Field Theory on and of Spacetime

Kasia Rejzner

University of York

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1. Introduction

2. pAQFT and QG
   - Gauge-invariant observables
   - QFT on causal sets
   - fRG and pAQFT

On the road to QG: where does QFT come in?

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- Some things to consider: Lorentzian signature, background independence, gauge invariant observables, locality vs non-locality.

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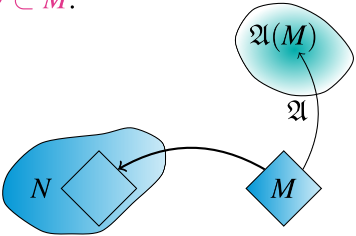

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Main advantage

Construction of observables $\mathfrak{A}(\mathcal{O})$ is independent from the construction of states. Entanglement and superposition are properties of states (always non-local) not of observables (often local).

Locality and how to break it

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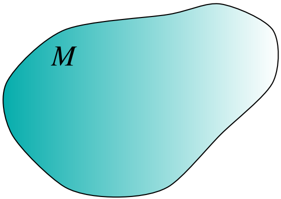

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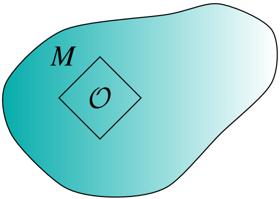

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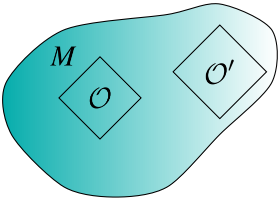

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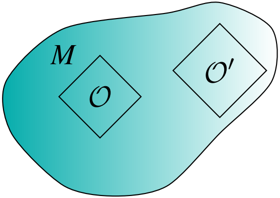

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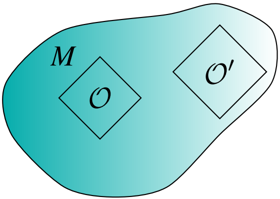

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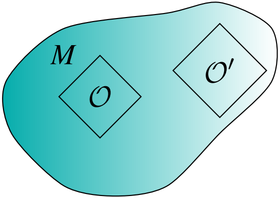

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- Good news: some of the methods for constructing AQFT models still work, even if these two types of locality are broken.

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- For a review see the book: *Perturbative algebraic quantum field theory. An introduction for mathematicians*, KR, Springer 2016.

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- The choice of action functional I specifies the dynamics. We use a modification of the Lagrangian formalism (fully covariant).

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- Use the deformation quantization to construct the non-commutative algebra $\mathfrak{A}(M) = (\mathcal{F}(M)[[\hbar]], \star)$, such that

$$F \star G \xrightarrow{\hbar=0} FG \qquad \frac{1}{i\hbar}(F \star G - G \star F) \xrightarrow{\hbar=0} \{F,G\}\,.$$

Building models in pAQFT II

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- For a quadratic action I_0 that induces hyperbolic equations of motion (e.g. $-(\Box + m^2)\varphi = 0$), $\star$ can be constructed directly, starting from Δ and choosing a choice of a 2-point function $\Delta^+ = \frac{i}{2}\Delta + H$.

$$F \star_H G \doteq m \circ e^{\hbar\left\langle \Delta^+, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle} (F \otimes G)\,,$$

Time-ordered products

- Take an interaction $V \in \mathcal{F}_{\mathrm{loc}}(M)$ and define the formal S-matrix

$$\mathcal{S}(\lambda V) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\lambda}{\hbar}\right)^n V \cdot_{\mathcal{T}} \ldots \cdot_{\mathcal{T}} V \,,$$

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- We also introduce the time-ordering map $\mathcal{T}$, so that $F \cdot_{\mathcal{T}} G = \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$. It formally corresponds to path integrating with a Gaussian measure:

$$\mathcal{T}F(0) \sim \int F(\varphi) d\mu(\varphi)$$

Interacting fields and states I

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- A natural state on $\mathcal{F}(M)$ and hence $\mathfrak{A}(M)$ is given by evaluation at a given field configuration. For the scalar field we can take $\omega(F) = F(0)$.

Interacting fields and states II

- Wightman n-point functions of the free theory are

$$W_n(f_1,\dots,f_n)=(\Phi(f_1)\star\cdots\star\Phi(f_n))(0)\,,$$

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- Interacting correlation functions are obtained as:

$$(\Phi_{\text{int}}(f_1)\star\cdots\star\Phi_{\text{int}}(f_n))(0)\,,$$

similarly for other observables in the theory.

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- Formally (in terms of formal power series and on regular functionals), this product depends only on S, not on the splitting [Hawkins, KR LMP 2020]. Some obstructions could appear when renormalization is performed.

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Perturbative algebraic QFT (pAQFT) is a machinery to turn functionals of classical field configurations (classical observables) into quantum observables.

- It allows one to study some aspects of observables in QG that are accessible to QFT methods and to learn more about the algebraic structure they define.
- The ultimate goal is to break away from the classical picture and have an intrinsically quantum formulation.

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- Fix a background Γ_0 such that the map

$$X_{\Gamma_0} : x \mapsto (X^0_{\Gamma_0}, \dots, X^3_{\Gamma_0})$$

is injective.

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- Take another local field $A_\Gamma(x)$ (e.g. a metric scalar). Then

$$\mathcal{A}_\Gamma := A_\Gamma \circ \alpha_\Gamma$$

is invariant under diffeomorphisms.

