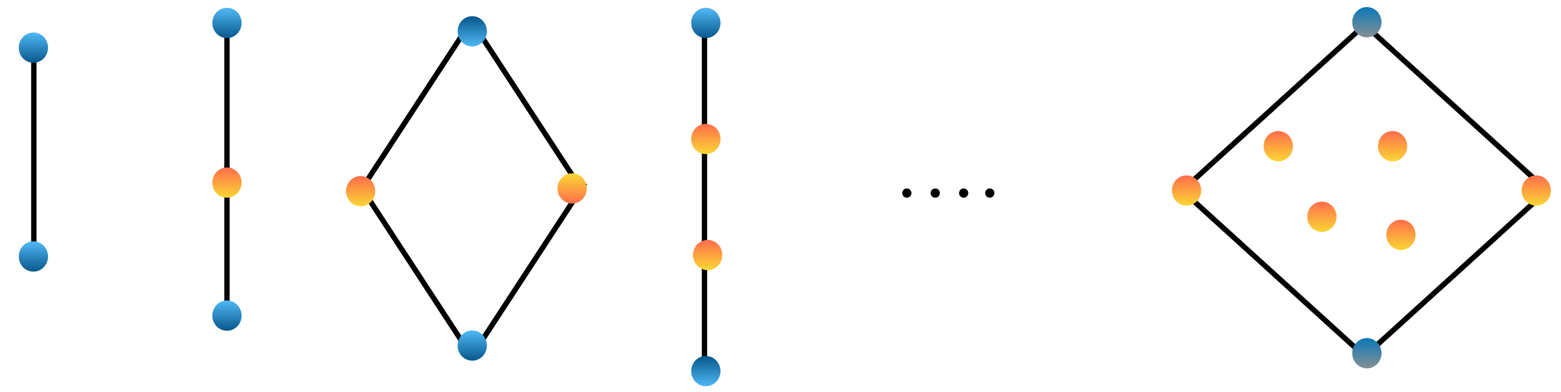
Do manifold-like causal sets stand a chance?

The Benincasa-Dowker-Glaser Action

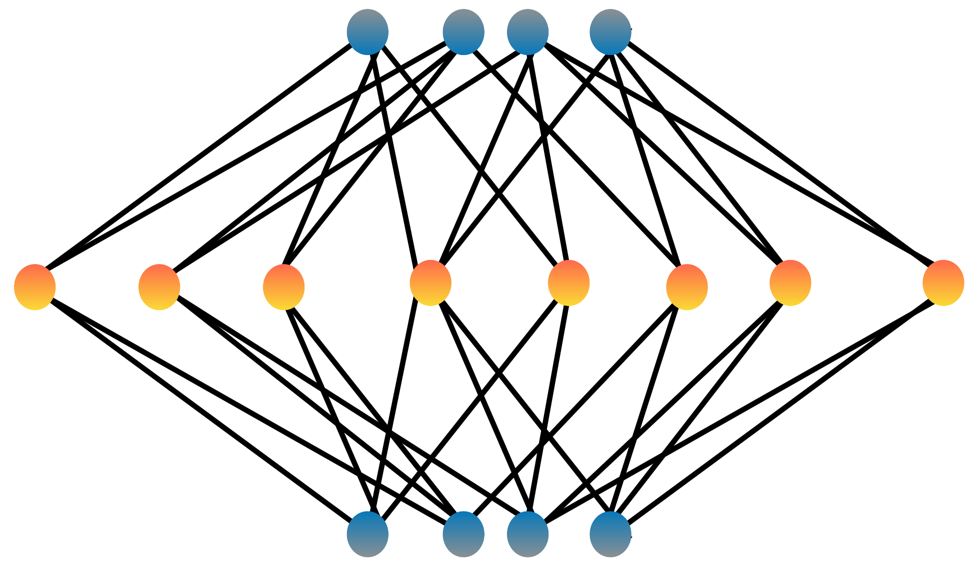
— *Benincasa & Dowker 2010,*
— *Dowker & Glaser 2011*
— *Glaser 2012*

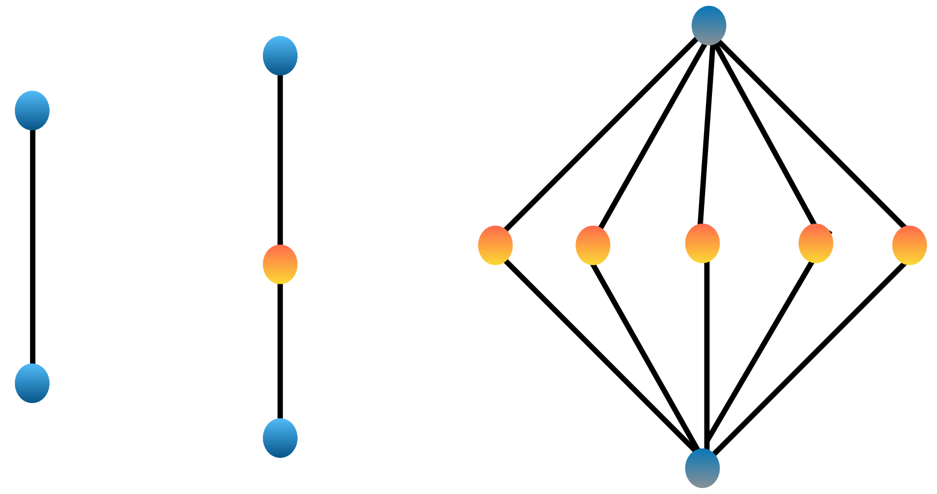
$$S^{(d)}_{BDG}(C) = \mu\left(n + \sum_{j=0}^{j_{max}} \lambda_j N_j\right)$$

$$S^{(4)}_{BDG} = \frac{4}{\sqrt{6}}\left(n - N_0 + 9N_1 - 16N_2 + 8N_3\right)$$



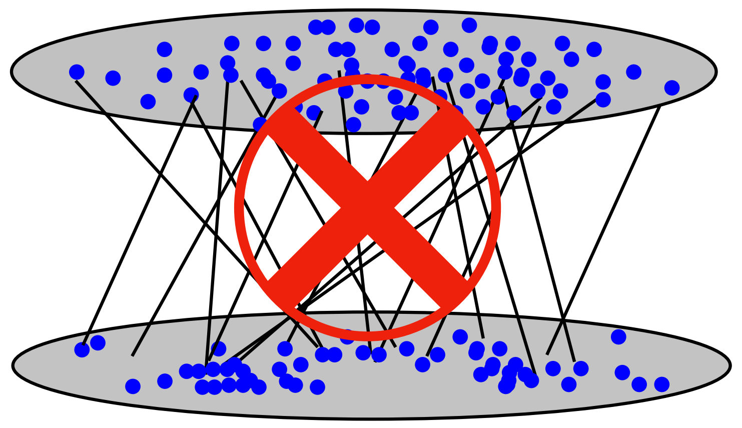
For the KR poset:





Do manifold-like causal sets stand a chance? Yes!

