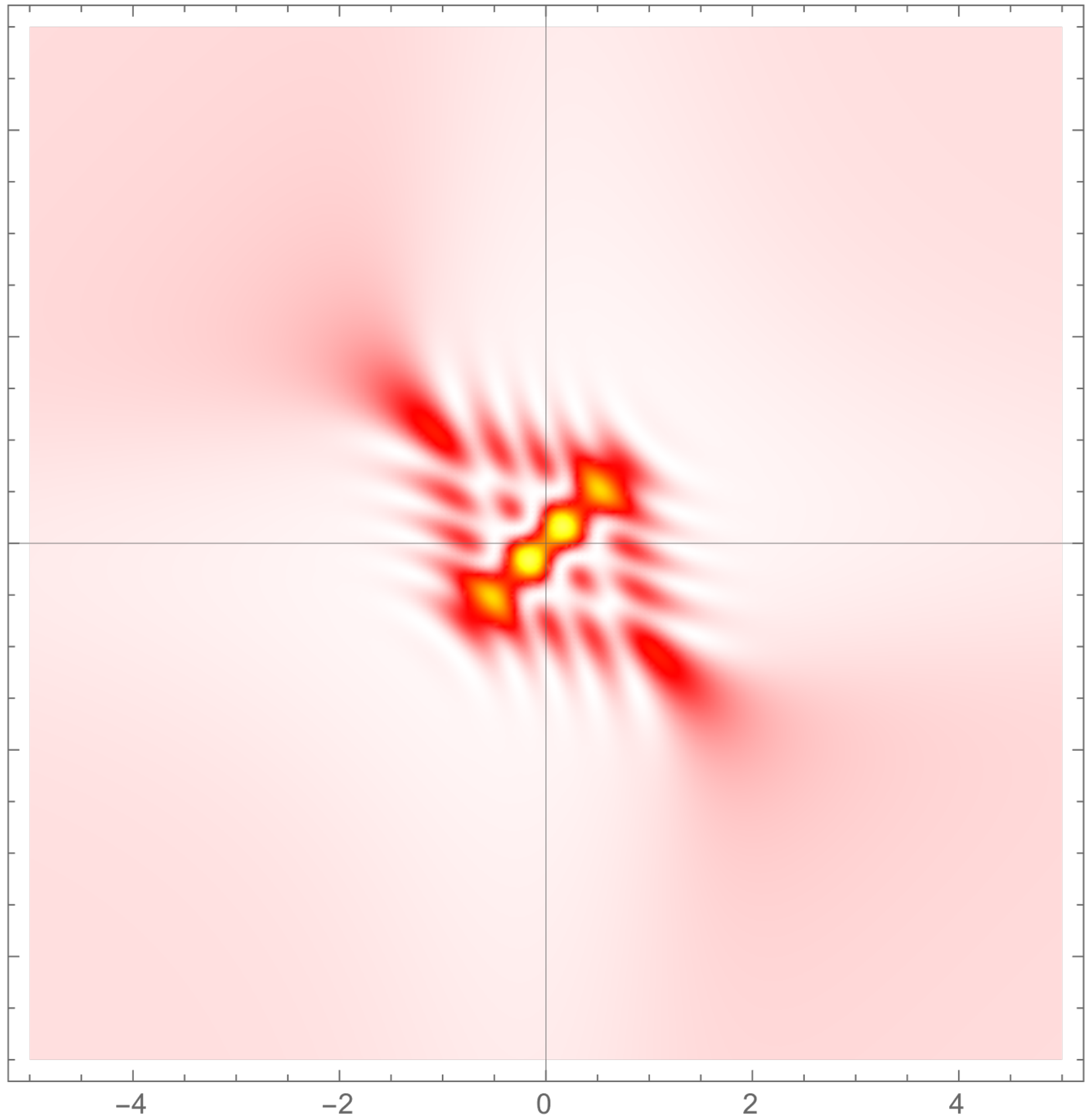
On the existence of real-time path integrals



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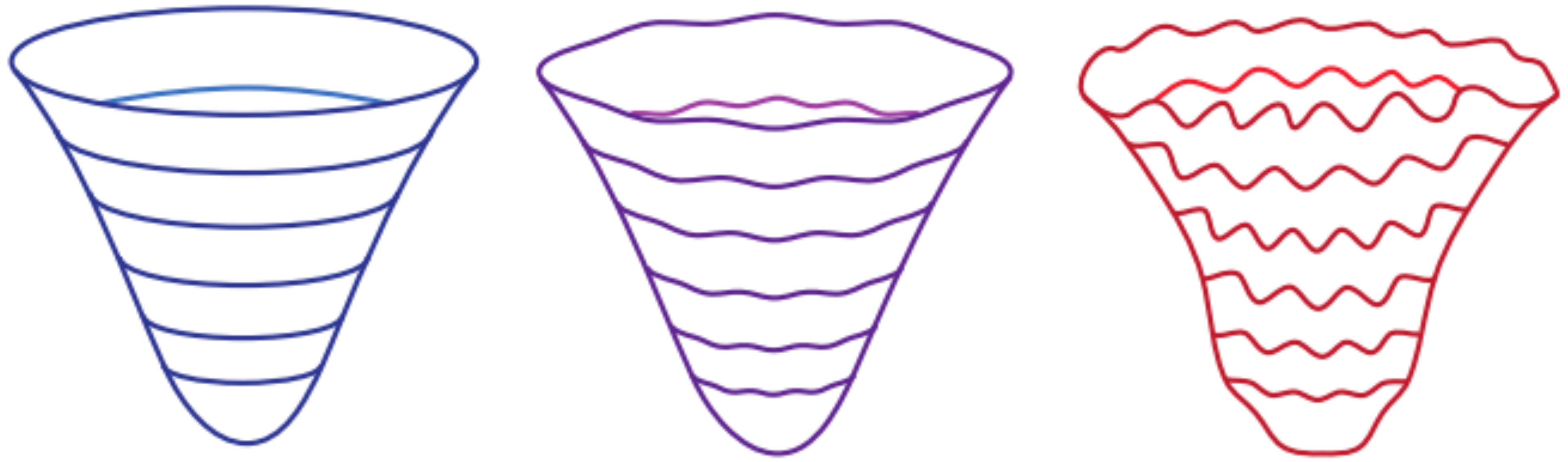
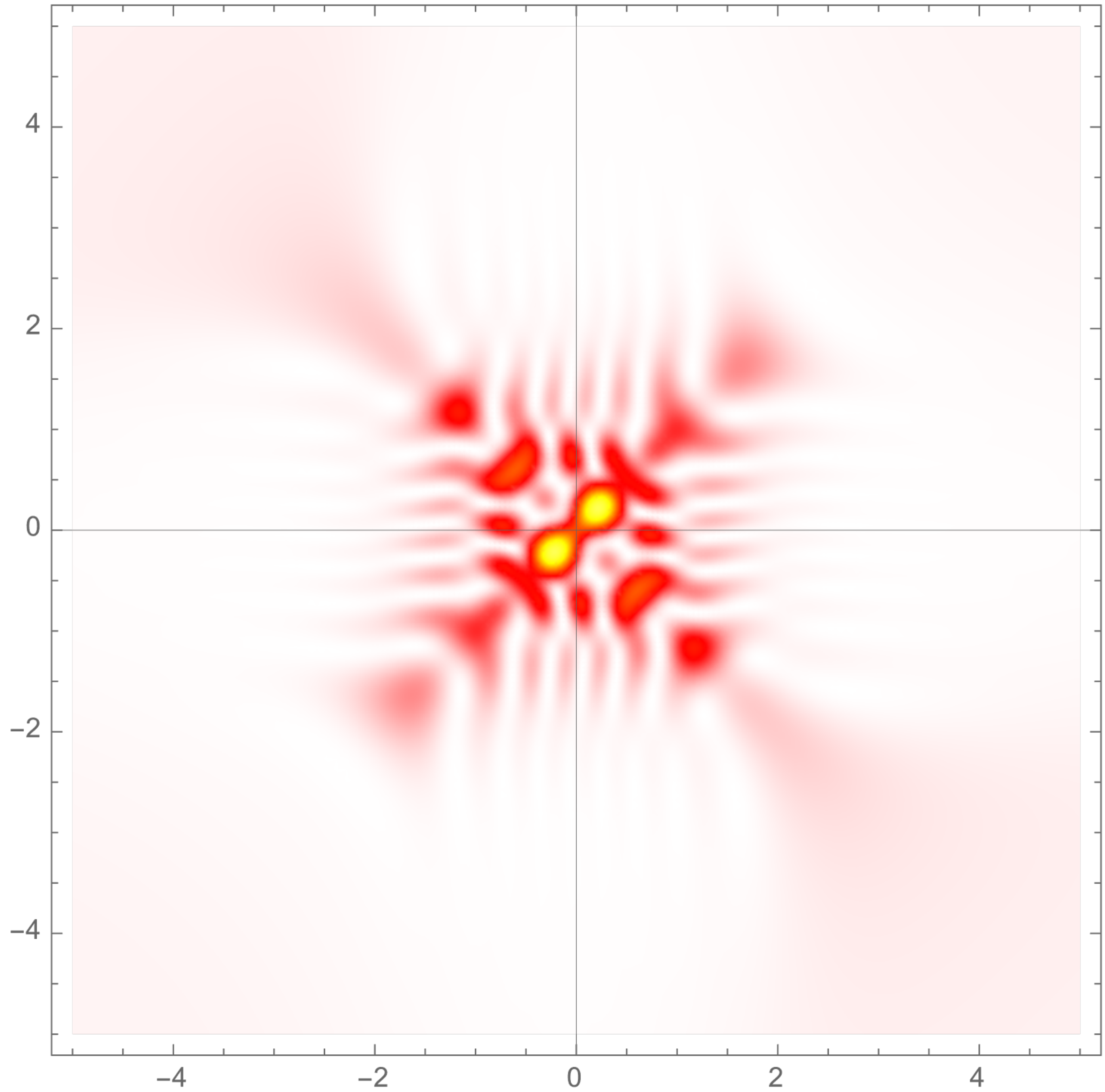


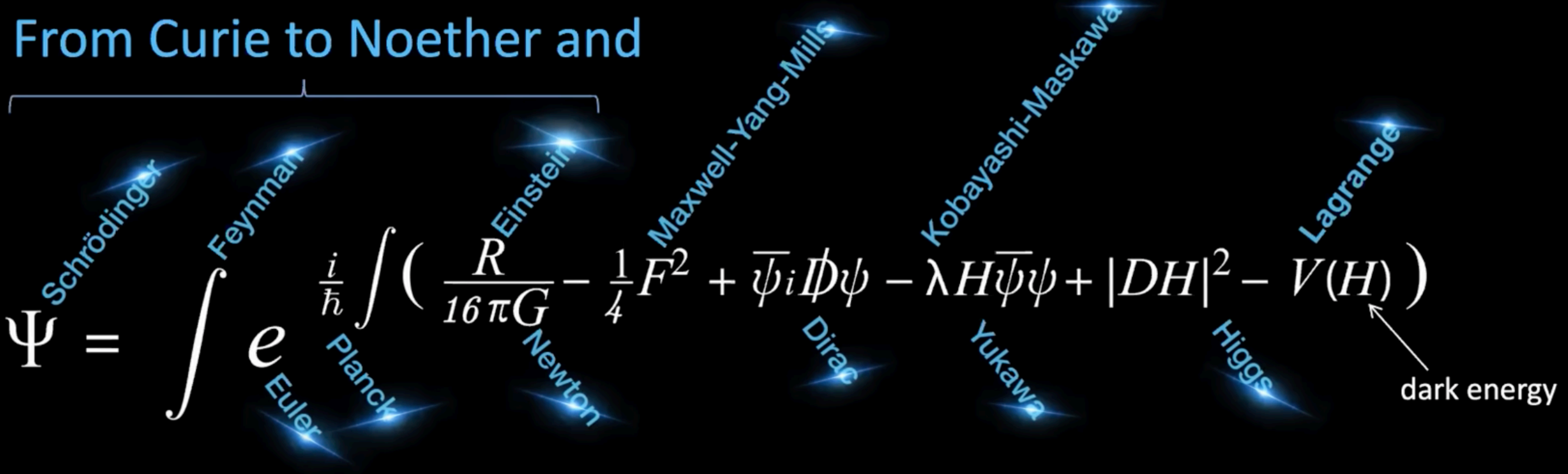
11th of July 2023

Quantum gravity 2023



Path integral

Interference gives our cleanest description of the Universe as formalized by the Feynman path integral. But how is this infinite-dimensional conditionally convergent oscillatory integral defined?



$$\Psi = \int e^{\frac{i}{\hbar}\int\left(\frac{R}{16\pi G} - \frac{1}{4}F^2 + \overline{\psi} i \not{D}\psi - \lambda H\overline{\psi}\psi + |DH|^2 - V(H)\right)}$$

$$\psi = (u_L, d_L, u_R, d_R, u_L, d_L, u_R, d_R, u_L, d_L, u_R, d_R, e_L, \nu_L, e_R, \nu_R)\times 3$$

Path integral

In **quantum gravity**, the **path integral for gravity** has influenced many explorations

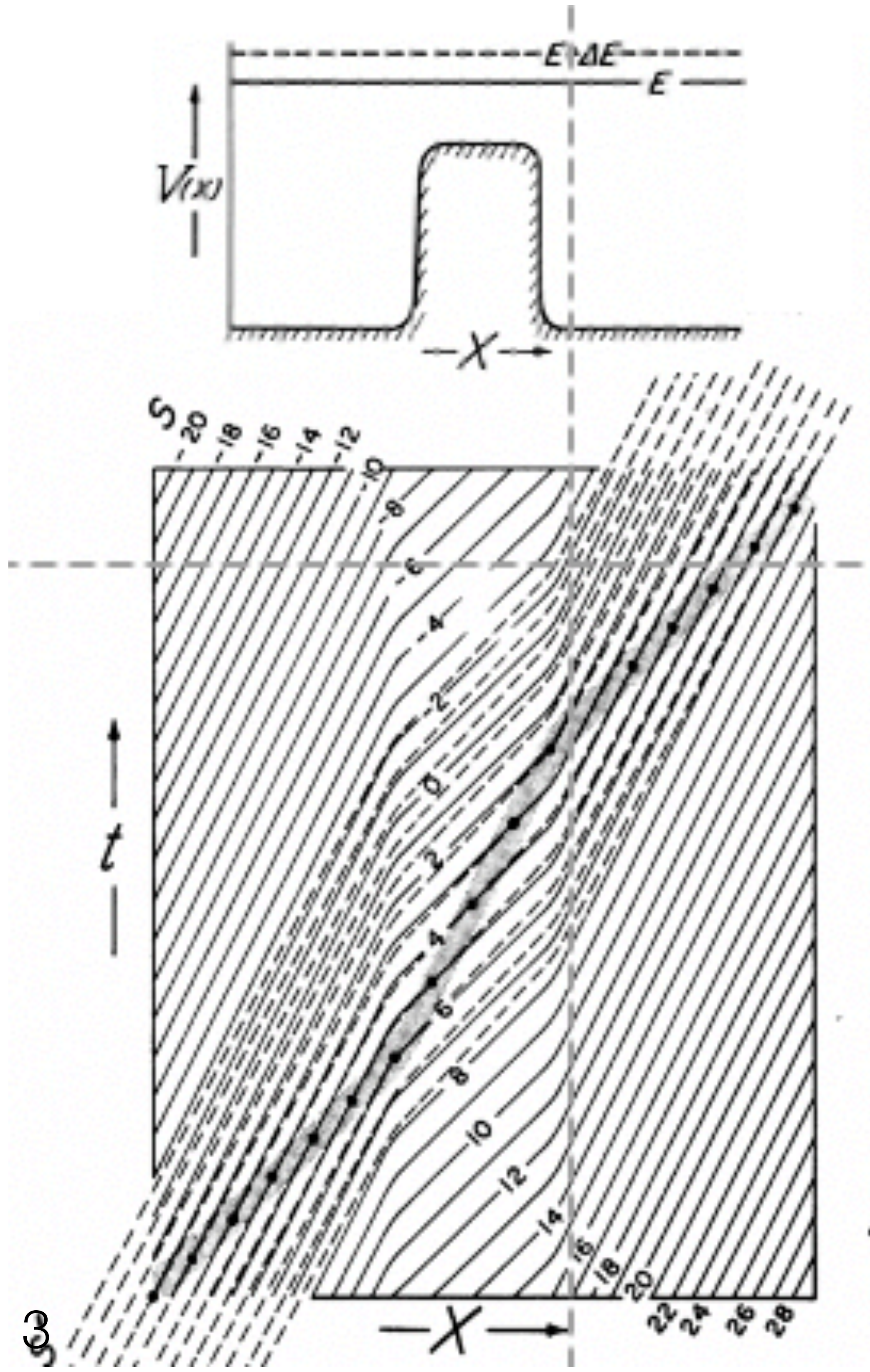
Wheeler: A classical trajectory emerges as an interference phenomena in quantum mechanics. Classical spacetime spacetime should emerge as an interference effect in superspace