Relational observables III

Physical interpretation

Fields X^{μ}_{Γ} are configuration-dependent coordinates such that $[A_{\Gamma} \circ X_{\Gamma}^{-1}](Y)$ corresponds to the value of the quantity A_{Γ} provided that the quantity X_{Γ} has the value $X_{\Gamma} = Y$.

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for a test density f. This functional depends on the choice of observable A and "observer" X.
- If X^μ_Γ and A_Γ are all local fields themselves, then $F_{\mathcal{A}}$ is non-local with local derivatives, hence amenable to EG renormalization.

Examples:

- On generic backgrounds g_0, without matter, one can use traces of the powers of the Ricci operator:

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- See also papers by Fröb et. al. [JCAP 2017, CQG 2018].

Algebraic QFT on causal sets

- One can replace the smooth manifold with a discrete set of points equipped with the causal order relation and the main ideas of pAQFT carry over.

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- Let $(\mathcal{C}, \preceq)$ be a discrete set of points $\mathcal{C}$ with a relation $\preceq$:

$$\begin{aligned} x \preceq y \preceq z \implies x \preceq z, & \qquad \textit{transitivity} \\ x \preceq y \text{ and } y \preceq x \implies x = y, & \qquad \textit{acyclicity} \\ |I(x,y)| < \infty, & \qquad \textit{local finiteness} \end{aligned}$$

where

$$I(x,y) = \{z \in \mathcal{C} \mid x \preceq z \preceq y\}$$

and we write $x \prec y$ if $x \preceq y$ and $x \neq y$.

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- Free dynamics: a discretized retarded Green function Δ^R (ideally coming from a discretization of some normally hyperbolic operator).

Some results from pAQFT

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- Further investigation of interacting correlation functions (work in progress).

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- We propose new flow equations that can be realized on arbitrary globally hyperbolic manifolds in any Hadamard state (examples: deSitter, thermal states).

Generating functions

- For an arbitrary but fixed Hadamard state ω, define:

$$Z(j) := \omega(\mathcal{S}_V(J)) = \omega[\mathcal{S}(V)^{-1} \star S(V+J)] = \omega[R_V \mathcal{S}(J)]\,.$$

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- It is a generating function for time-ordered interacting correlators:

$$\frac{\delta^n Z}{i^n \delta j(x_1)\ldots\delta j(x_n)}\Big|_{j=0} = \omega \circ R_V\left(\chi(x_1) \cdot_{\mathcal{T}} \ldots \cdot_{\mathcal{T}} \chi(x_n)\right)$$
$$= \omega_V(\chi(x_1) \cdot_{\mathcal{T}} \cdots \cdot_T \chi(x_n)) = \omega \circ R_V(\chi(x_1) \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} \chi(x_n))\,,$$

where $\omega_V \doteq \omega \circ R_V$ is the interacting state.

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- The effective action $\tilde{\Gamma}$ is $\tilde{\Gamma}(\phi) = W(j_\phi) - J_\phi(\phi)$, where $j_\phi \in C_c^\infty(M)$ is the current defined by

$$\frac{\delta W}{\delta j}\bigg|_{j=j_\phi} = \phi\,,$$

for $\phi \in \mathcal{E}$.

Choice of the regulator

- We use a local regulator

$$Q_k = -\frac{1}{2}\int dx\, q_k(x)\chi(x)^2\,,$$

and chose $q_k(x) = k^2 f(x)$, where f is a compactly supported smooth function (to be taken to 1). Compare: *Spectral functions of gauge theories with Banks-Zaks fixed points*, Yannick Kluth, Daniel F. Litim, Manuel Reichert, Phys.Rev.D 2023.

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- Modify the free theory: $I_{0k} = I_0 + Q_k$. The regularised generating functional Z_k is
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- We also have $W_k(j) = -i\log Z_k(j)$, $\tilde{\Gamma}_k(\phi) = W_k(j_\phi) - J_\phi(\phi)$ and finally we can translate $\tilde{\Gamma}_k$ to get the *average effective action*,

$$\Gamma_k(\phi) = \tilde{\Gamma}_k(\phi) - Q_k(\phi)\,.$$

Flow equations I

- By definition:

$$\partial_k W_k(j) = -\frac{1}{2}\int dx \partial_k q_k(x) \frac{1}{Z_k(j)} \omega(S(V)^{-1} \star [S(V+J+Q_k) \cdot_{\mathcal{T}} \mathcal{T}\chi^2(x)]) .$$

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- After a short computation:

$$\begin{aligned}
&\partial_k \Gamma_k(\phi) \\
&= -\frac{1}{2}\int dx \partial_k q_k(x) \left[\frac{1}{Z_k(j_\phi)} \omega\big(R_V(S(J_\phi+Q_k)\cdot_{\mathcal{T}} \mathcal{T}\chi^2(x))\big) - \phi^2(x)\right] \\
&\qquad\qquad = \lim_{y\to x} \frac{i}{2}\int dx \partial_k q_k(x) \left[\frac{\delta^2 W_k(j)}{\delta j(x)\delta j(y)} - i\widetilde{H}_F(x,y)\right] ,
\end{aligned}$$

where we use an appropriate distribution $\tilde{H}_F$. This corresponds to a choice of normal ordering. Hence...

Flow equations II

Wetterich-form equation

$$\partial_k \Gamma_k = -\frac{i}{2} \int dx \partial_k q_k(x) : \left[\Gamma_k^{(2)} - q_k\right]^{-1} :_{\widetilde{H}_F} (x) \,,$$

Thank you very much for your attention!