—Loomis and Carlip, 2017
— A.Anand Singh, A.Mathur and Surya, 2021
—P. Carlip, S. Carlip and S. Surya, 2022
— P. Carlip, S. Carlip and S. Surya, in preparation



Bilayer Posets :

$$S_{BDG}^{(4)} = \mu\left(n + \lambda_0 N_0\right)$$

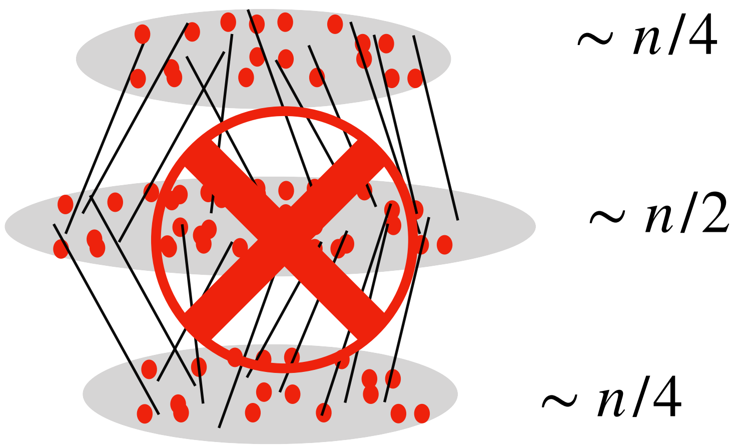
$$Z_{bilayer}[\mu, \lambda_0] \sim \int_0^{1/2} \mathrm{d}p\, |\mathscr{C}_{p,n}| \exp(iS_L(p)) = e^{i\mu n} \int_0^{1/2} \mathrm{d}p \exp\left[n^2\left(i\mu\lambda_0 p/2 + h(2p)/4\right) + o(n^2)\right]$$

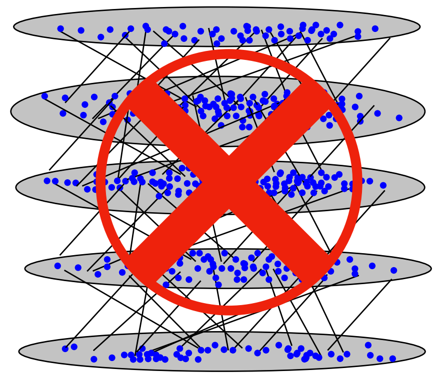
Suppression for:

$$\tan(-\frac{\mu\lambda_0}{2}) > \sqrt{3}$$

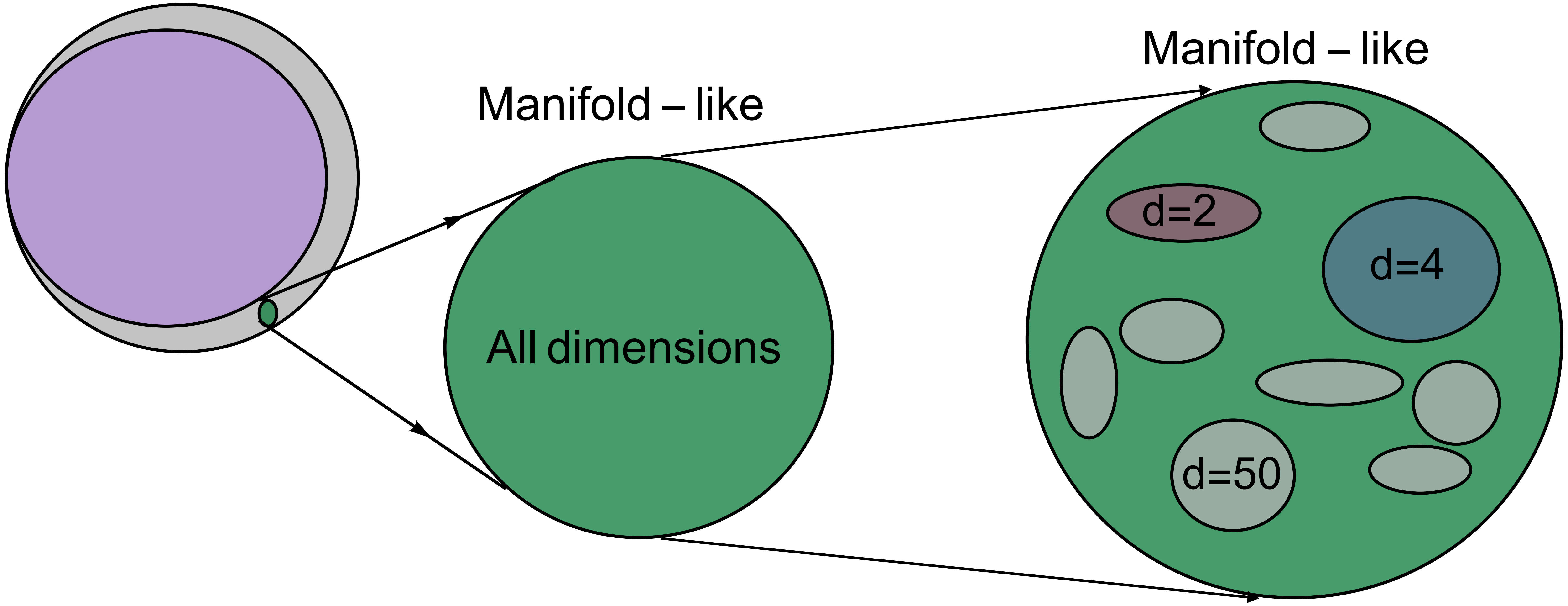
$$d = 4, \quad \mu = \left(\frac{l}{l_p}\right)^2 \Rightarrow l \approx 1.452 l_p$$

The discrete Einstein Hilbert action in any dimension suppresses all k-layer orders for $k << n$: Action wins over Entropy





Do manifold-like causal sets emerge?



Why $d = 4$ spacetime?

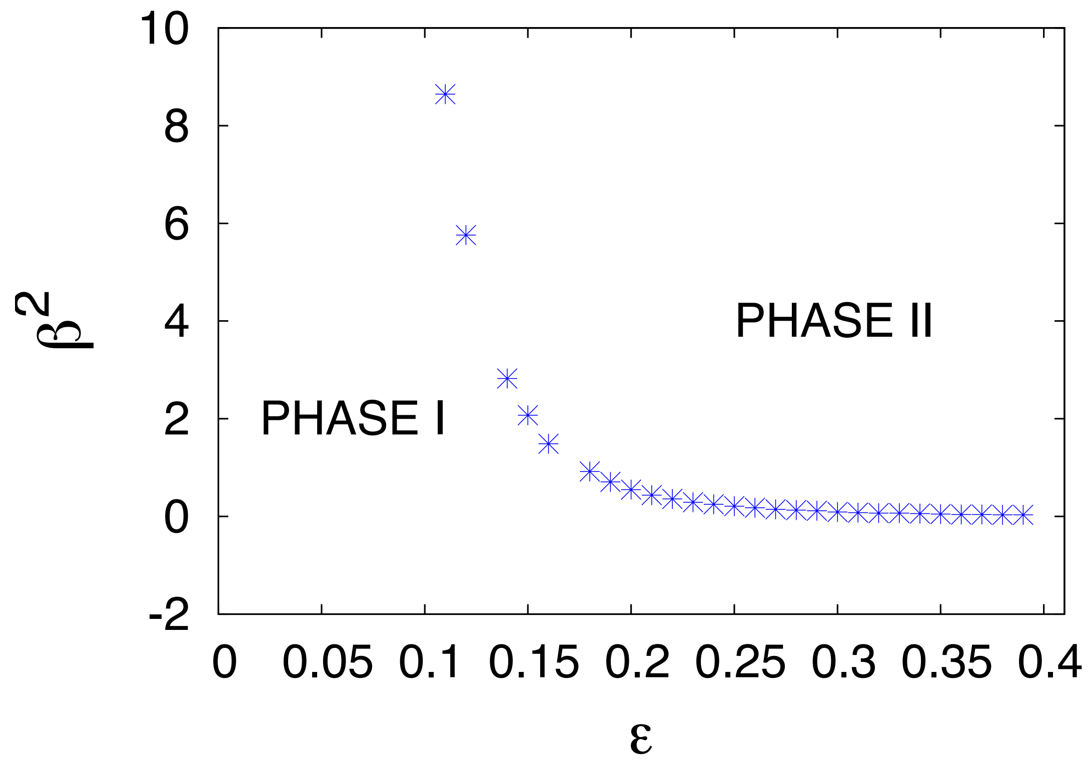
Does the d dimensional discrete Einstein Hilbert action suppress $d' \neq d$?
(Some hints..)