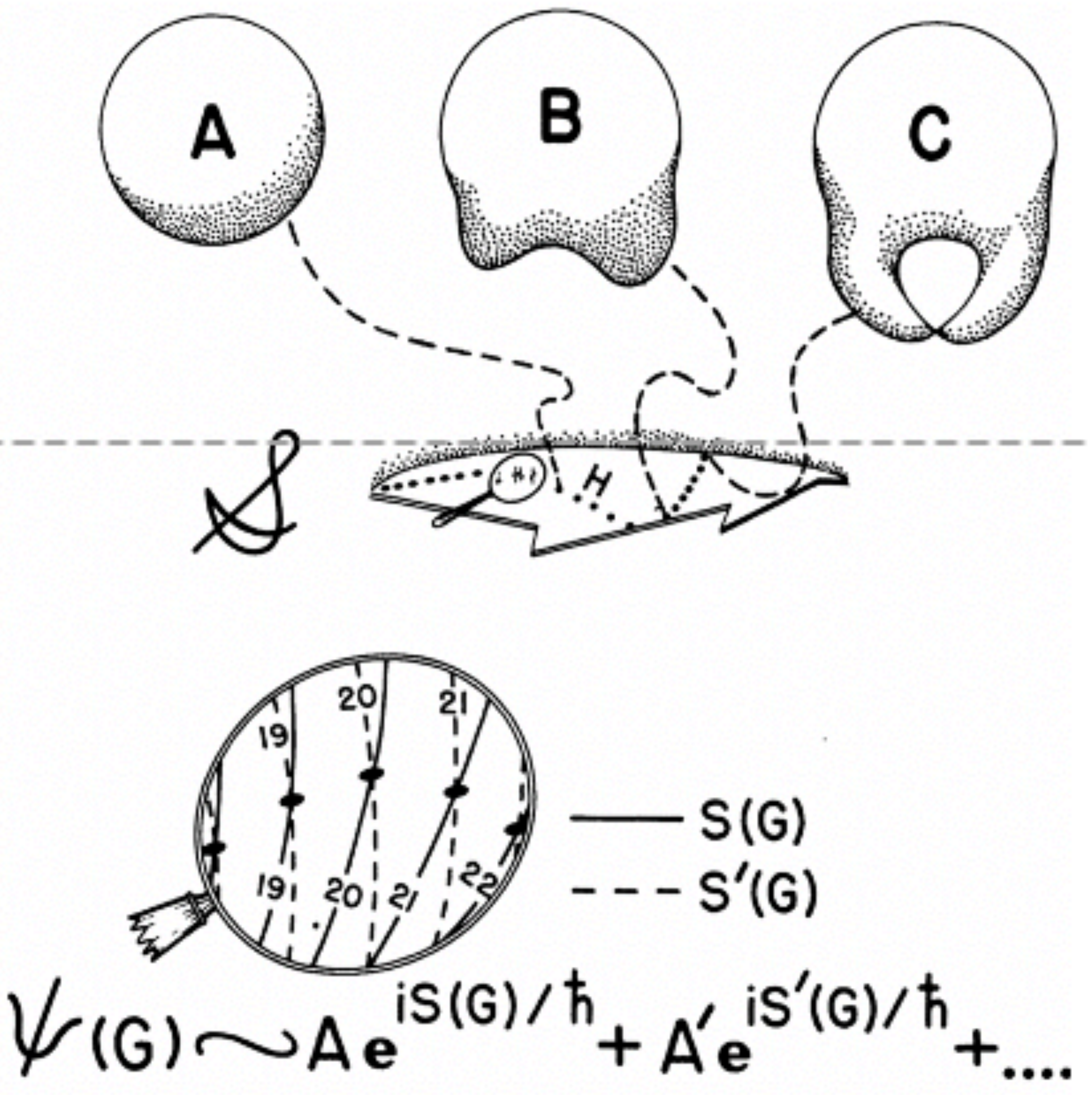
- The Wheeler-DeWitt equation

$$\hat{\mathscr{H}}_0 \Psi[\mathscr{G}^{(3)}] = 0 \qquad \hat{\mathscr{H}}_i \Psi[\mathscr{G}^{(3)}] = 0$$

- The path integral over spacetimes

$$K[\mathscr{G}_1^{(3)}, \mathscr{G}_0^{(3)}] = \int_0^\infty \int_{\mathscr{G}_0^{(3)}}^{\mathscr{G}_1^{(3)}} e^{iS_{EH}[\mathscr{G};N]/\hbar} \, \mathscr{D}\mathscr{G} \mathrm{d}N$$





Problems with the Feynman Path integral

Path integral

What is the problem? And why should we care?

- **Feynman and Hibbs**: *"...we feel that the possible awkwardness of the special definition of the sum over all paths may **eventually require new definitions to be formulated**. Nevertheless, **the concept of the sum over paths, like the concept of an ordinary integral, is independent of a special definition** and **valid in spite of the failure of such definitions**"*
- **Terence Tau**: *"**The point of rigour is not to destroy all intuition;** instead, it should be used to **destroy bad intuition while clarifying and evaluating good intuition**. It is **only with a combination of both rigorous formalism and good intuition** that one can **tackle complex mathematical problems.**"*

Conditional convergence

Alternating sums occur in many places, ranging from classical systems, and wave optics, to quantum physics

- Absolutely v.s. conditionally convergent sums

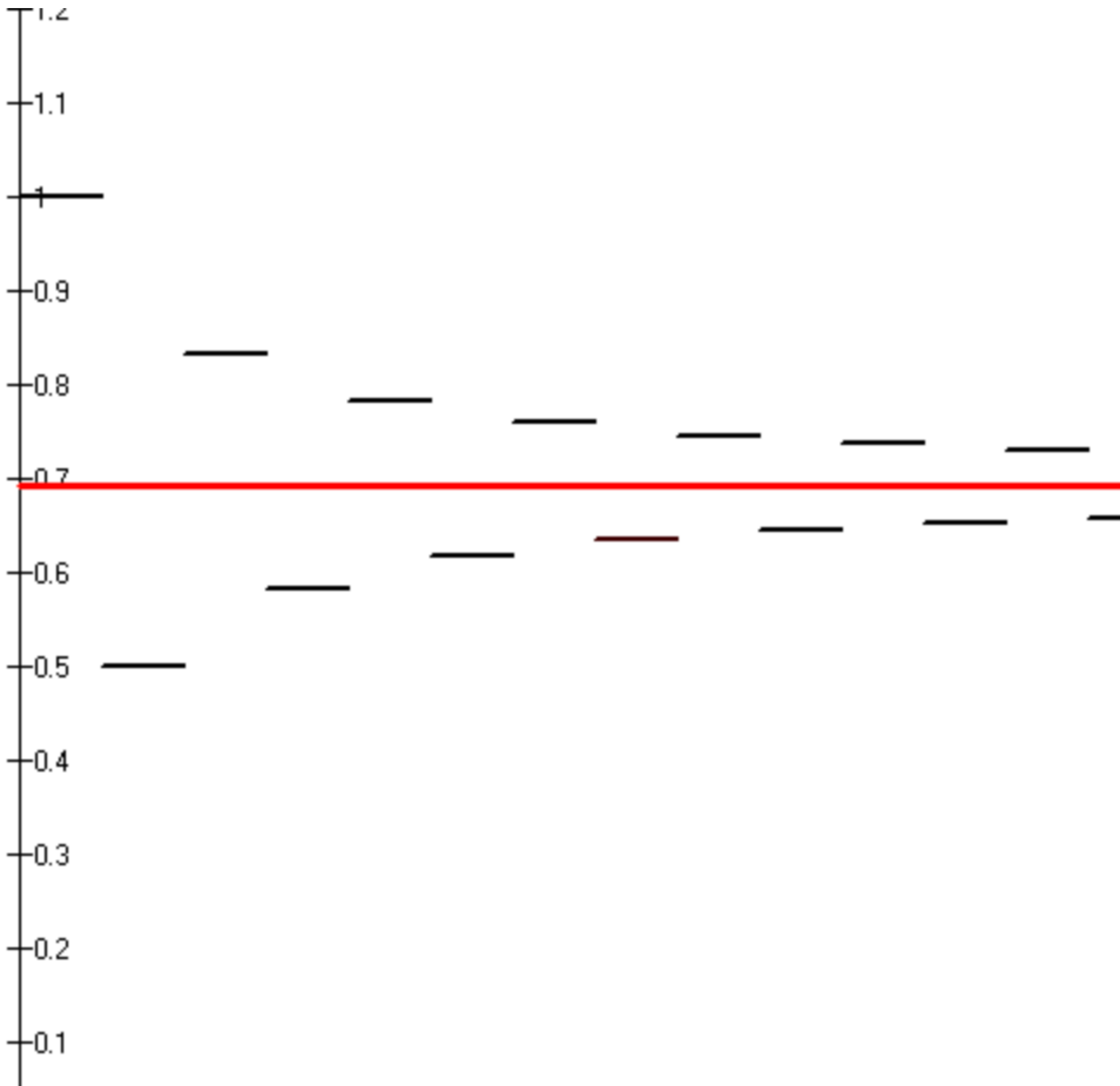
$$S = \sum_{i=1}^{\infty} a_i \qquad \sum_{i=1}^{\infty} |a_i| < \infty$$

- Conditional series depend on the ordering

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln 2$$

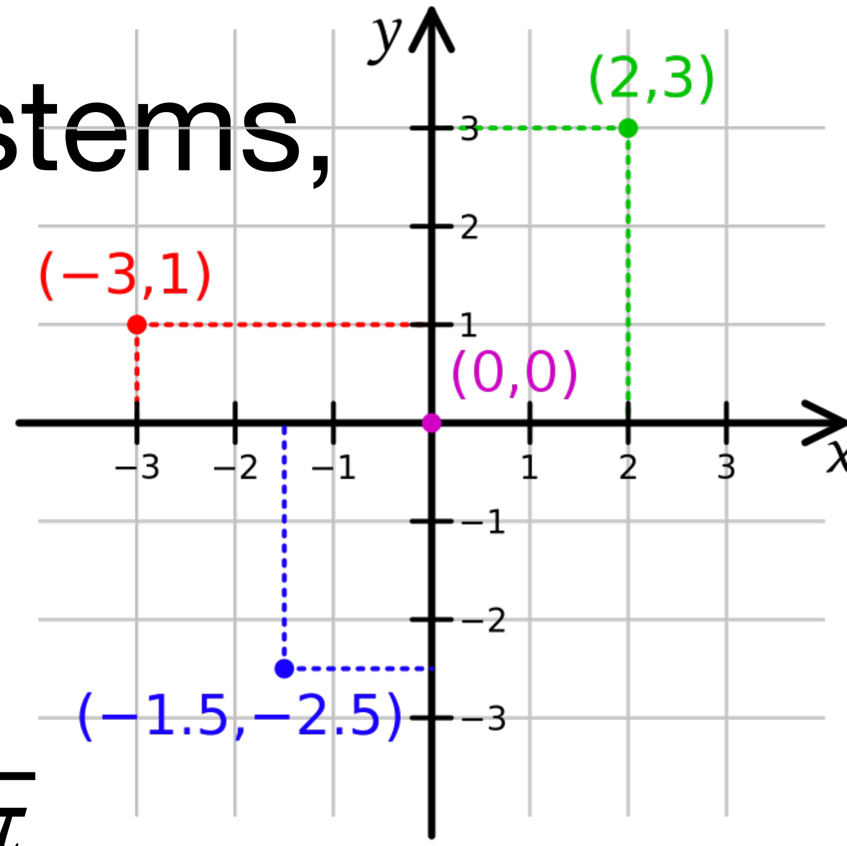
$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \ldots\right) = \frac{1}{2}\ln 2$$



Oscillatory integrals

Oscillatory integrals occur in many places, ranging from classical systems, and wave optics, to quantum physics