State Sum Models: Lorentzian Statistical Geometry

$$Z_\beta = \sum_{c\in\Omega} \exp(i\beta S_{BDG}(c)),$$

Inverse "temp": $\beta \to i\beta$

$$Z_\beta \to \tilde{Z}_\beta = \sum_{c\in\Omega} \exp(-\beta S_{BDG}(c))$$

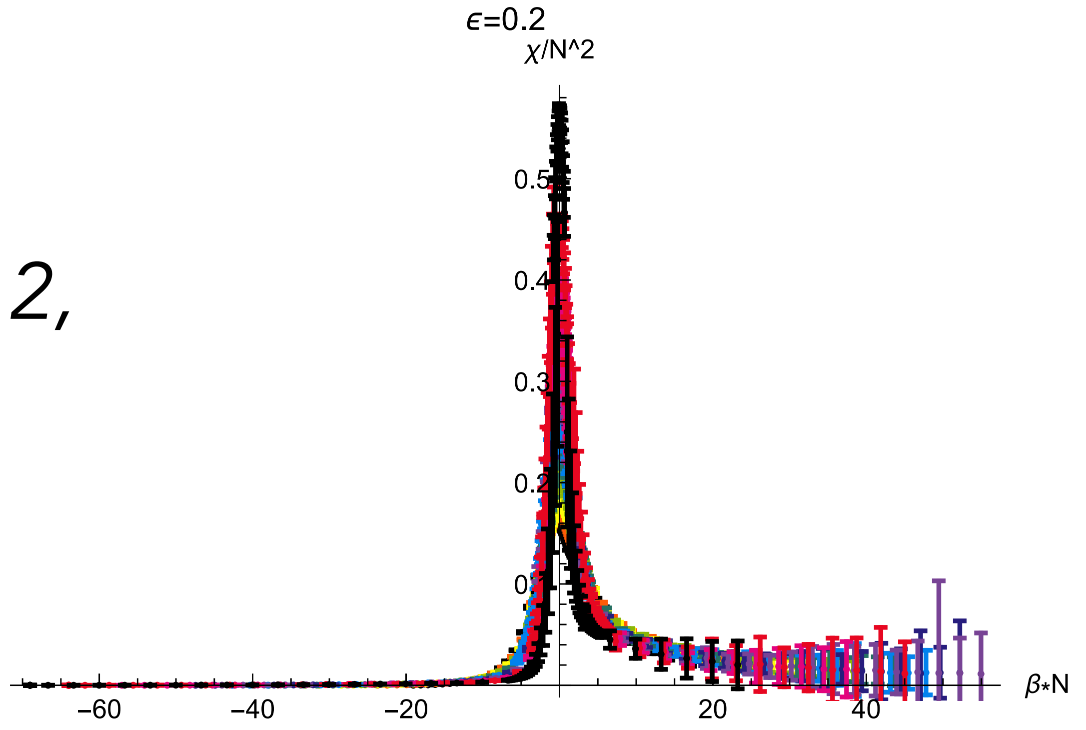


2d Orders

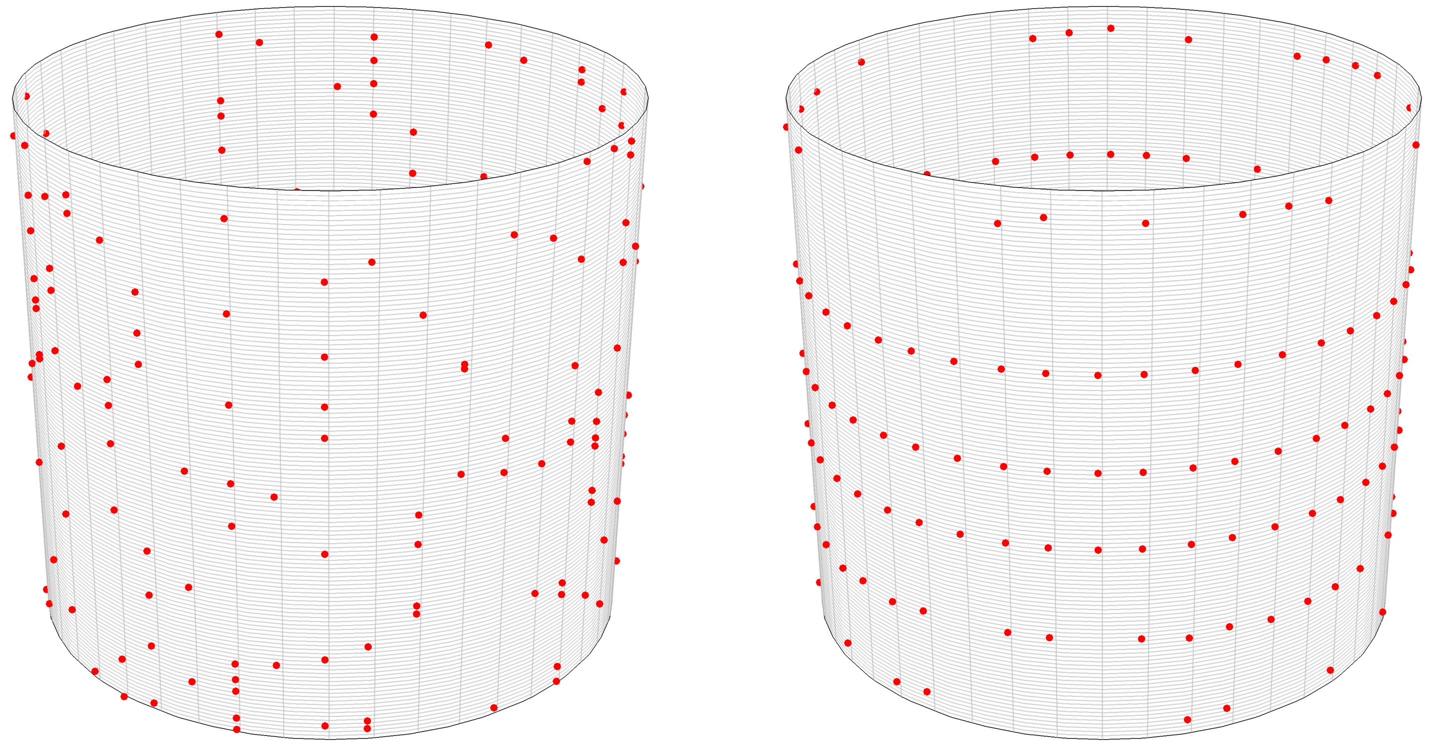
– *Surya 2012,*

$\beta < \beta_c$: Continuum phase

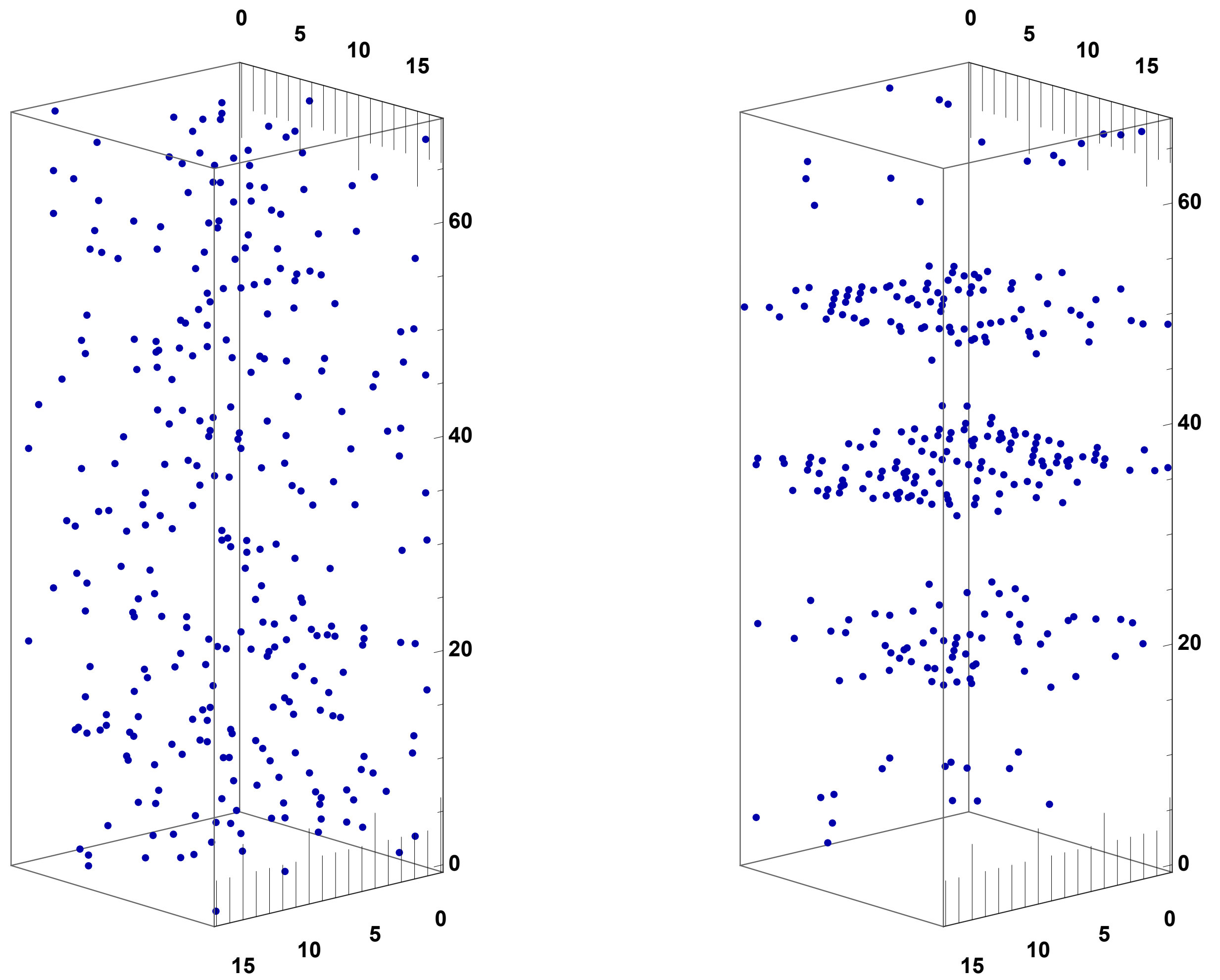
$\beta > \beta_c$: ``Connected/Quantum" Phase