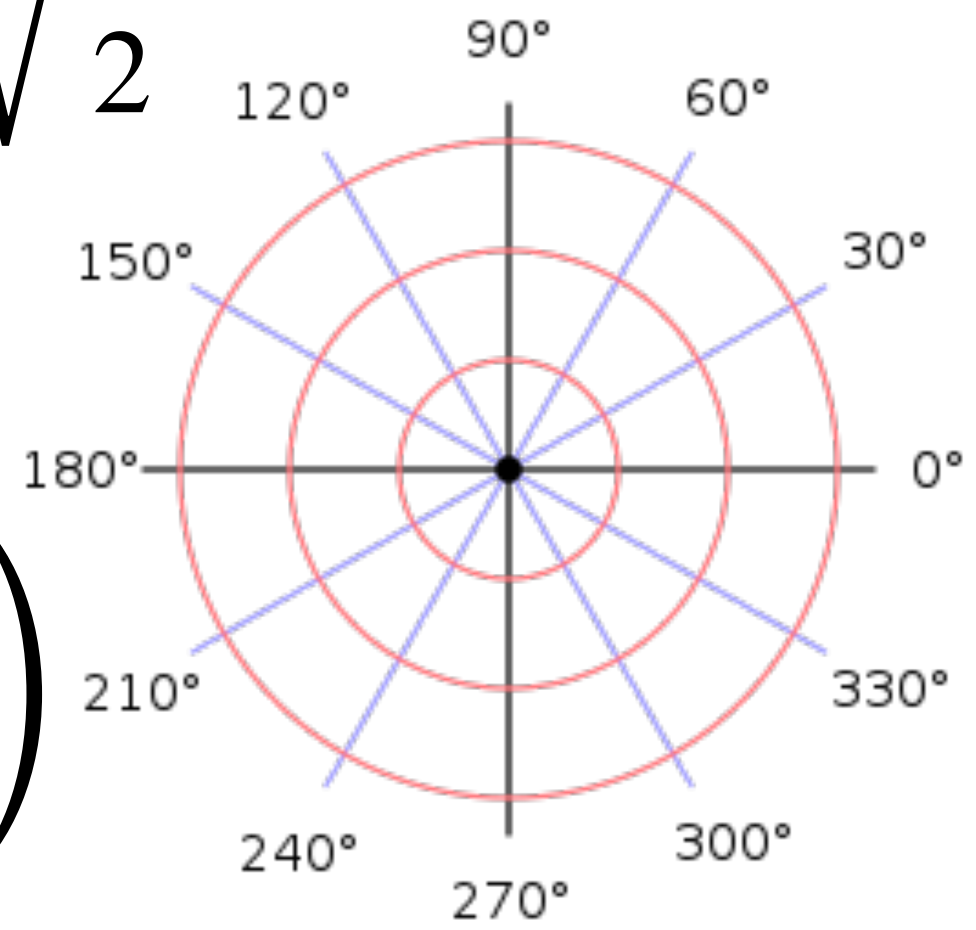


- Fresnel integral can be defined with the typical regularization

$$\int_{-\infty}^{\infty} e^{ix^2}\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} e^{ix^2}\mathrm{d}x = -(1+i)\sqrt{\frac{\pi}{2}}\lim_{R\to\infty}\mathbf{erf}\left(\frac{i-1}{\sqrt{2}}R\right) = (1+i)\sqrt{\frac{\pi}{2}}$$

- Higher dimensional generalizations run into problems

$$\int\prod_{l=1}^{N} e^{iy_l^2}\mathrm{d}y_l = \lim_{R\to\infty}\frac{2\pi^{N/2}}{\Gamma(N/2)}\int_0^R e^{ir^2}r^{N-1}\mathrm{d}r = \lim_{R\to\infty}(i\pi)^{N/2}\left(1-\frac{\Gamma(N/2,-iR^2)}{\Gamma(N/2)}\right)$$



oscillates around the box cutoff regulator for N = 2 and diverges for N > 2.

Infinite dimensional integrals

Integration theory is an application of measure theory

sigma-algebra $\mathscr{A}$ on the space Ω

1. $\Omega \in \mathscr{A}$
2. $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$
3. $A_n \in \mathscr{A}, n \in \mathbb{N} \Rightarrow \cup_{n=1}^{\infty} A_n \in \mathscr{A}$

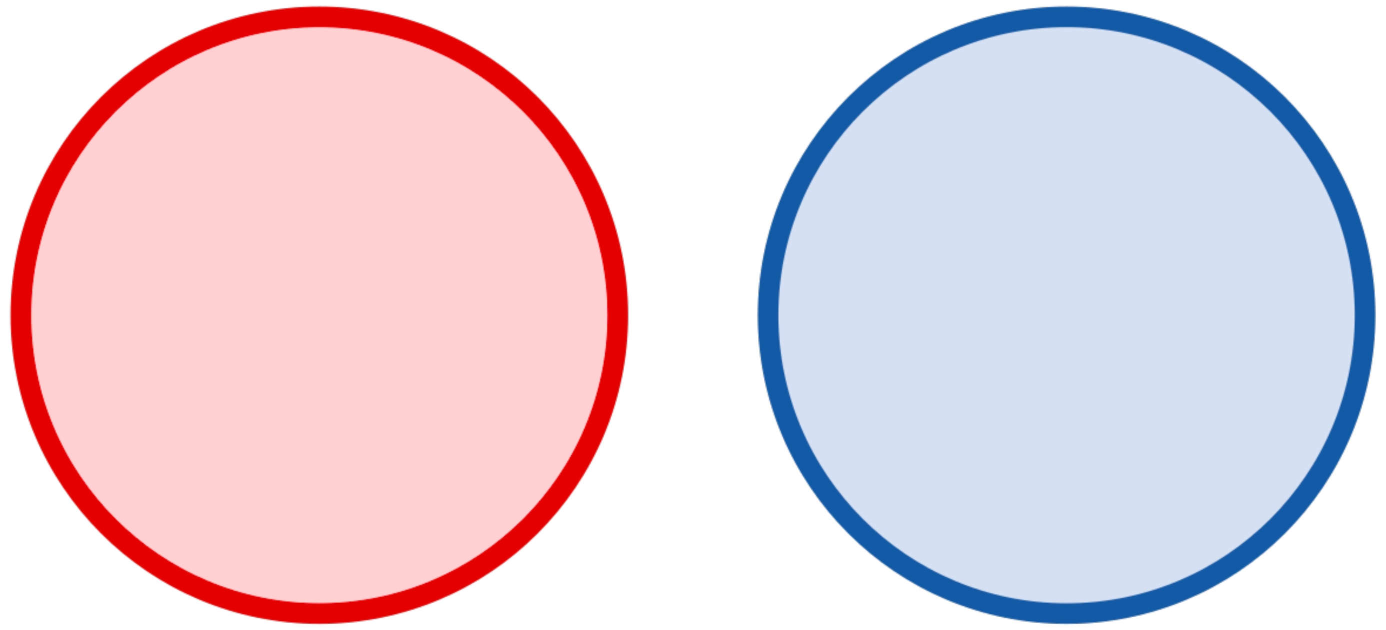
sigma-measure $\mu : \mathscr{A} \to [0,\infty]$

1. $\mu(\emptyset) = 0$
2. $A_n \in \mathscr{A}, n \in \mathbb{N}$, **pairwise disjoint**

$$\Rightarrow \mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

We define integrals of **positive simple functions** as a finite sum

$f = \sum_{i=1}^{r} \alpha_i 1_{A_i}$ **integrates to** $\int_\Omega f \,\mathrm{d}\mu = \sum_{i=1}^{r} \alpha_i \, \mu(A_i)$



leading to the general integral for **positive functions**

$\int_\Omega f \,\mathrm{d}\mu = \sup \left\{ \int_\Omega g \,\mathrm{d}\mu \,\middle|\, \right.$ **where** g **is simple and** $\left. 0 \le g \le f \right\}$

Infinite dimensional integrals

The infinite product of Lebesgue measures is not a sigma measure.

- Lebesgue formalized the standard measure on geometric spaces
  $\mu([a,b]) = b - a$
- Unfortunately, the infinite product is **not a measure** due to **translation invariance** $\mathscr{D}x \overset{!}{=} \prod_{i=1}^{\infty} \mathrm{d}x_i$

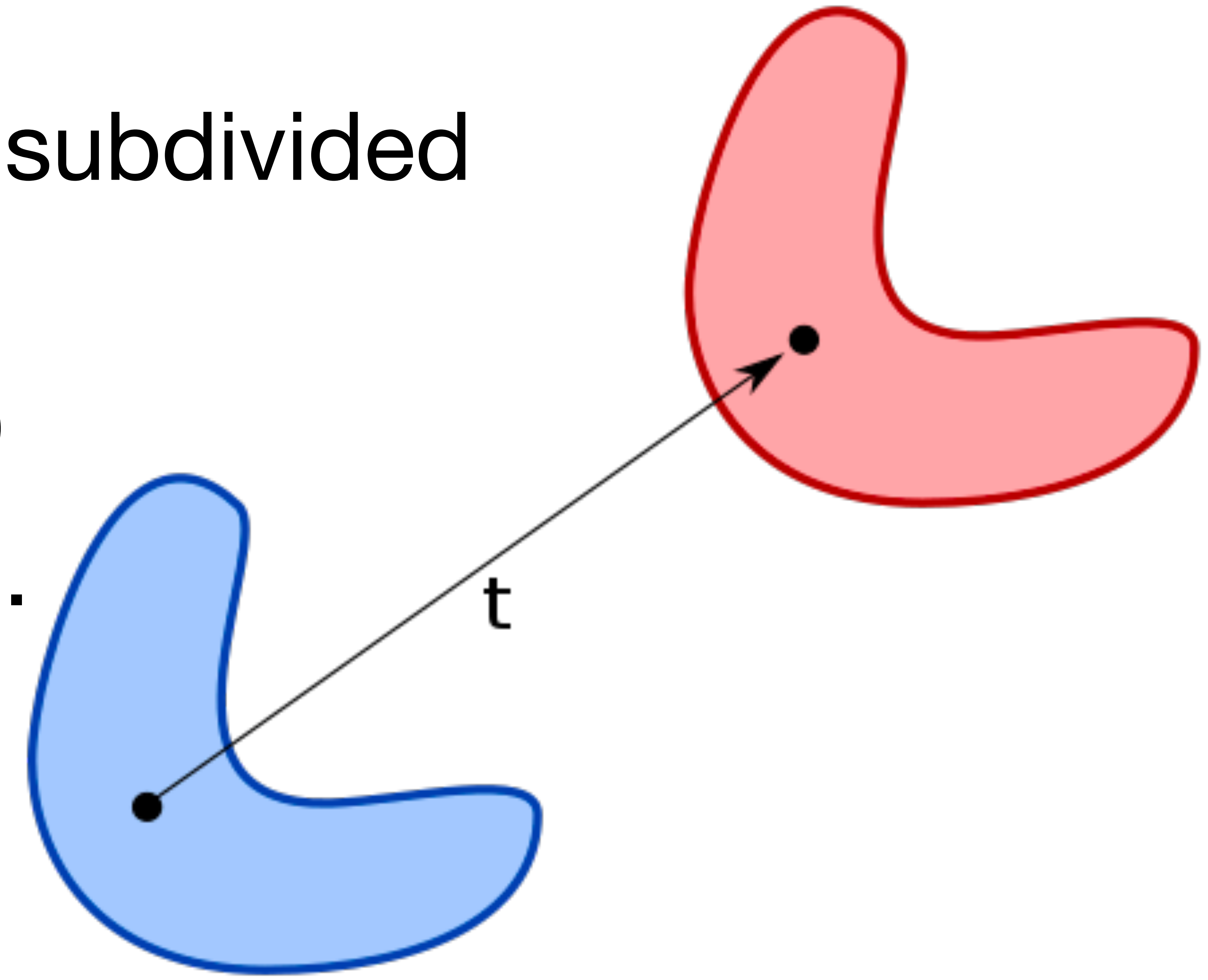
The measure of the n-dimensional hypercube can be subdivided

$1 = \mu([0,1]^n) = 2^n \mu([0,1/2]^n)$ In the limit $n \to \infty$

the subcube has a vanishing measure $\mu([0,1/2]^\infty) = 0$

and so does any subset that we construct from them.

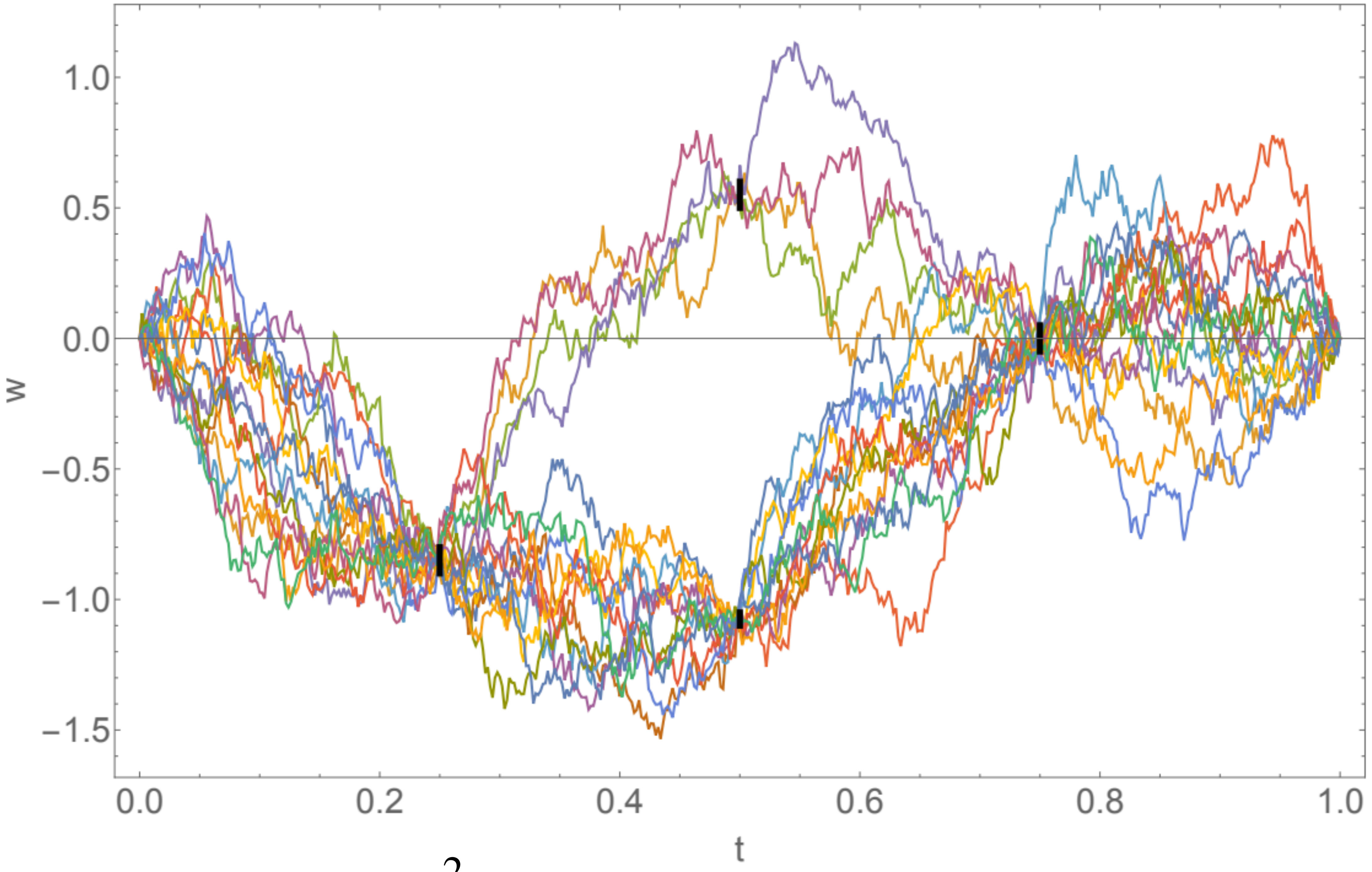
Such measures are useless in physics!



Infinite dimensional integrals

There exist infinite-dimensional measures that that are not translation invariant

Restricted Brownian motion moving between two points, leads to the Brownian bridge measure. When applied to a space of N slits, the measure forms an N-dimensional integral



$$Q = \{w \in \Omega \,|\, a_i < w(t_i) < b_i,\ 0 < t_1 < \ldots < t_N < 1\}$$

$$\mu_B(Q) = \left(\frac{\sqrt{2\pi} W}{\prod_{i=1}^{N+1} (W\sqrt{2\pi}(t_i - t_{i-1}))} \right) \int_{a_1}^{b_1} \ldots \int_{a_N}^{b_N} e^{-\sum_{i=1}^{N+1} \frac{(w_i - w_{i-1})^2}{2W^2(t_i - t_{i-1}}} \mathrm{d}w_1 \ldots \mathrm{d}w_N$$

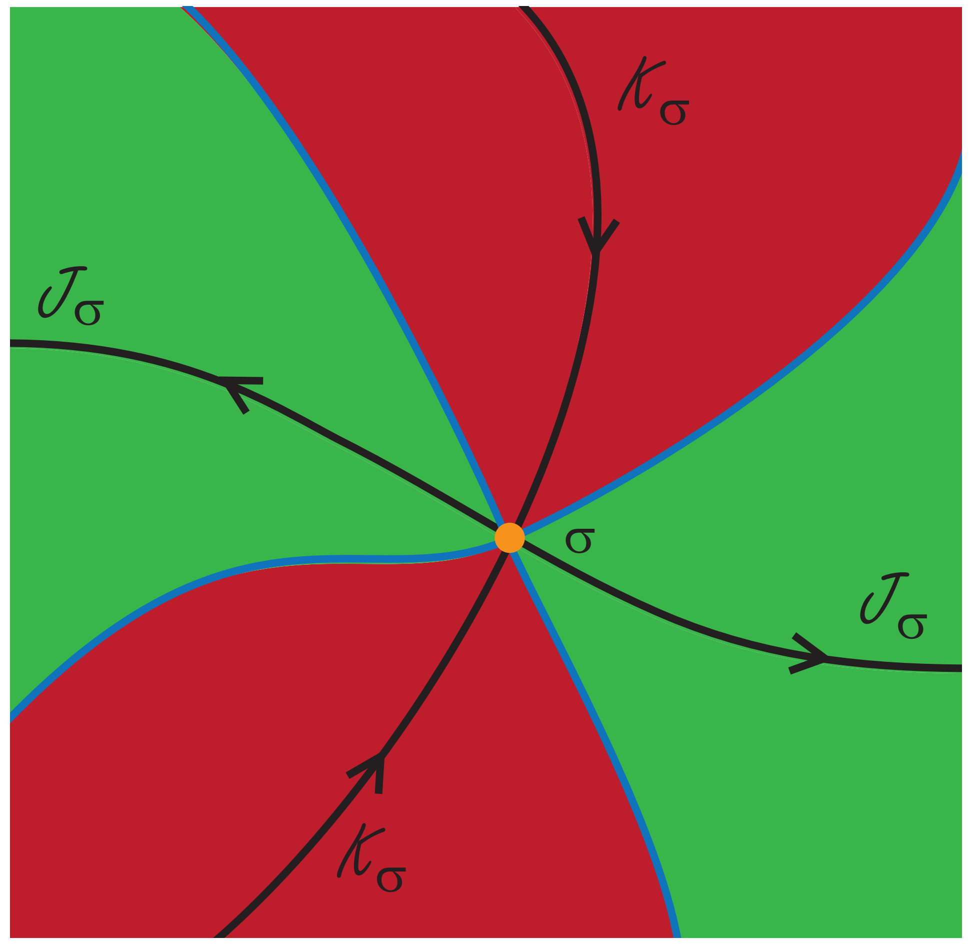
with stiffness, W. Note that the paths are **not differentiable!**

New definition of the Feynman Path integral

Picard-Lefschetz theory

Picard-Lefschetz theory suggests another route for theories with **analytic actions.** One-dimensional case:

$$I = \int_{\mathbb{R}} e^{if(x)} \mathrm{d}x$$



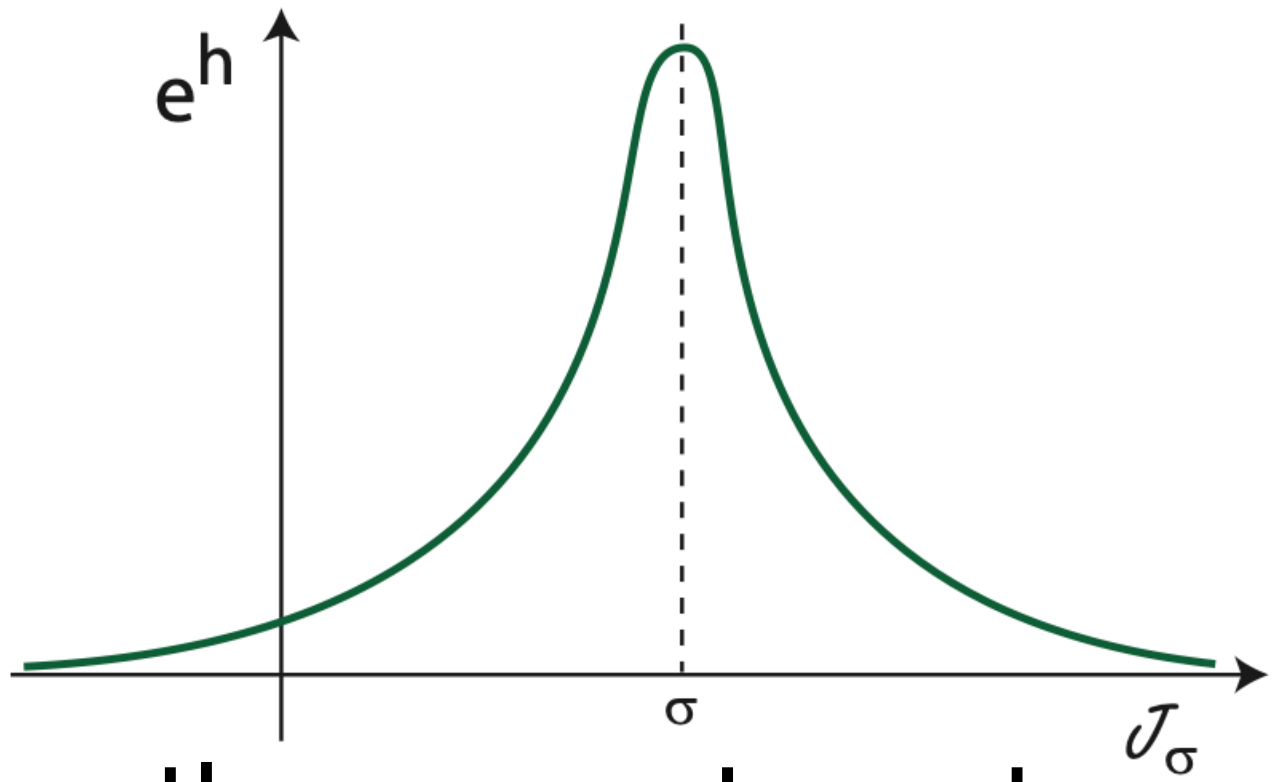
- **Analytically continue** the integrand into the complex plane

$$if(x) = h(x) + iH(x)$$

- Find all **saddle points**

$$I = \sum_i n_i e^{iH(x_i)} \int_{\mathcal{J}_i} e^{h(x)} \mathrm{d}x$$

Absolutely convergent



- Find the **steepest ascent** and **descent contours** associated with the real part of the exponent

- Deform the integration domain to the **relevant descent thimbles**

Thimble is relevant when the ascent contour **intersects** the original integration contour

New proposal for real-time QM

When applying Picard-Lefschetz theory to the real-time path integral, can we deform the paths and define the integral using the Brownian bridge measure for each relevant instanton?

$$\int_{\mathbb{R}^n} e^{f(\mathbf{x})}\mathrm{d}\mathbf{x} \equiv \lim_{R\to\infty} \int g_R(\mathbf{x}) e^{if(\mathbf{x})}\mathrm{d}\mathbf{x} = \lim_{R\to\infty} \sum_i \int_{\mathcal{J}_i} g_R(\mathbf{x}) e^{if(\mathbf{x})}\mathrm{d}\mathbf{x} = \sum_i \int_{\mathcal{J}_i} e^{if(\mathbf{x})}\mathrm{d}\mathbf{x}$$

For a regulator g, that converges to 1 as $R \to \infty$, is analytic in the complex plane, decays rapidly enough that no contributions from infinity are introduced. Extreme paths cancel out and we obtain a unique result:

$$G[x_1, x_0; T] = \int_{x(0)=x_0}^{x(T)=x_1} e^{iS[x]/\hbar}\mathscr{D}x \equiv \sum_{n_C} e^{iS[x_C]/\hbar} \int_{\mathcal{J}_{n_C}} e^{i\theta_{n_C}(\delta x)}\mathrm{d}\mu_{n_C}(\delta x)\ \Theta(T)$$

sum over relevant classical solutions

contour in space of complexities paths associated with the relevant instanton

phase, reduces to Maslow phase in semiclassical limit

Real, positive probability measure

New proposal for real-time QM

The structure of the path integral is completely organized by the classical paths. Note that this formula is exact and not the saddle point approximation. For more details see arXiv:2207.12798 (JF and Neil Turok)

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Real, positive probability measure

where the instantons are defined by

$m\ddot{x} = -V'(x),$ **with** $x(0) = x_0,$ **and** $x(T) = x_1$

This formula should also apply to gravity!

New proposal for real-time QM

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Real, positive probability measure

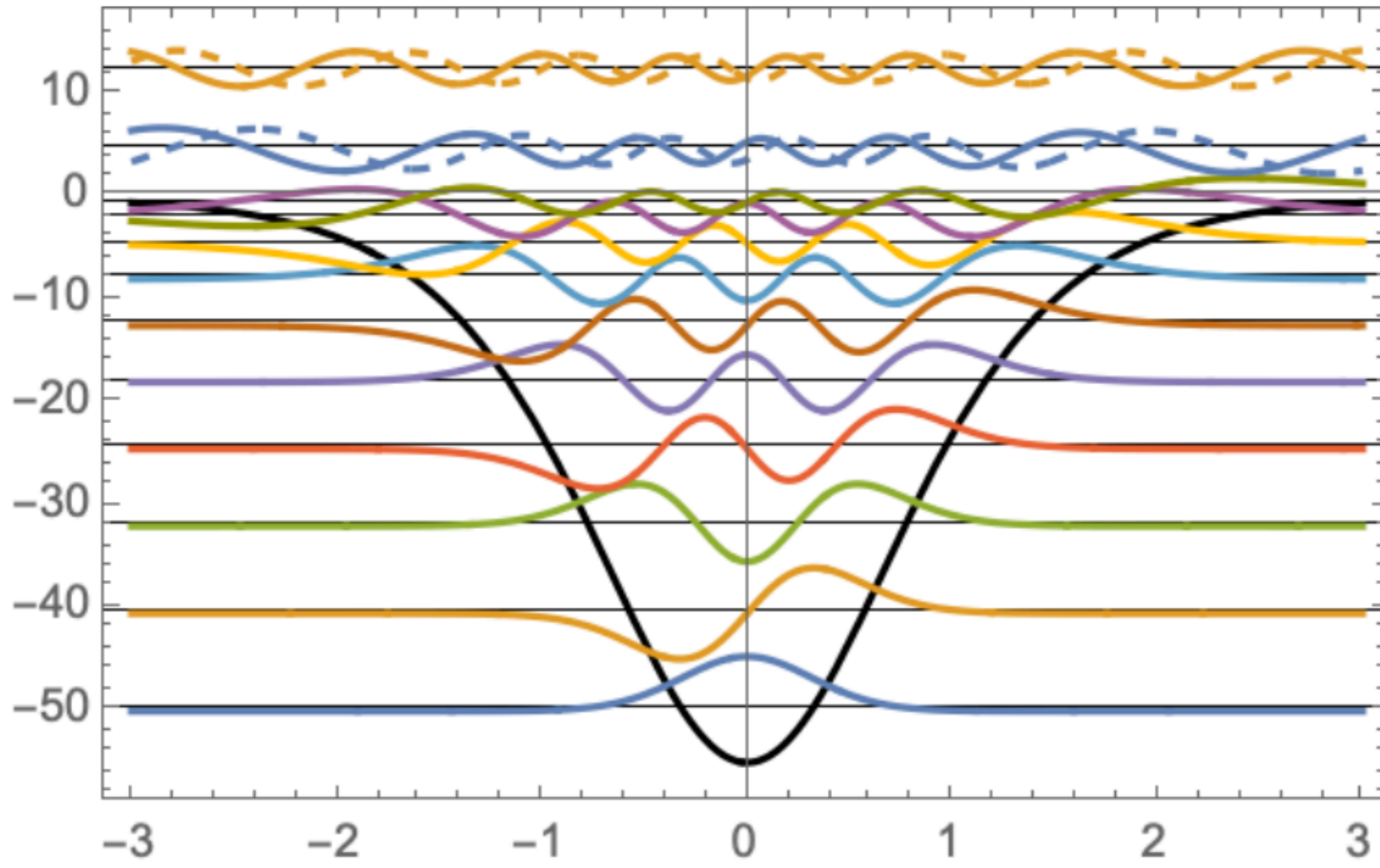
An instanton is relevant if and only if **there exists a steepest ascent deformation of the saddle point to a real path**

The propagator in action

$G[x_1, x_0; T]$ $S[x(t)]$

Pöschl–Teller potential

Enough formalism! How to study this all in practice?
The harmonic oscillator is fully solvable but too simple!
The Teller potential is both generic and solvable.