– *Glaser, O'Connor and Surya, 2018*



$S^1 \times \mathbb{R}$



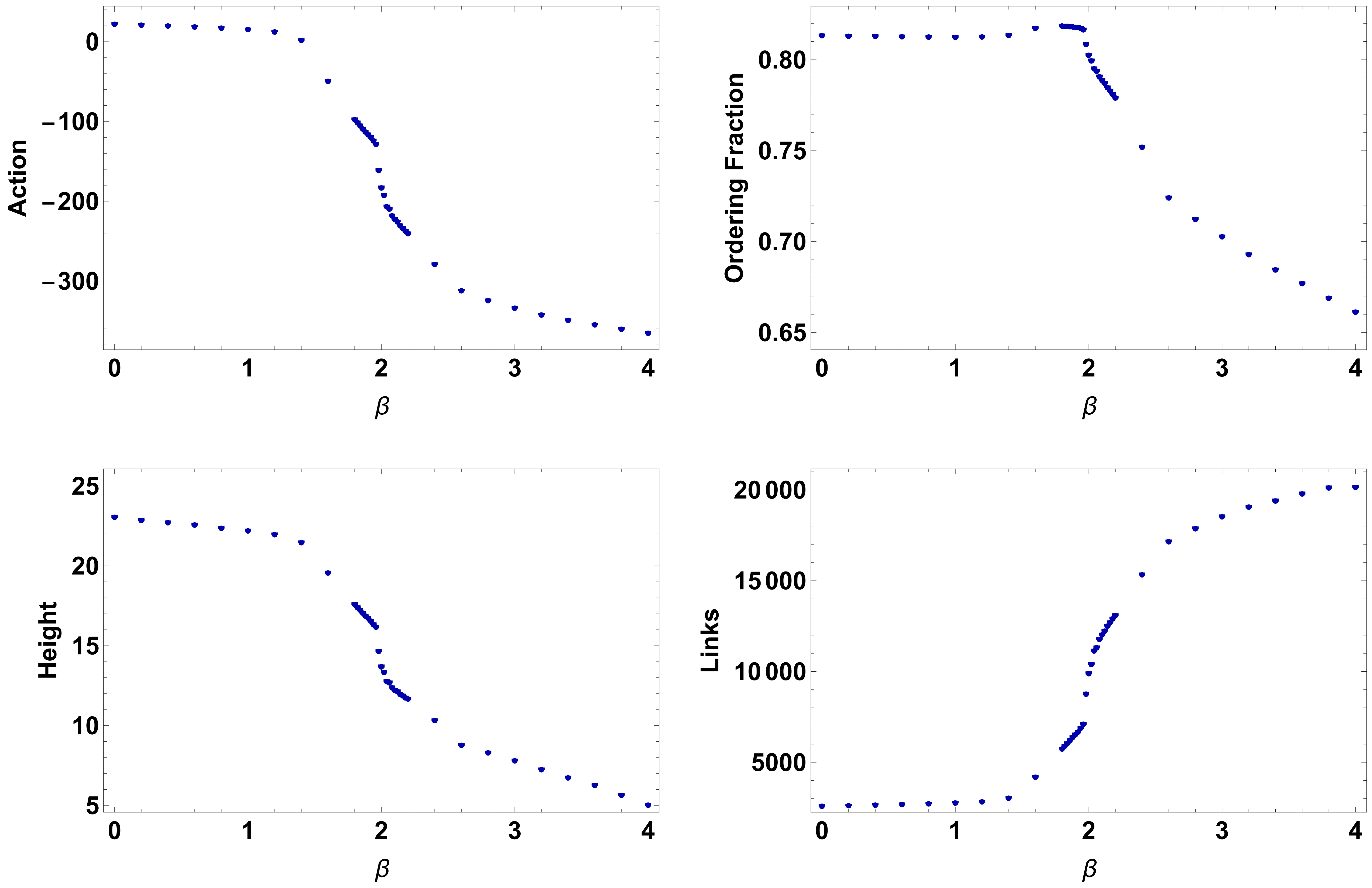
$T^2 \times \mathbb{R}$

– *Cunningham & Surya, 2020*

Track Observables $\mathcal{O}$ for continuum-non-continuum phases

- Myrheim-Myer dimension estimator
- Interval Abundance
- Action
- Height

Finite size scaling to know if $\langle \mathcal{O} \rangle \lim\limits_{n\to\infty} \mathbb{G}$



Computationally very expensive!!