$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \qquad V(x) = \frac{V_0}{\cosh(x)^2} \qquad \hat{H}\phi_E(x) = E\phi_E(x)$$

with the propagator

$$G[x_1, x_0; T] = \left[\sum_{0 \le n < N} \phi_n(x_1)\phi_n^*(x_0)e^{-iE_nT/\hbar} + \int_{-\infty}^{\infty} \phi_k(x_1)\phi_k^*(x_0)e^{-\frac{i\hbar k^2 T}{2m}}\frac{\hbar^2 k \mathrm{d}k}{m}\right]\Theta(T)$$

with $\phi_E(x)$ proportional to the Legendre polynomials

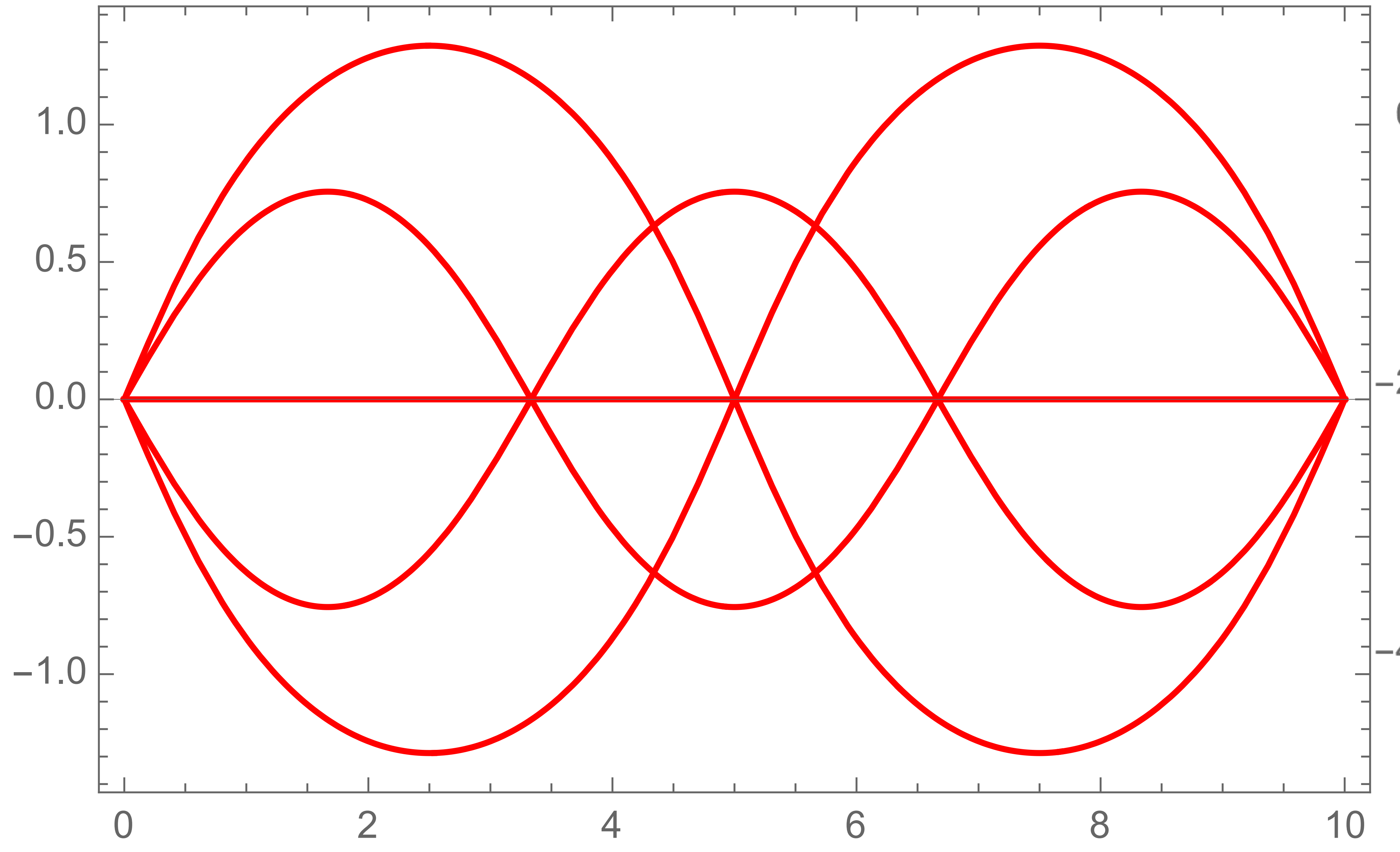
Caustics in the propagator

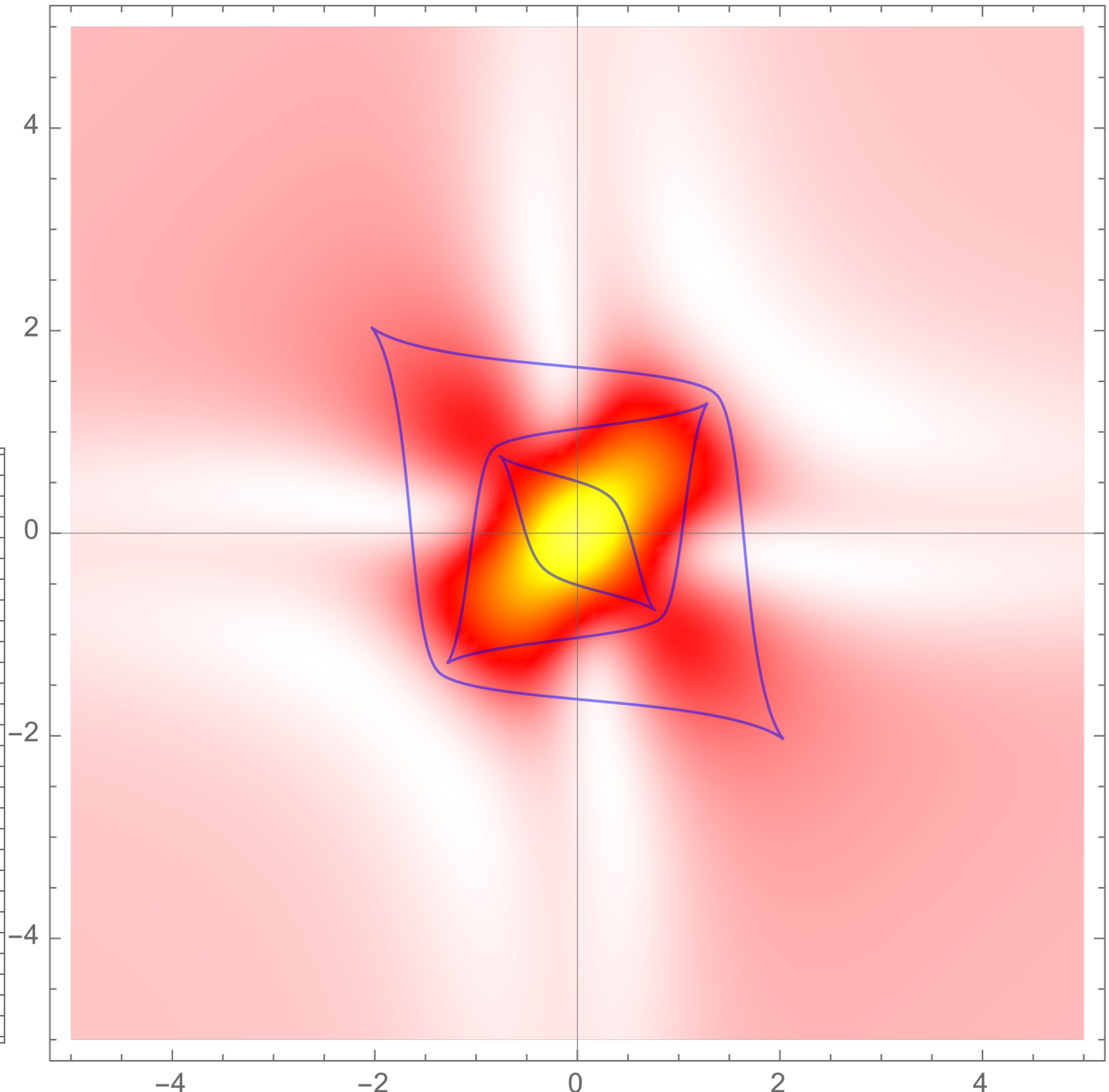
The propagator consists of an **interference pattern** structured by **caustics**!

Caustics in the propagator

Caustics separate regions with distinct numbers of real solutions to the boundary value problem

$$m\ddot{x}(t) = \frac{2V_0 \tanh(x(t))}{\cosh^2(x(t))} \quad \text{with} \quad x(0) = x_0,\ x(T) = x_1$$