- Cunningham & Surya, 2020

Is the Continuum Unique?

Calibrating the Continuum Approximation: new techniques

- $(M, g) \sim_\rho (M', g')$: what does this mean?
- Lorentzian spacetimes via GH convergence

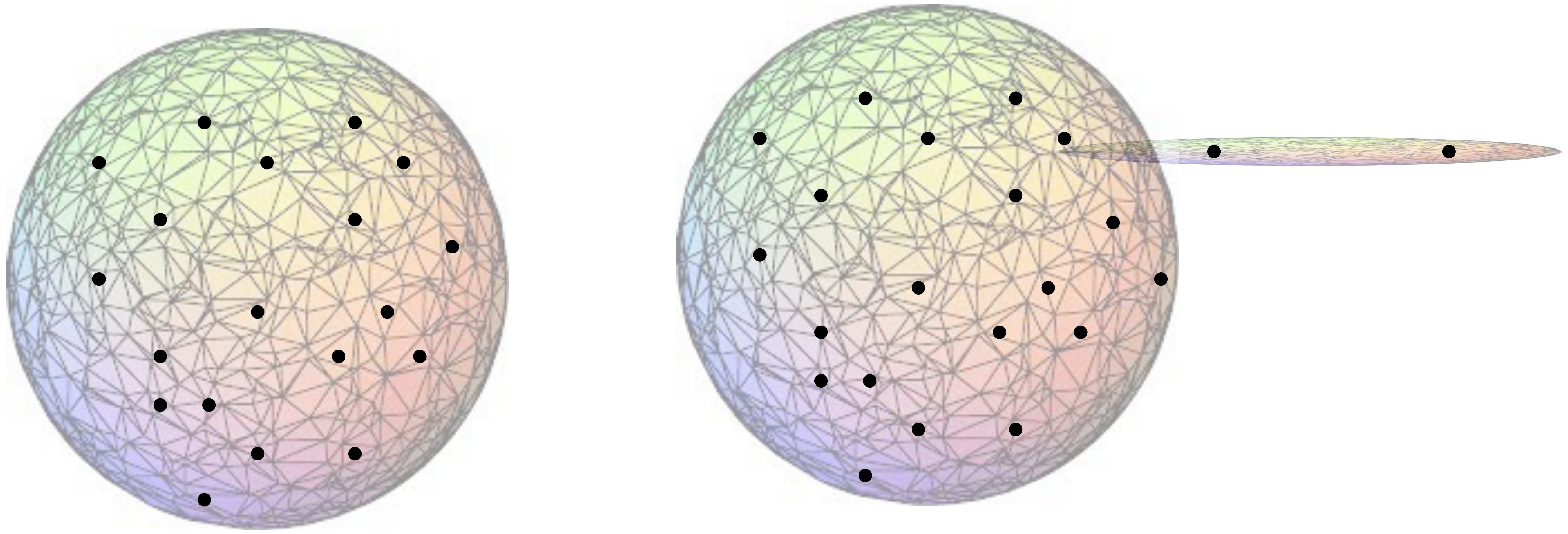
— Bombelli, 2000, Bombelli and Noldus, 2004
—Burtscher and Allen, 2021,
--Kunzinger and Steinbauer, 2021

- Can we define ρ-closeness?
- Convergence in $\rho \to \infty$ limit

-- Bombelli and Meyer, 1989
-- Minguzzi and Suhr , 2022
-- Muller, 2022

Uniform Random Sampling Method

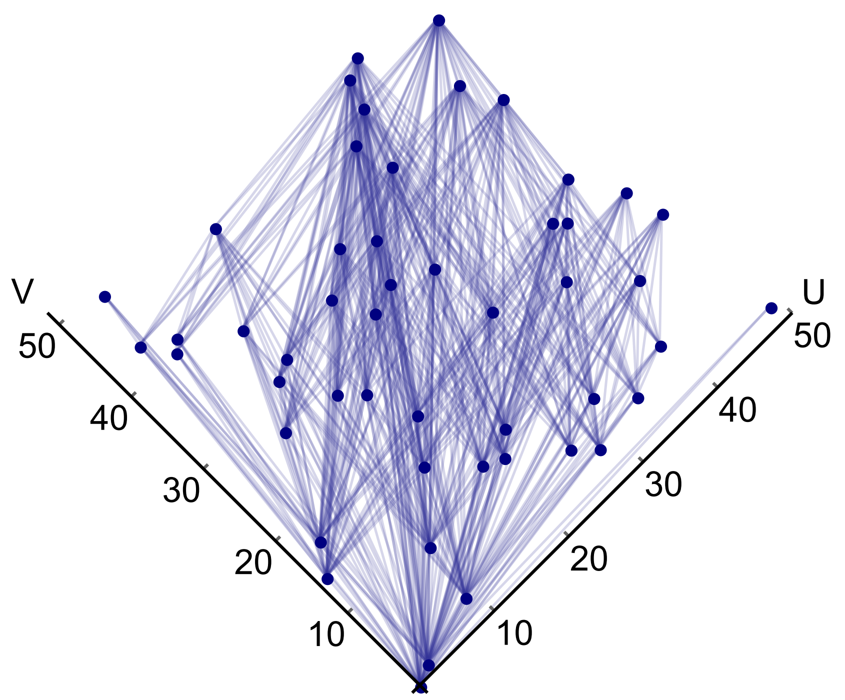
$$P_V(n) = \frac{(\rho V)^n}{n!} e^{-\rho V}, \quad \langle n \rangle = \rho V$$



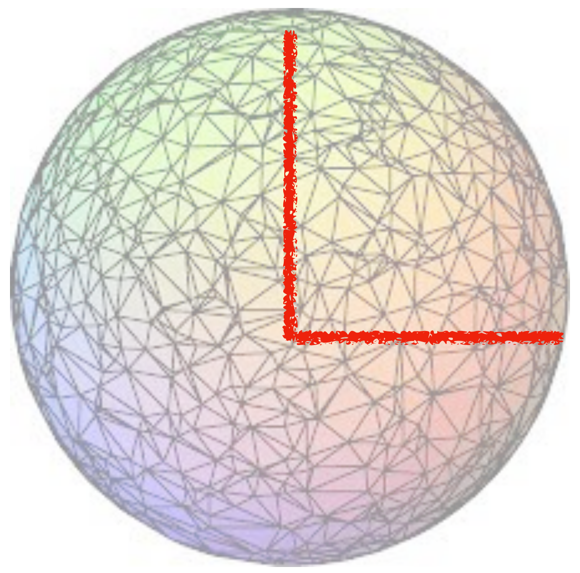
→ String is a set of measure zero $\Rightarrow d_{RS}(A, B) \approx 0$ upto scale ρ^{-1}