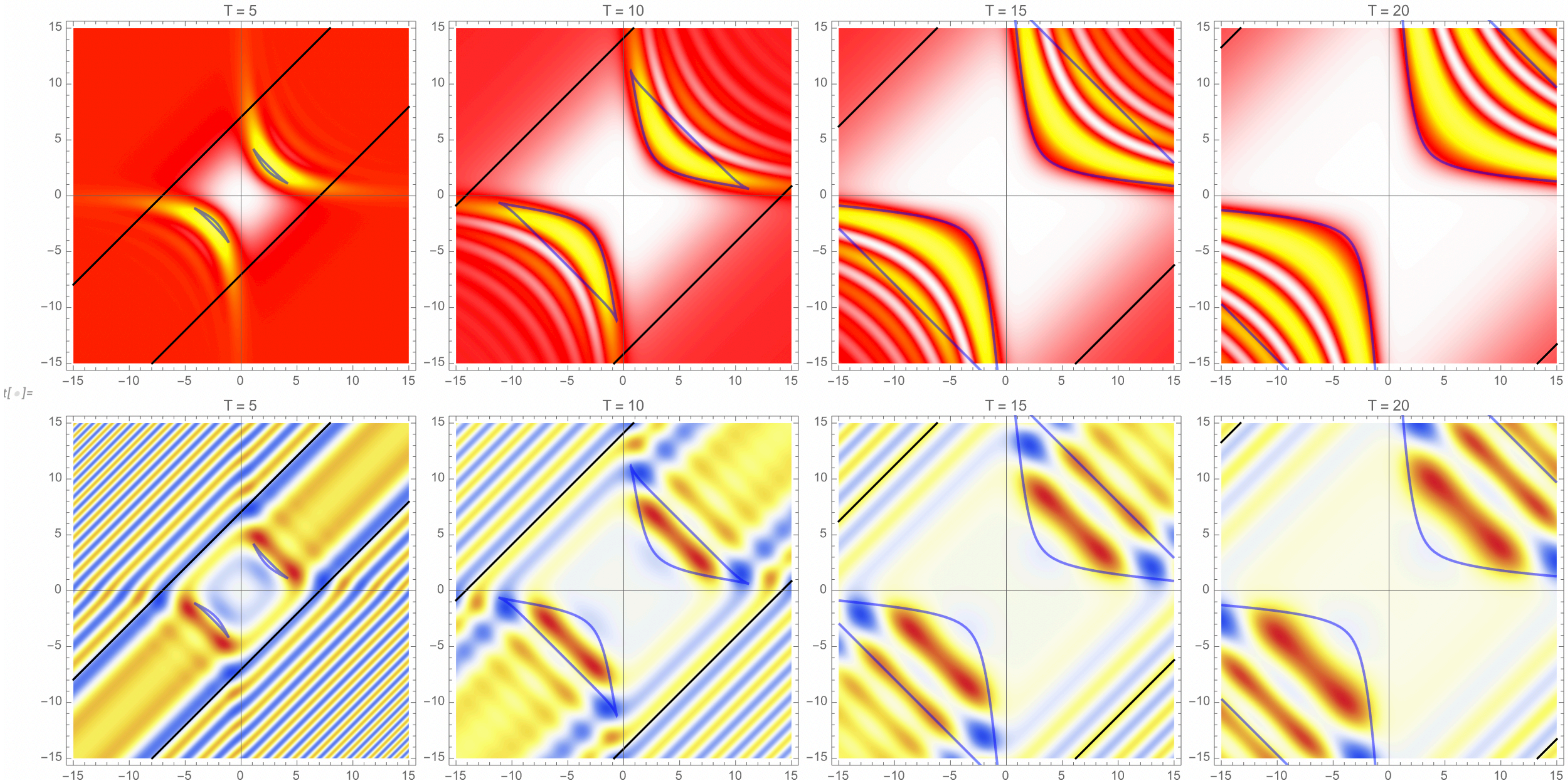


Caustics in the propagator

The potential barrier: there are always either 1 or 3 real classical paths

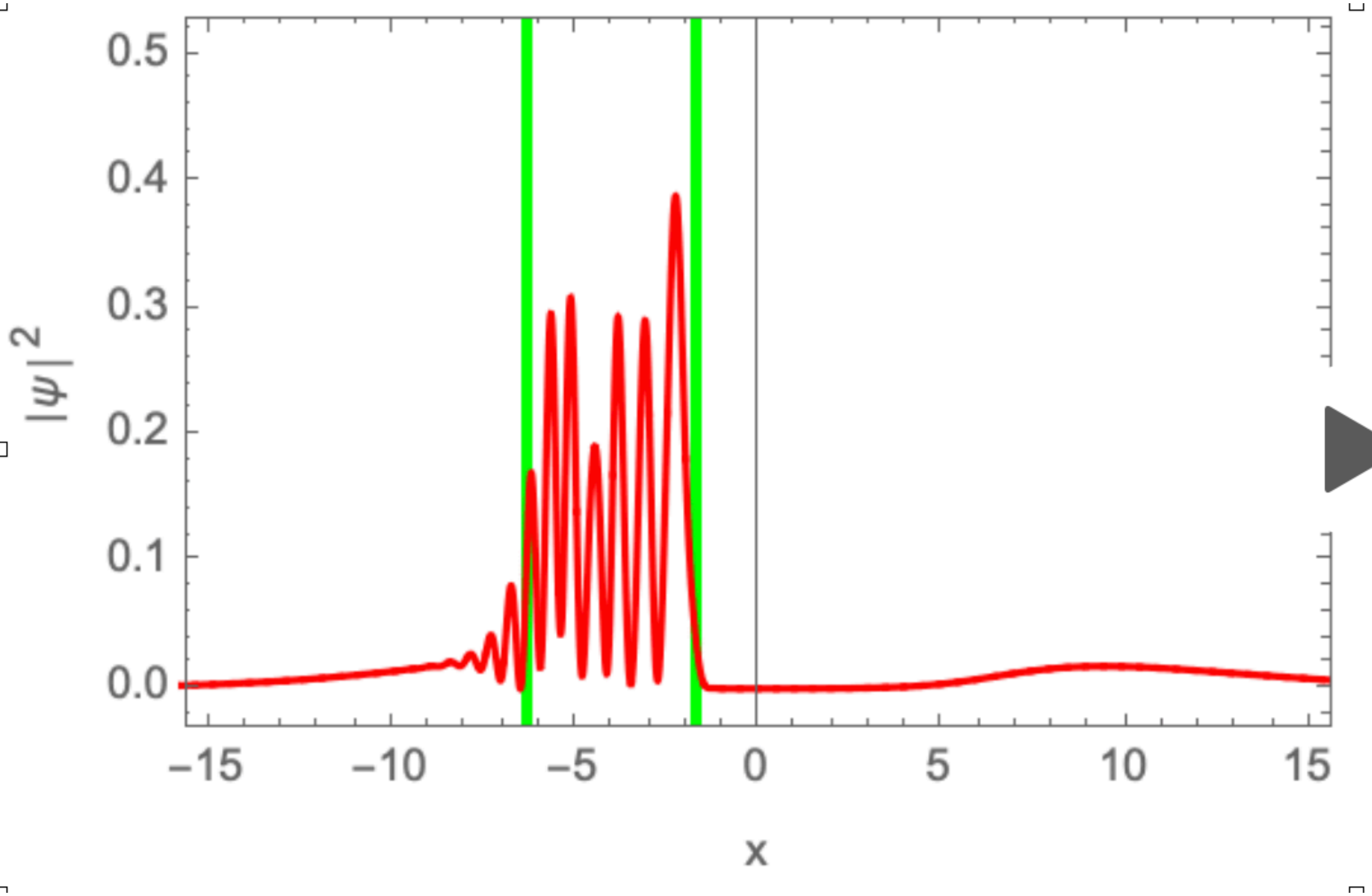
Caustics in the propagator

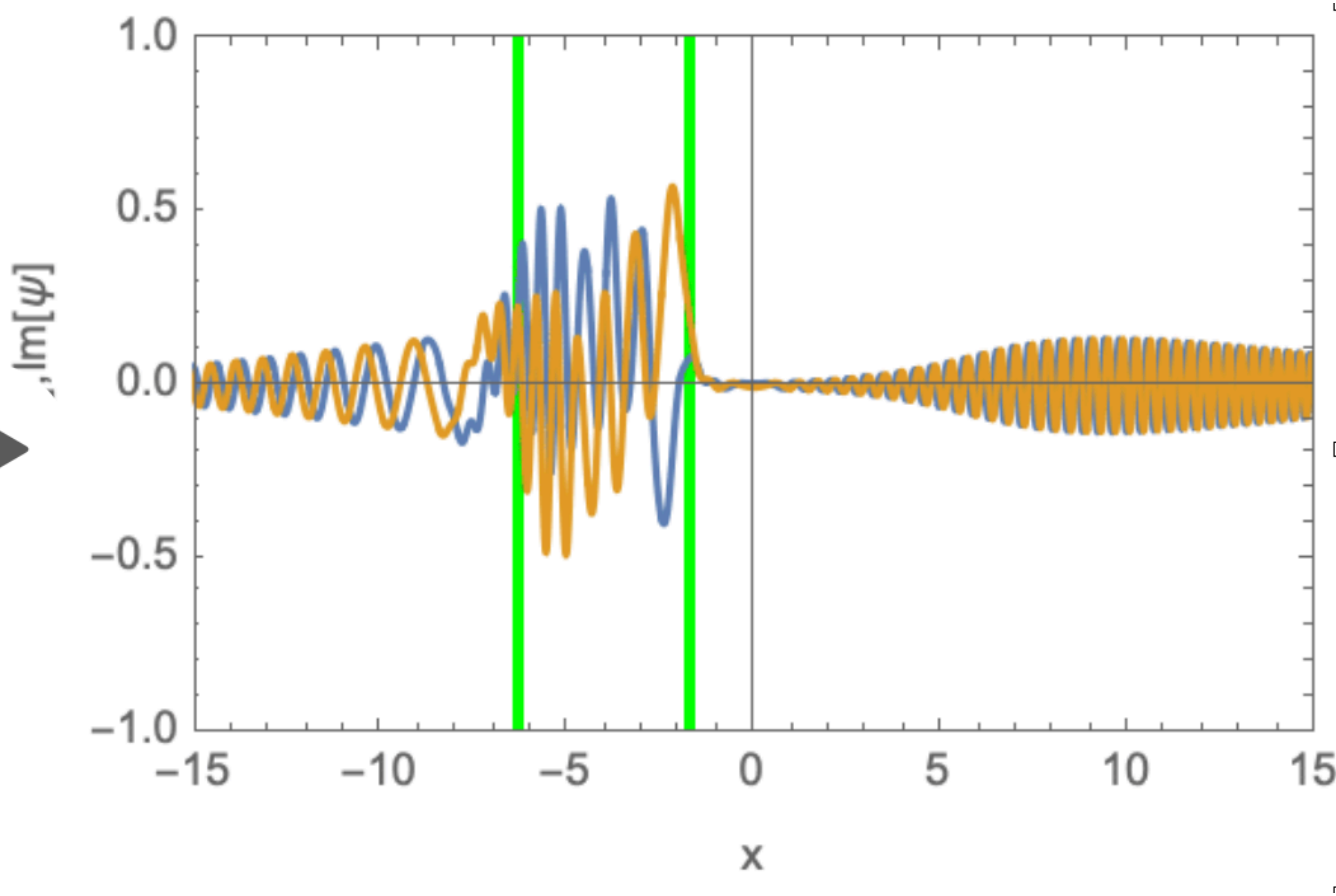
The potential barrier



Caustics in the propagator

Not only in the propagator but also in the Schrödinger equation



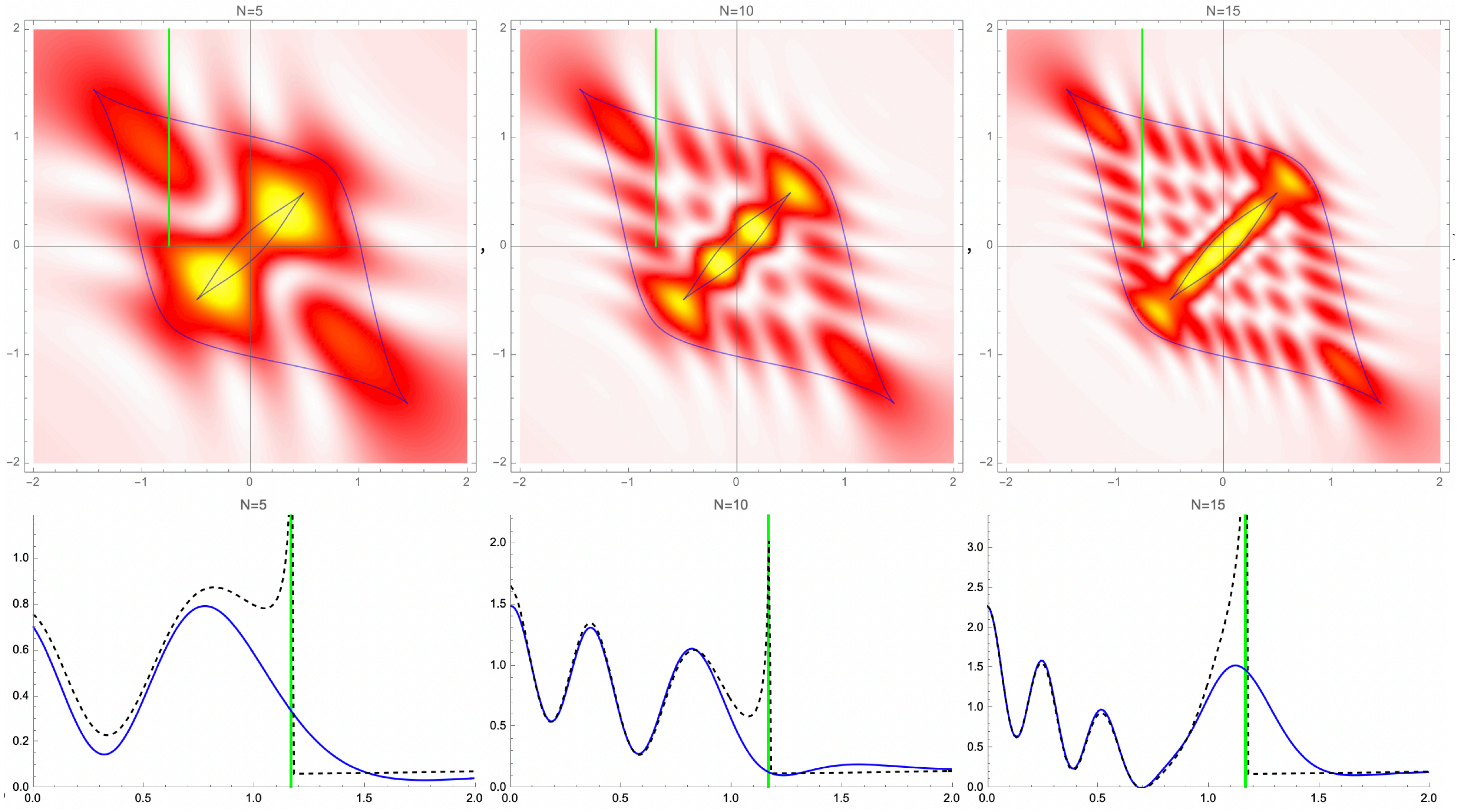


Relevant classical paths

How about complex saddle points?

The real WKB approximation does a good job in the triple-image but fails in the single-image region.

We must include complex classical paths solving the boundary value problem

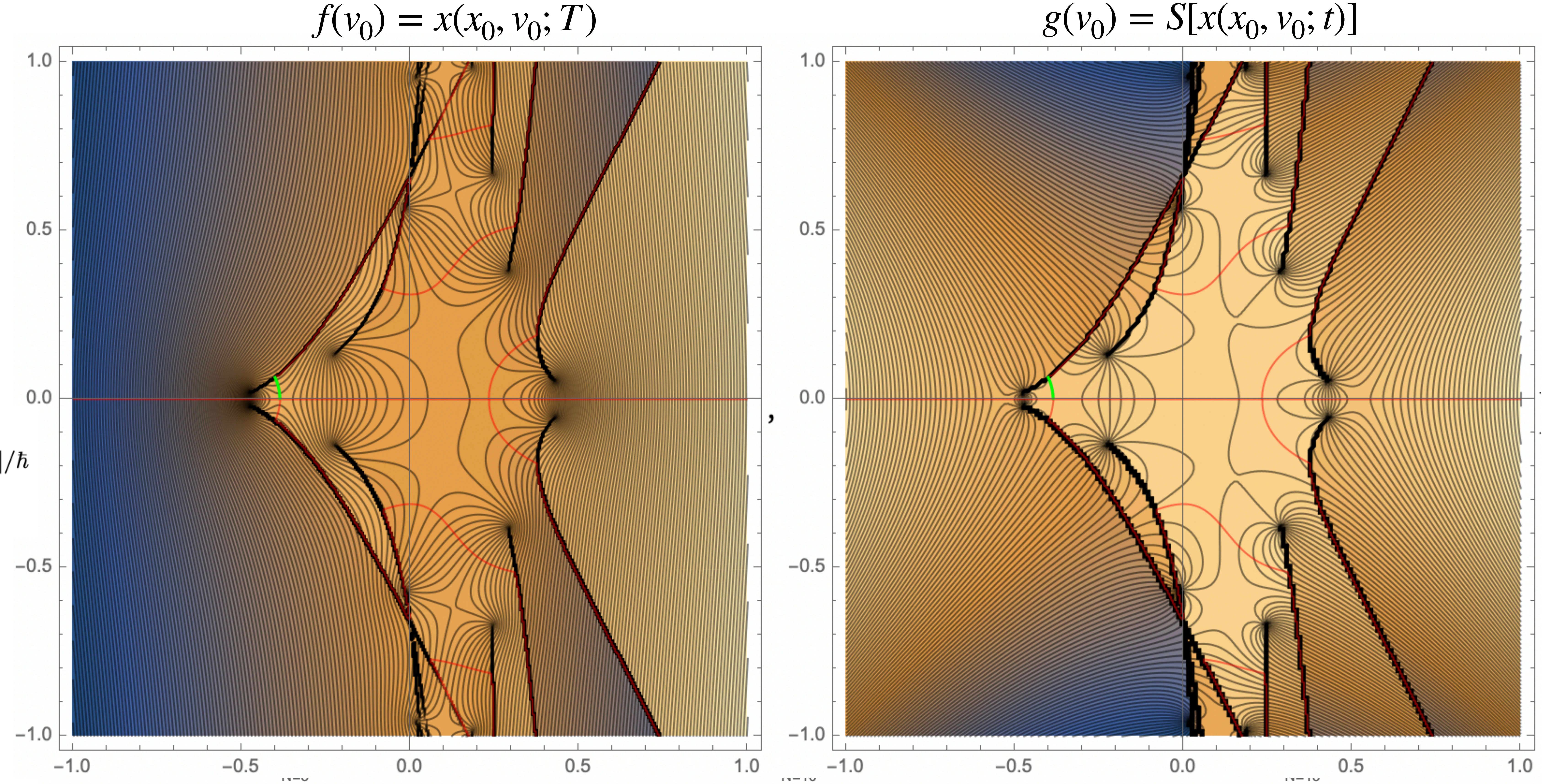


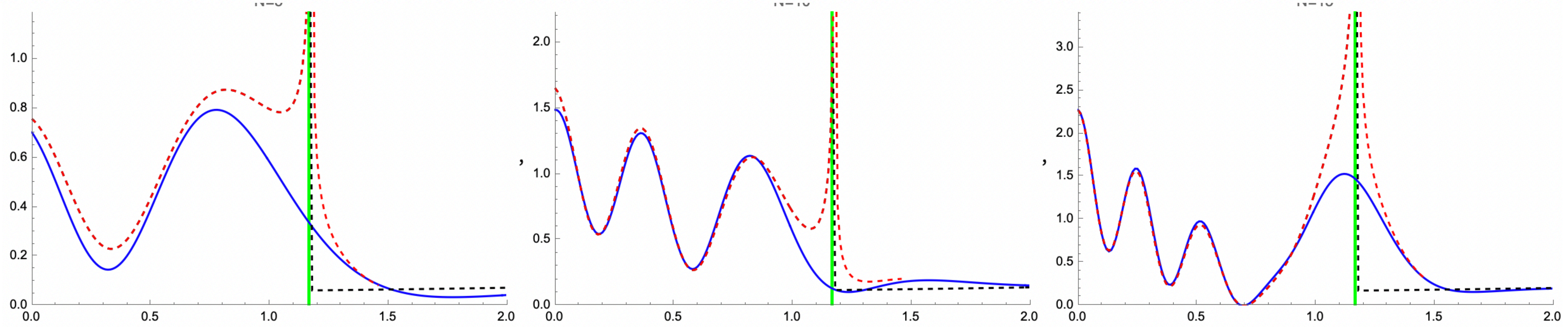
$$G_{WKB}[x_1, x_0; T] = \sum_{x_c} \sqrt{\frac{1}{2\pi i\hbar} \det\left(-\frac{\partial^2 S[x_c]}{\partial x_0 \partial x_1}\right)} e^{iS[x_c]/\hbar}$$

Relevant classical paths

Complex WKB approximation works great, till the complex saddle point seizes to exist!

$$G_{WKB}[x_1, x_0; T] = \sum_{x_c} \sqrt{\frac{1}{2\pi i\hbar} \det\left(-\frac{\partial^2 S[x_c]}{\partial x_0 \partial x_1}\right)} e^{iS[x_c]/\hbar}$$