Lorentzian Uniform Random Sampling - Causal Sets

Bombelli, 2000



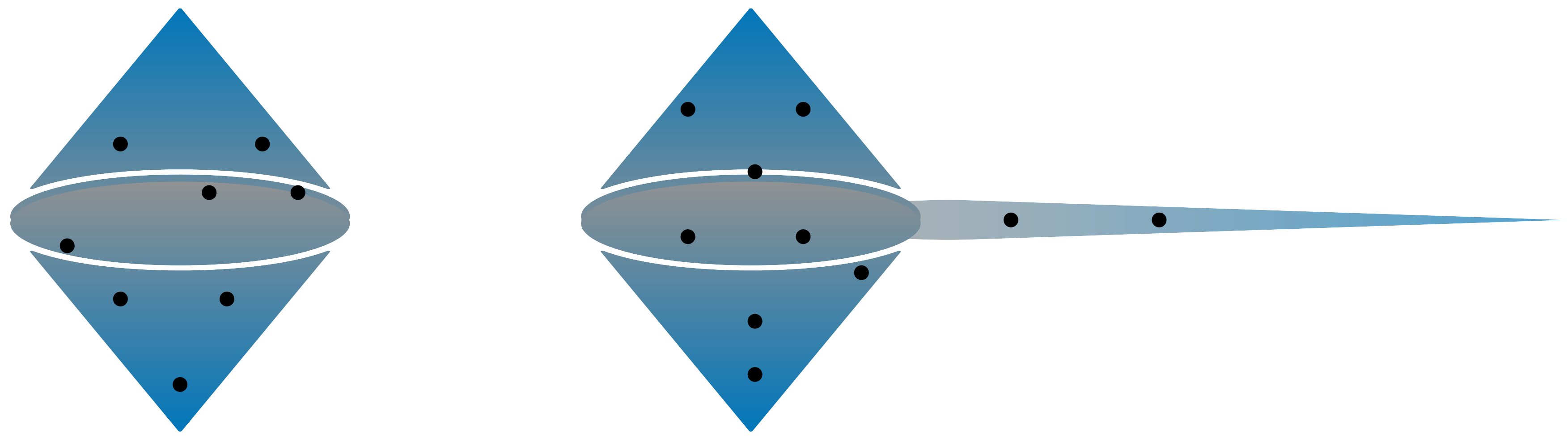
- Consider two spacetimes (M, g), (M', g') of volume $\sim V$
- Random sampling produces a causal set c by using causality relation $\prec$ in (M, g)
- Ω_n : ensemble of $n-$element causal sets
- $P_n(c\,|\,M)$, $c \in \Omega_n$ is a probability distribution.



$\mathbb{S}^{|\Omega_n|-1}$

- Since $\sum_{c\in\Omega_n} P_n(c\,|\,M) = 1$, $\sqrt{P_n(c\,|\,M)}$ form coordinates on positive part of the sphere in $\mathbb{R}^{|\Omega_n|}$
- $d_n(M, M') = \frac{2}{\pi}\cos^{-1}\left(\sum_{c\in\Omega_n}\sqrt{P_n(c\,|\,M)}\sqrt{P_n(c\,|\,M')}\right)$
- Closeness function but not a distance function

$d_n(M, M') \simeq 0,$ **upto** ρ^{-1}



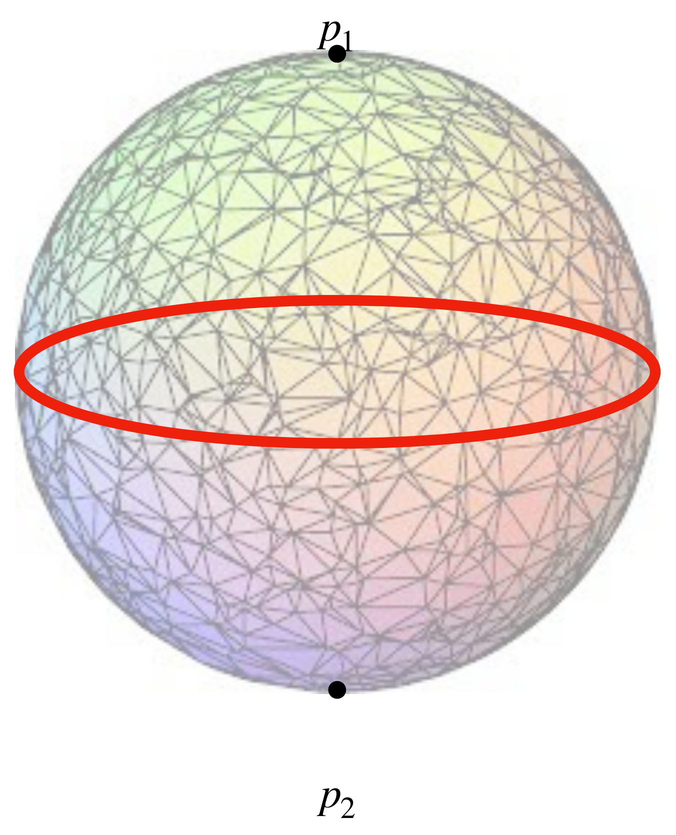
A B

Distance between Abstract Metric Spaces $(A, d_A), \quad (B, d_B)$

- Let (M, d) be a metric space such that $\Phi : A \hookrightarrow M, \quad \Psi : B \hookrightarrow M$ are two **isometric embeddings**
- Gromov-Hausdorff Distance: **shortest** Hausdorff distance over all possible isometric embeddings

$$d_{GH} \equiv \inf_{(M,d),\Phi,\Psi} d_H(\Phi(A), \Psi(B))$$

- Calculating this quantity explicitly is very hard!



$$d_{GH}(\mathbb{S}^m, \mathbb{S}^n) \leq \frac{\pi}{2}$$

$$d_{GH}(\mathbb{S}^0, \mathbb{S}^n) = \frac{\pi}{2}$$

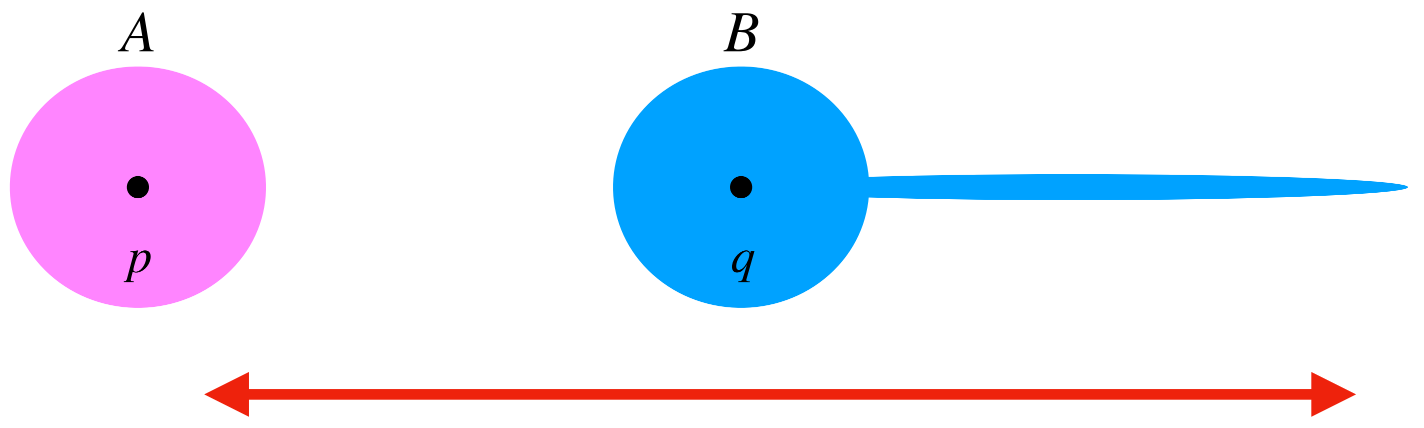
$$d_{GH}(\mathbb{S}^m, \mathbb{S}^\infty) = \frac{\pi}{2}$$

Lim, Memoli &Smith, 2022

Hausdorff distance

$$d(p, B) \equiv \inf_{q \in B} d(p, q)$$

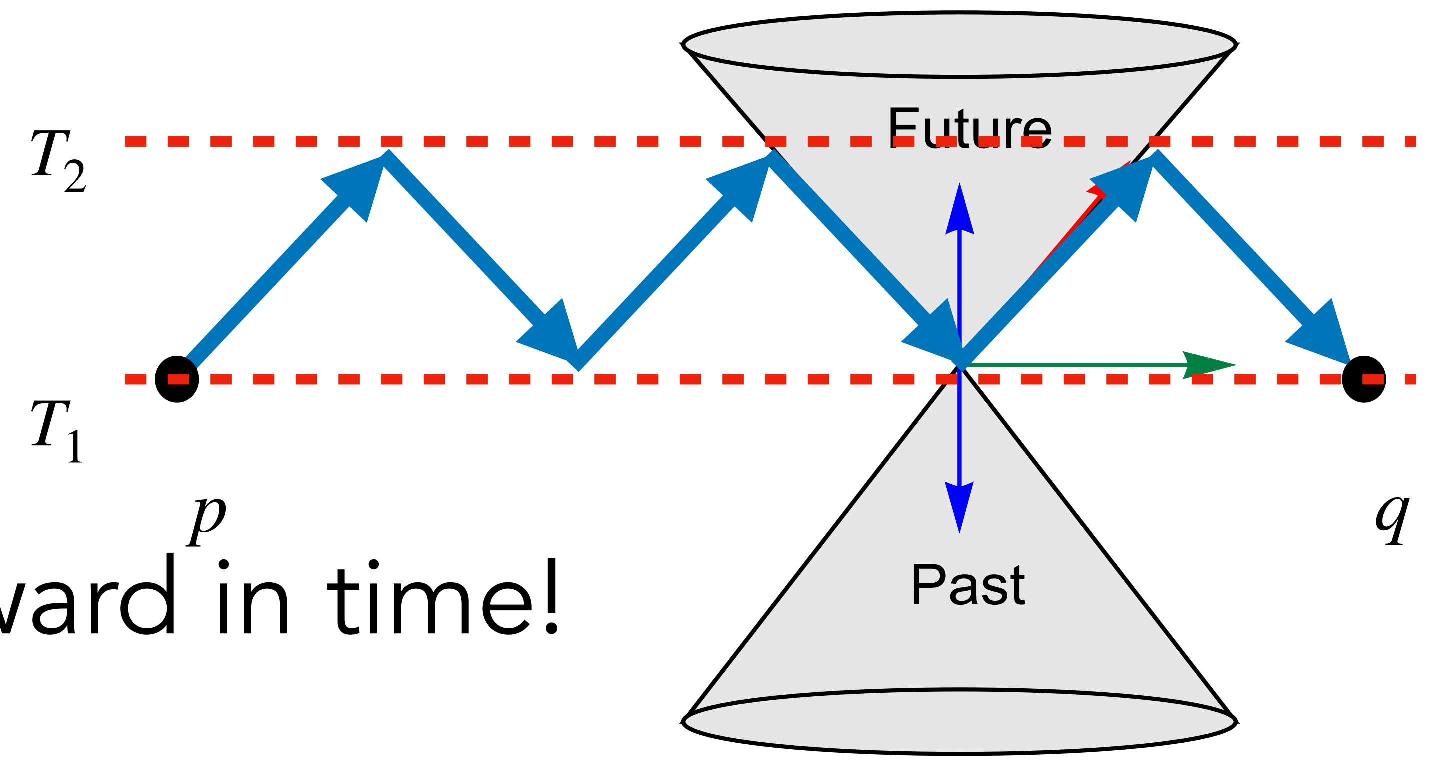
$$d_H(A, B) \equiv \max \left(\sup_{p \in A} d(p, B), \sup_{q \in B} d(A, q) \right)$$



Null Distance Function for GH Convergence

Sormani and Vega, 2016

Allen and Burtscher, 2020

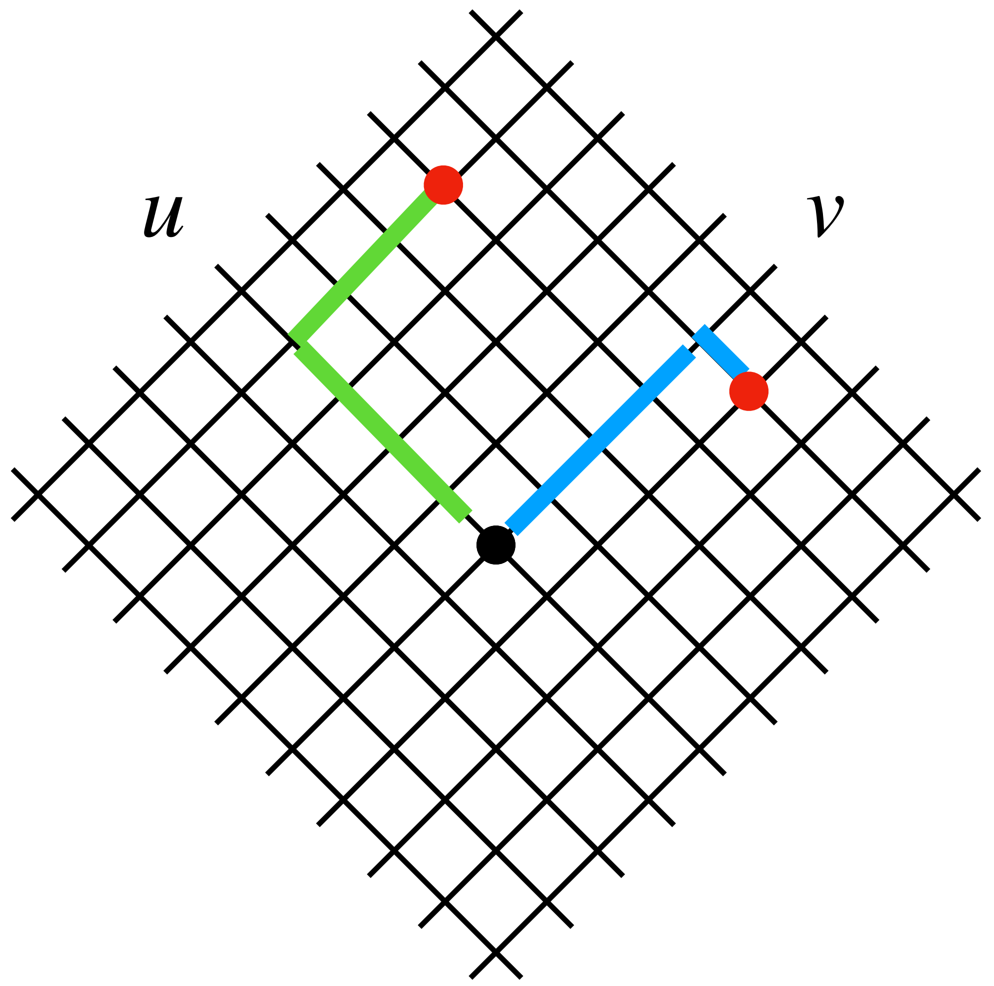


- Time function $T : (M, d) \to \mathbb{R}$ -- monotonic, etc.
- $\gamma(p, q)$: piece-wise causal curve: allow it to go backward and forward in time!
- The length along the curve is defined as $L_T(\gamma(p,q)) = \sum_i |T(s_i) - T(s_{i-1})|$
- Null Distance function: $d_T(p,q) \equiv \inf_{\gamma(p,q)} L_T(\gamma(p,q))$ IS a genuine distance function
- Can be used to study convergence of FRW-type spacetimes *Allen and Burtscher, 2020*