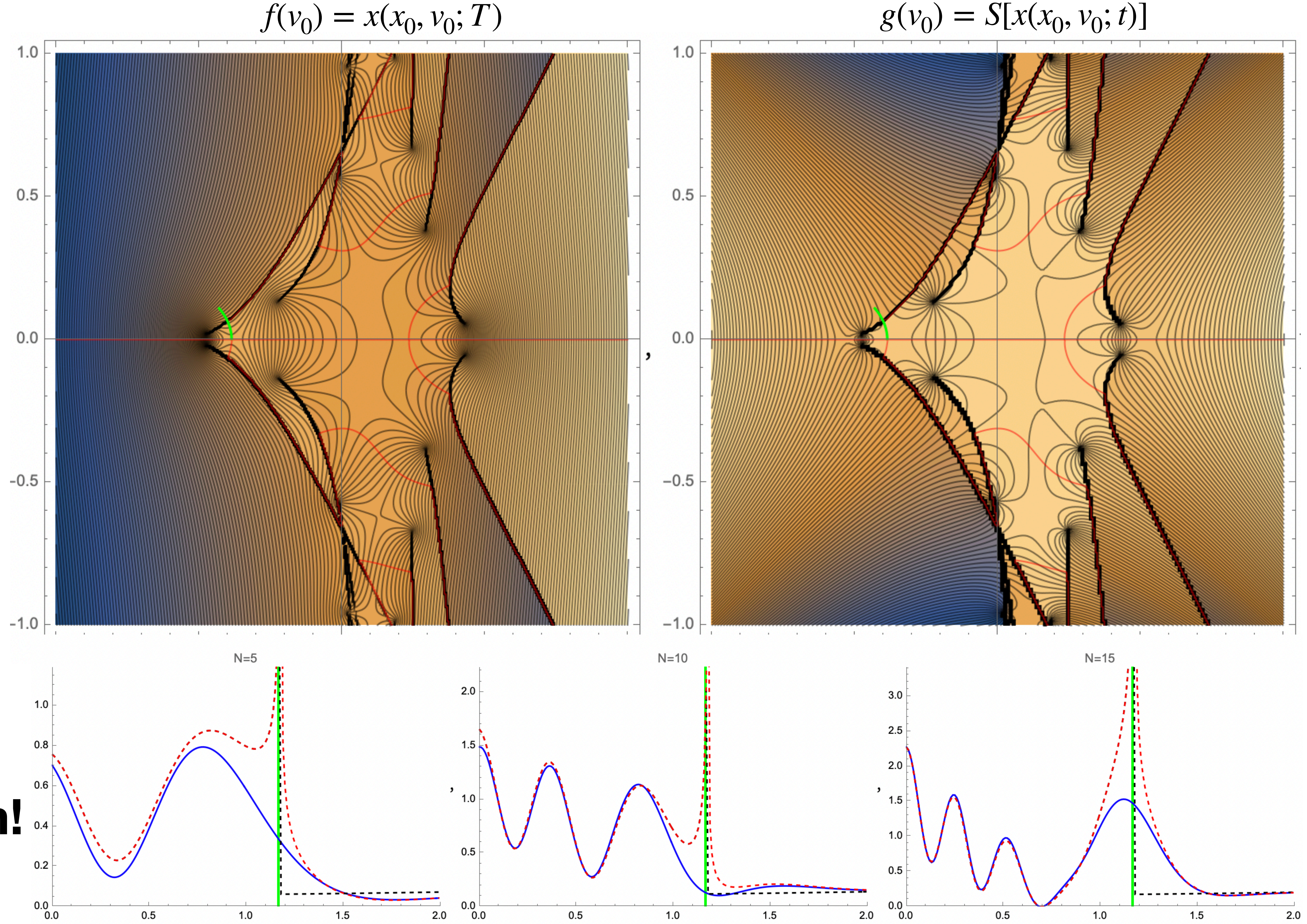


Relevant classical paths

Complex WKB approximation works great, till the complex saddle point seizes to exist!

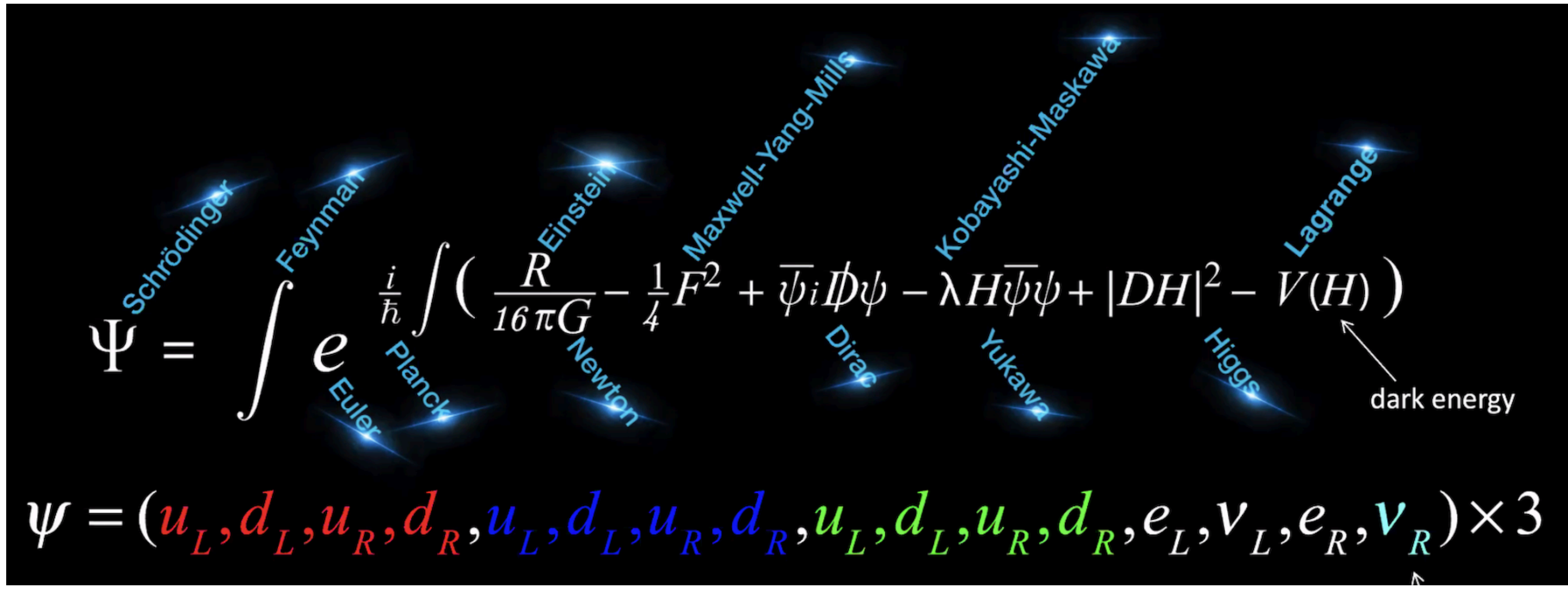
With analytic continuations we can move past the **pole-crossing**, and complete the WKB approximation.

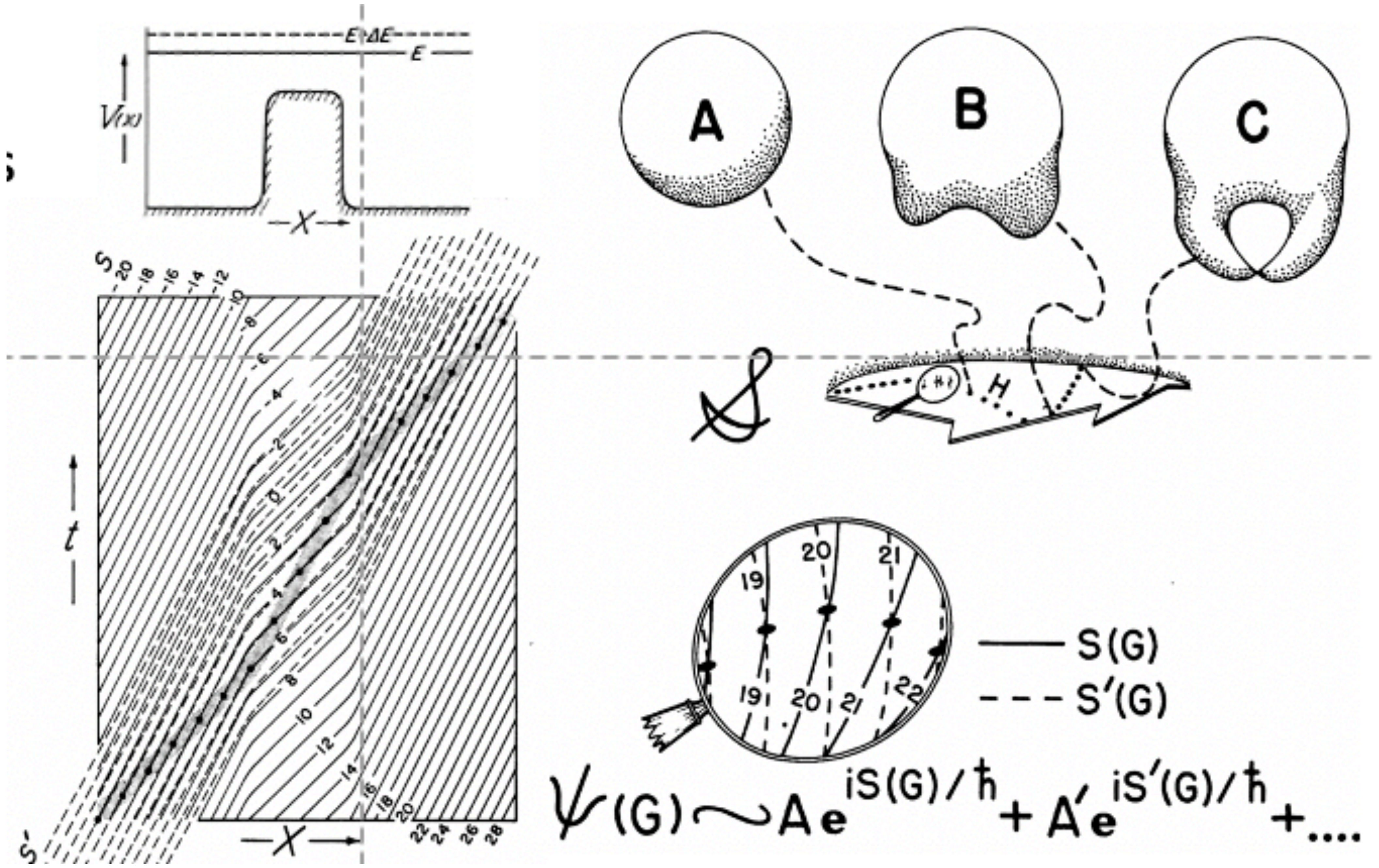
No solution to the boundary value problem!



Summary

- **Interference** is central to our understanding of the **quantum universe**
- We **propose a new definition** of the real-time path integral using **Picard-Lefschetz theory**
- Instantons go **beyond** complex classical paths!
- We hope that this will be useful in **quantum mechanics, quantum field theory, and Lorentzian quantum cosmology**





Oscillatory integrals

Oscillatory integrals occur in many places, ranging from classical systems, and wave optics, to quantum physics

- Absolutely *v.s.* conditionally convergent integrals

$$I = \int f(x)\mathrm{d}x \qquad \int |f(x)|\,\mathrm{d}x < \infty$$

$$\int_{-\infty}^{\infty} e^{if(x)}\,\mathrm{d}x \quad \text{or} \quad \int_{x(0)=x_0}^{x(1)=x_1} e^{iS[x(t)]}\,\mathscr{D}x(t)$$

- Fubini's theorem

$$\iint f(x,y)\,\mathrm{d}(x,y) = \int\left[\int f(x,y)\,\mathrm{d}x\right]\mathrm{d}y = \int\left[\int f(x,y)\,\mathrm{d}y\right]\mathrm{d}x \quad \text{when} \quad \iint |f(x,x)|\,\mathrm{d}(x,y) < \infty$$

- Dominated convergence theorem

$$\lim_{n\to\infty}\left[\int f_n(x)\mathrm{d}x\right] = \int\left[\lim_{n\to\infty} f_n(x)\right]\mathrm{d}x \quad \text{when} \quad |f_n(x)| \leq g(x) \quad \forall n \quad \text{with} \quad \int |g(x)|\,\mathrm{d}x < \infty$$

Feynman-Kac formula

When using a Wick rotation: **interference -> statistical physics**

$$\int e^{\frac{i}{\hbar}\int (m\dot{x}^2/2 - V(x))\mathrm{d}t}\mathscr{D}x \to \int e^{-\frac{1}{\hbar}\int (m\dot{x}^2/2 + V(x))\mathrm{d}t}\mathscr{D}x$$

which is still mathematically **ill-defined**. However, we can define the set of symbols in terms of the **Brownian bridge measure**

$$\frac{\int e^{-\frac{1}{\hbar}\int (m\dot{x}^2/2 + V(x))\mathrm{d}t}\mathscr{D}x}{\int e^{-\frac{1}{\hbar}\int m\dot{x}^2/2\mathrm{d}t}\mathscr{D}x} \equiv \int e^{-\frac{1}{\hbar}\int V(x)\mathrm{d}t}\mathrm{d}\mu_B(x)$$

The smoothing due to the kinetic term and wildness of the infinite product "measure" are beautifully balanced in the Brownian bridge.

Picard-Lefschetz theory

- Fresnel integral

$$\int_{-\infty}^{\infty} e^{ix^2}\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} e^{ix^2}\mathrm{d}x = (1+i)\sqrt{\frac{\pi}{2}}$$

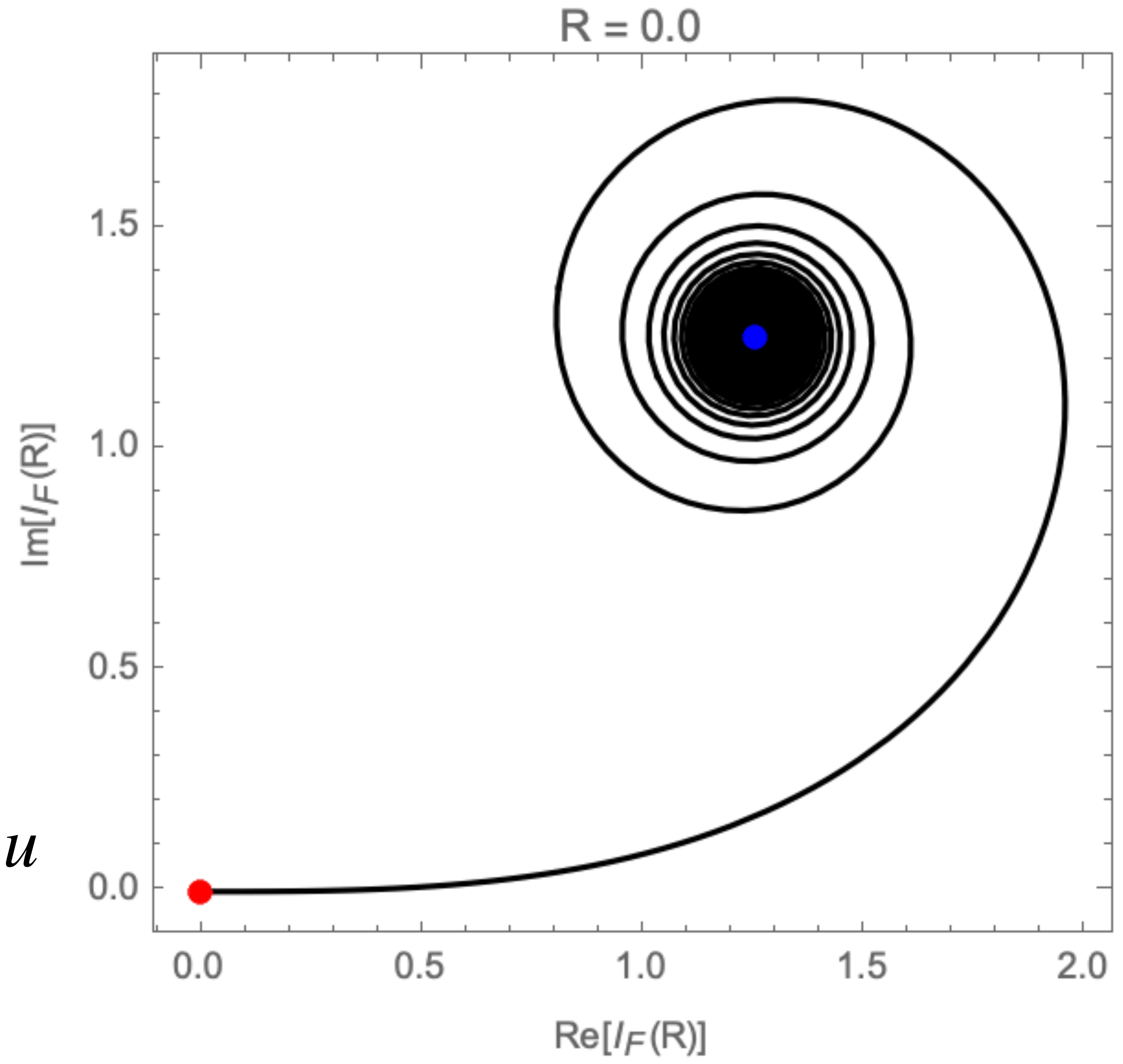
- Multi-dimensional extension

$$\iint_{\mathbb{R}^2} e^{i(x^2+y^2)}\mathrm{d}x\mathrm{d}y = \lim_{R\to\infty} 2\pi\int_0^R r\, e^{ir^2}\mathrm{d}r = \lim_{R\to\infty}\left[i\pi - \pi e^{iR^2}\right]$$