GH-like distance for 2d orders using a lattice embedding

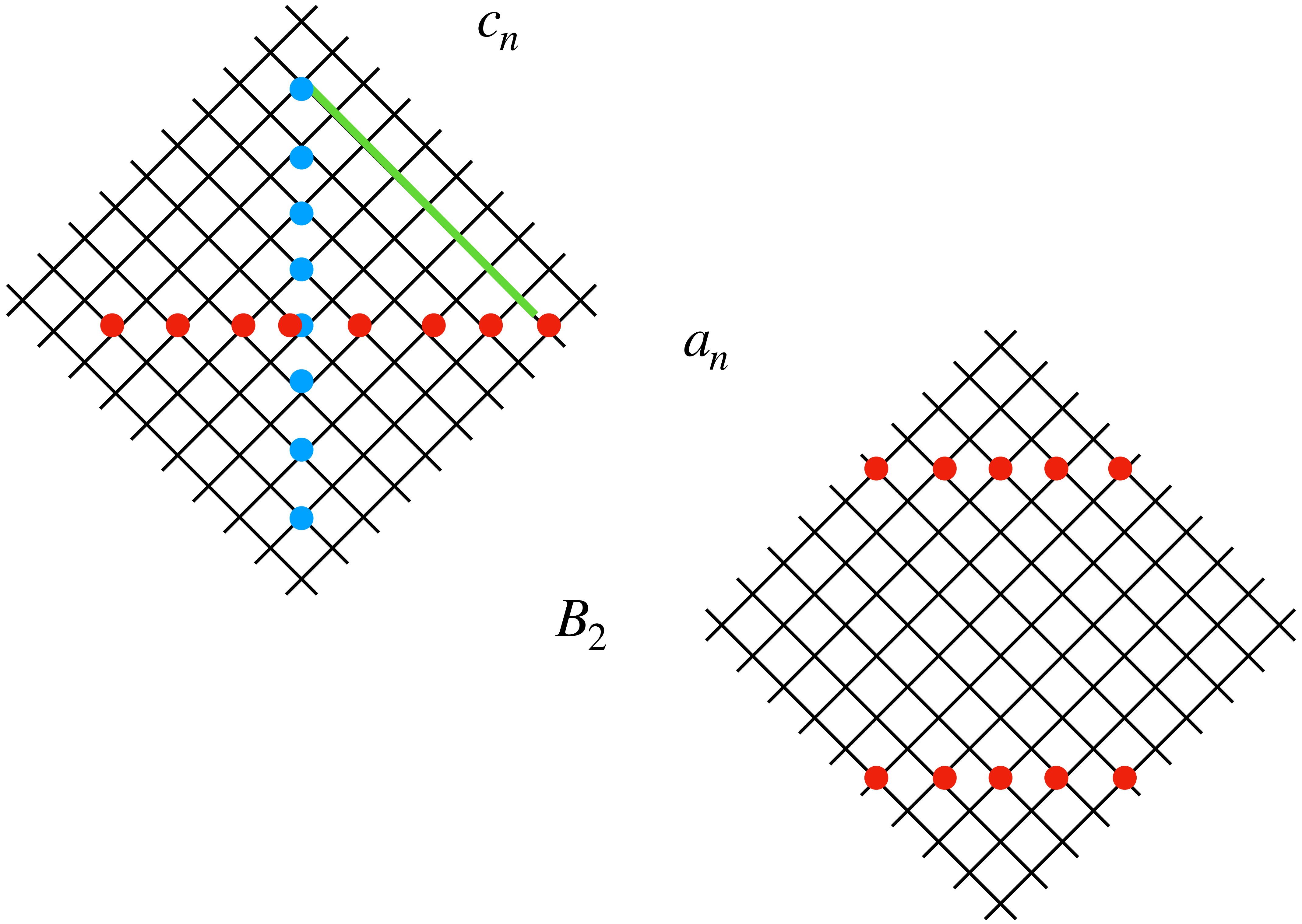
-Work in progress with Alan Daniel Santhosh

- $S = (1,2,\ldots n)$, $U = (u_1, \ldots u_n)$, $V = (v_1, \ldots v_n)$, $u_i \in S, v_i \in S$
- 2d order $C = U \cap V : e_i = (u_i, v_i) \prec e_j = (u_j, v_j) \Leftrightarrow u_i < u_j, v_i < v_j$
- Examples:
    - $u_1 < u_2 \ldots < u_n, v_1 < v_2 \ldots < v_n \Rightarrow C$ is a chain
    - $u_1 < u_2 \ldots < u_n, v_n < v_{n-1} \ldots < v_1 \Rightarrow C$ is an antichain
    - U, V randomly sampled : random 2d order $\sim (\mathbb{D}^2, \eta)$
- Every 2d order can be embedded as a 2d order into the light cone lattice $\mathscr{L}$
- The null distance function on $\mathscr{L}$: $d_t(a,b) = \frac{1}{2}(|u_b - u_a| + |v_b - v_a|)$
- $A, B \subseteq \mathscr{L}, \quad d_H(A,B) = \sup_{a \in A} \inf_{b \in B} d_t(a,b)$
- Let $c_1, c_2 \in \Omega_{2d}, \quad \mathscr{E}_i : c_i \hookrightarrow \mathscr{L}, \; d_{GH}(c_1,c_2) \equiv \inf_{\mathscr{E}_i} d_H^{\leftrightarrow}(\mathscr{E}_1(c_1), \mathscr{E}_2(c_2))$



$$t = \frac{1}{2}(v+u), \quad x = \frac{1}{2}(v-u)$$

Preliminary calculations...



- $d_{GH}(a_n, a_{n+1}) = 1, \quad d_{GH}(c_n, c_{n+1}) = 1$
- $d_{GH}(a_n, c_n) = m, \quad n = 2m$ or $n = 2m + 1$
- $d_{GH}(B_2, c_n) = \dfrac{n}{4}, \quad d_{GH}(B_2, a_n) = \dfrac{n}{4} + \dfrac{1}{2}$
- $d_{GH}(KR, c_n) \leq \dfrac{n}{2}\,, \ d_{GH}(KR, a_n) = \dfrac{n}{4}$
- $d_{GH}(L_4, c_n) \leq \dfrac{n}{8}, \quad d_{GH}(L_4, a_n) \leq \dfrac{3n}{8}$
- Distance between Antichain a_n and Chain c_n grows the fastest
- Distance between the K-layer poset -- does it get closer to c_n than a_n as K increases?
- Measuring distance between random orders : challenging and may need numerical work

In Conclusion...

- The Continuum Approximation could be relevant to many approaches to quantum gravity:
    - $\mathbb{QG} \sim_V (\mathbf{M}, \mathbf{g})$
    - Uniqueness: If $\mathbb{QG} \sim_V (\mathbf{M}, \mathbf{g})$ AND $\mathbb{QG} \sim_V (\mathbf{M}', \mathbf{g}')$ then $(\mathbf{M}, \mathbf{g}) \sim_V (\mathbf{M}', \mathbf{g}')$
- In CST, the continuum approximation is recognised using geometric order invariants $\mathcal{O}$
- New Lorentzian geometric tools to calibrate how close Lorentzian $\mathbb{QG}$ is to a smooth spacetime..
- Can they help us determine this up to a scale: $(\mathbf{M}, \mathbf{g}) \sim_V (\mathbf{M}', \mathbf{g}')$?

Thank you!