- Complex analysis

$$\int_{\mathbb{R}} e^{ix^2}\,\mathrm{d}x = \frac{1+i}{\sqrt{2}}\int_{\mathbb{R}} e^{-u^2}\,\mathrm{d}u = (1+i)\sqrt{\frac{\pi}{2}} \qquad x = \frac{1+i}{\sqrt{2}}u$$

$$\iint_{\mathbb{R}^2} e^{i(x^2+y^2)}\,\mathrm{d}(x,y) = i\int_{\mathbb{R}^2} e^{-(u^2+v^2)}\,\mathrm{d}(u,v) = i\pi$$



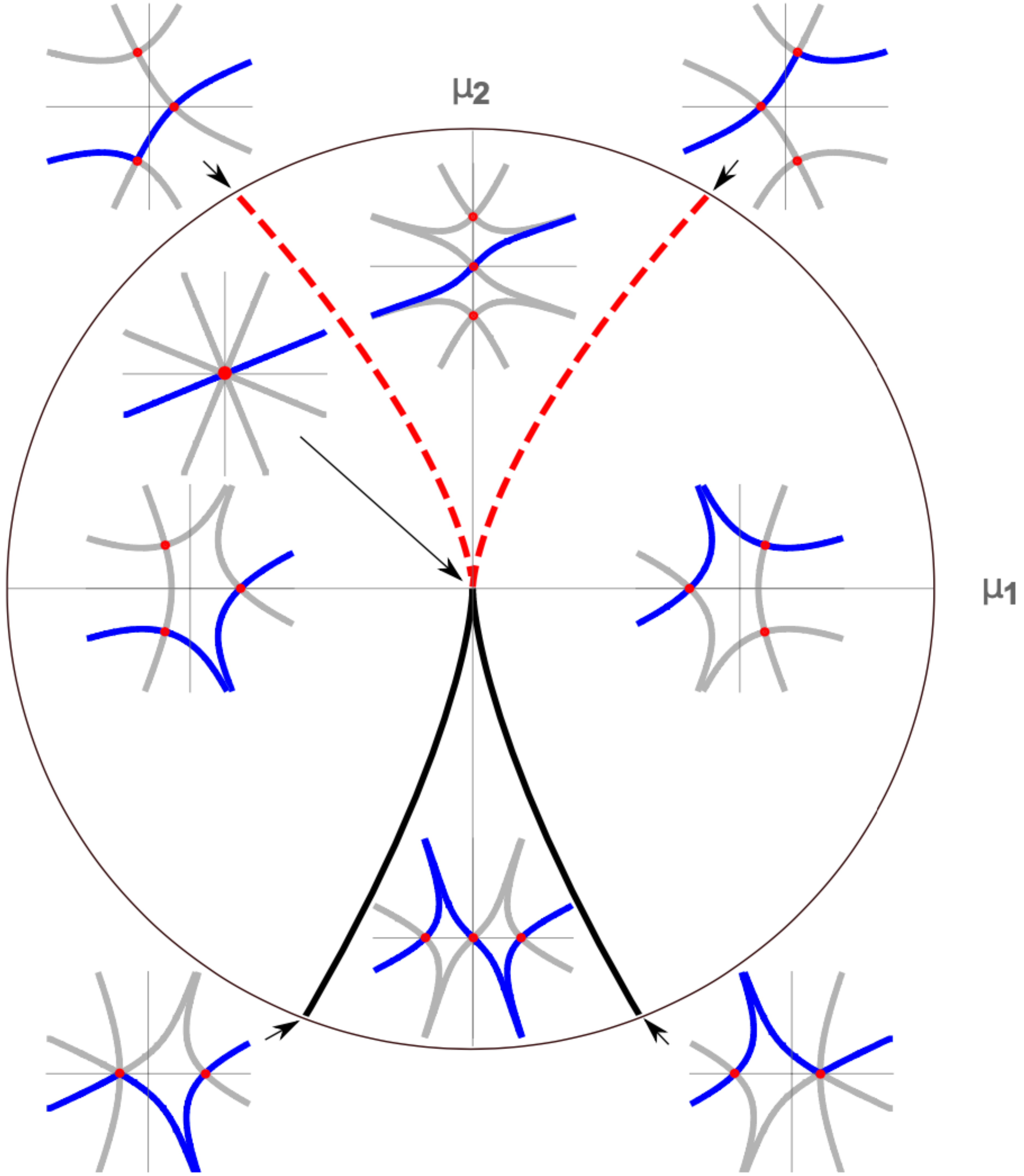
Single plane lensing

$$\Psi(\mu, \nu) = \left(\frac{\nu}{\pi}\right)^{N/2} \int e^{i\nu\phi(x,\mu)} \mathrm{d}x$$

Name	Symbol	K	N	$\phi(\boldsymbol{x}; \boldsymbol{\mu})$
Maximum/minimum	$A_1^{\pm}$	0	1	$\pm x^2$
Fold	A_2	1	1	$x^3/3 + \mu x$
Cusp	A_3	2	1	$x^4/4 + \mu_2 x^2/2 + \mu_1 x$
Swallowtail	A_4	3	1	$x^5/5 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x$
Butterfly	A_5	4	1	$x^6/6 + \mu_4 x^4/4 + \mu_3 x^3/3 + \mu_2 x^2/2 + \mu_1 x$
Elliptic umbilic	D_4^-	3	2	$x_1^3 - 3x_1 x_2^2 - \mu_3(x_1^2 + x_2^2) - \mu_2 x_2 - \mu_1 x_1$
Hyperbolic umbilic	D_4^+	3	2	$x_1^3 + x_2^3 - \mu_3 x_1 x_2 - \mu_2 x_2 - \mu_1 x_1$
Parabolic umbilic	D_5	4	2	$x_1^4 + x_1 x_2^2 + \mu_4 x_2^2 + \mu_3 x_1^2 + \mu_2 x_2 + \mu_1 x_1$

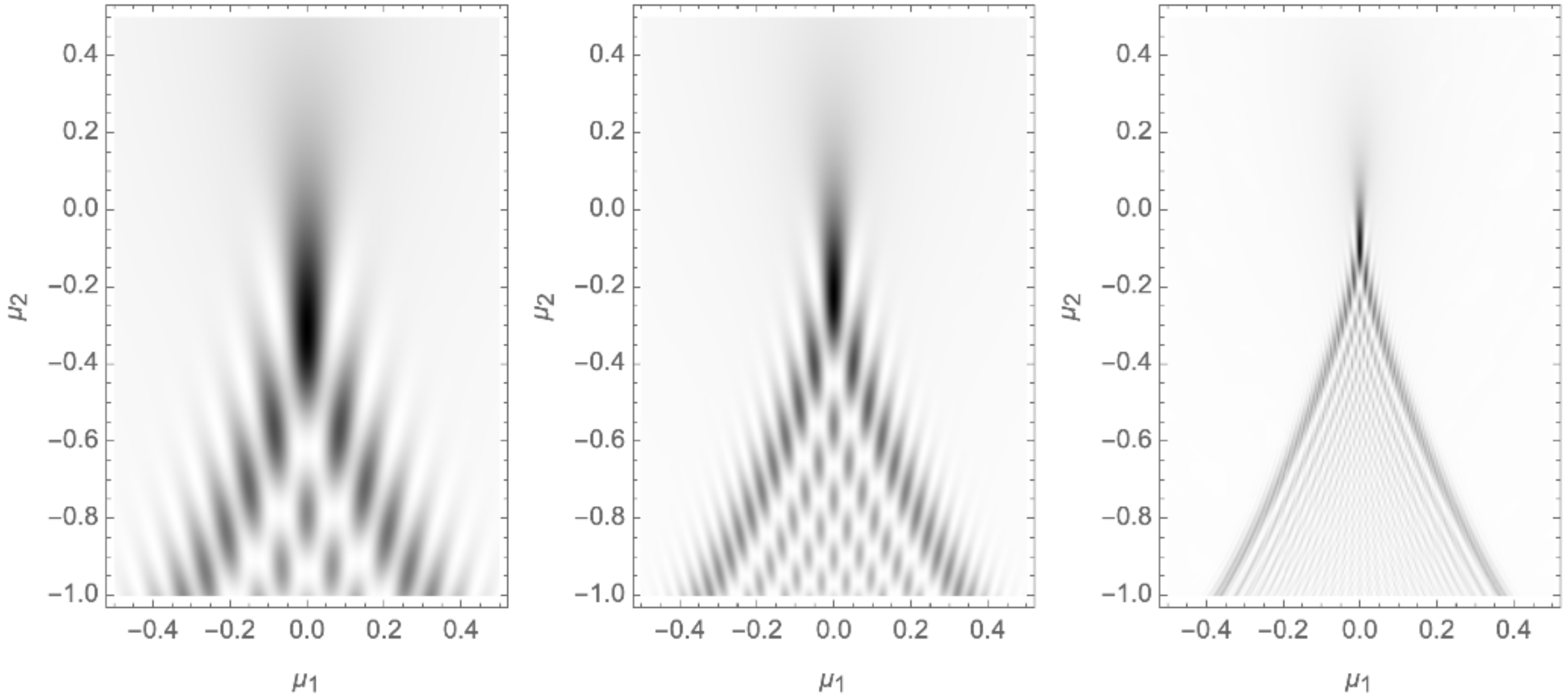
Table I: The unfoldings of the seven elementary catastrophes with codimension $K \leq 4$, with $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_K)$. The normal forms are defined as the unfolding at parameter $\boldsymbol{\mu} = \boldsymbol{0}$, *i.e.*, $\phi(\boldsymbol{x}; \boldsymbol{0})$.

Single plane lensing



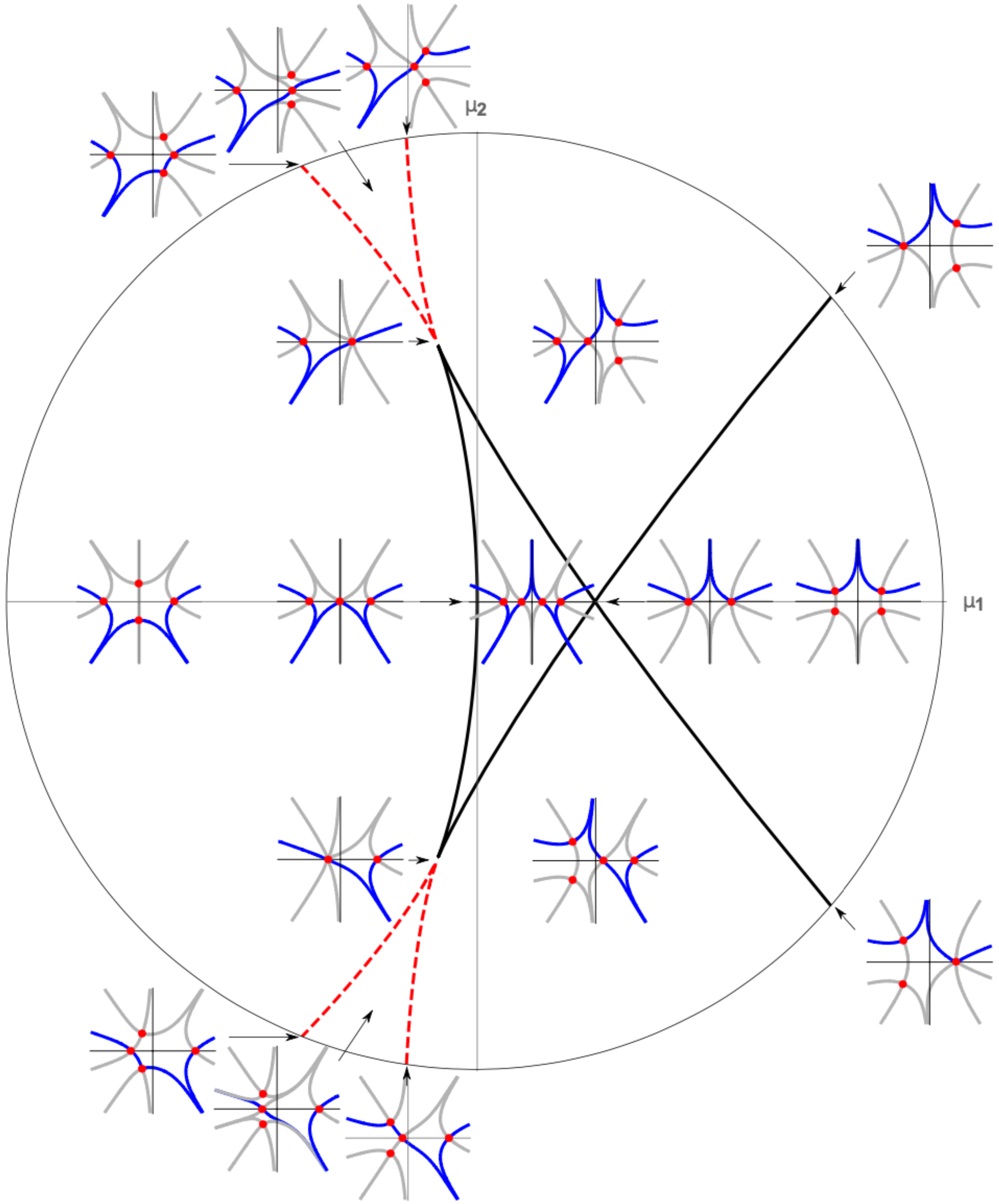
Caustics and Stoke's lines

$$\Psi(\mu,\nu) = \int e^{i\nu(x^4/4+\mu_2 x^2/2+\mu_1 x)} \mathrm{d}x$$



(a) $\nu = 50$ (b) $\nu = 100$ (c) $\nu = 500$

Single plane lensing



Caustics and Stoke's lines

$$\Psi(\mu,\nu) = \int e^{i\nu(x^5/5+\mu_3 x^3/3+\mu_2 x^2/2+\mu_1 x)} \mathrm{d}x$$