Asymptotic Quantum Correlations of Field-modes in Time-dependent Backgrounds

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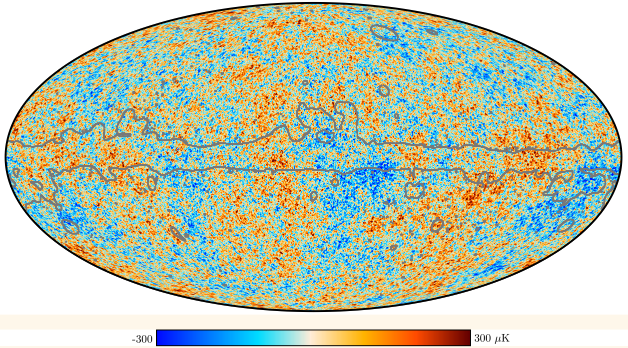
Inhomogeneities in the early-Universe

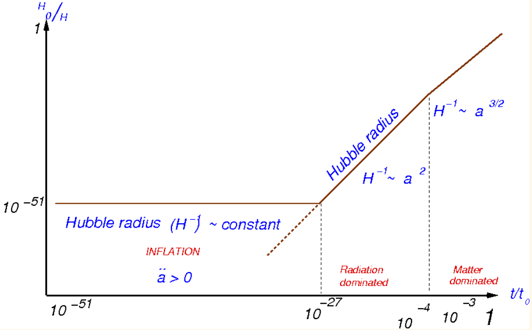


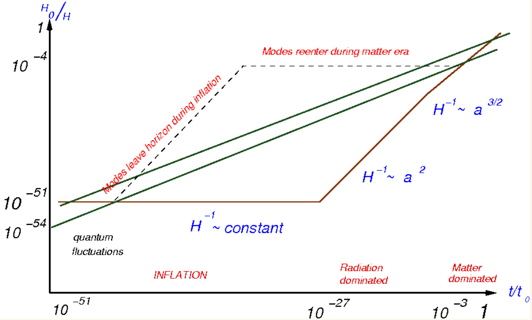
Temperature anisotropies of the CMB [Planck data 2018].

- Primordial inhomogeinities arise from vacuum quantum fluctuations.
- Fluctuations “freeze” after being stretched to cosmological scales during inflation. [Quantum-classical transition]

Quantum-classical transition of fluctuations

- Scale factor of expansion : $a(t)$. Hubble radius : $H^{-1} = a/\dot{a}$. During inflation, $a(t) \propto e^{Ht}$.
- Can entanglement help us understand quantum-classical transition of fluctuations?





Scalar field with time-dependent mass

- The Hamiltonian of a time-dependent massive scalar field (φ) in $(1+1)-$dimensions is:

$$\mathcal{H} = \frac{1}{2}\int dx \left[\pi^2 + (\nabla\varphi)^2 + m_f^2(t)\,\varphi^2\right] \tag{1}$$

- Upon lattice-regularization using $\varphi = jd$ (UV cutoff) and $L = (N+1)d$ (IR cutoff):

$$\mathcal{H} = \frac{1}{2}\sum_j \left[\pi_j^2 + \Lambda(t)\varphi_j^2 + (\varphi_j - \varphi_{j+1})^2\right] \quad ; \quad \Lambda(t) = d^2 m_f^2(t) \tag{2}$$

- Fluctuations propagating in time-dependent backgrounds can be reduced to the following form:

$$\mathcal{H} = \frac{1}{2}\left[\sum_{j=1}^{N} \pi_j^2 + \sum_{i,j=1}^{N} K_{ij}(t)\varphi_i\varphi_j\right] \tag{3}$$

Scalar field with time-dependent mass

- The normal modes of the system are as follows:

$$\omega^2_{k=1,..N}(t) = \begin{cases} \Lambda(t) + 4\sin^2 \frac{k\pi}{2(N+1)} & \text{Dirichlet } \varphi(0) = \varphi(L) = 0 \\ \Lambda(t) + 4\sin^2 \frac{(k-1)\pi}{2N} & \text{Neumann } \partial_x\varphi(0) = \partial_x\varphi(L) = 0 \end{cases} \quad (4)$$

- For each normal mode oscillator $\{y_j\}$, we obtain a form-invariant Gaussian state:

$$\Psi_{\mathrm{GS}}(\{y_j\}, t) = \prod_j \left(\frac{\omega_j(0)}{\pi b_j^2(t)}\right)^{1/4} \exp\left\{-\left(\frac{\omega_j(0)}{b_j^2(t)} - i\frac{\dot{b}_j(t)}{b_j(t)}\right)\frac{y_j^2}{2} - \frac{i}{2}\omega_j(0)\int \frac{dt}{b_j^2(t)}\right\} \quad (5)$$

- The wave-function can also be written as:

$$\Psi_{\mathrm{GS}}(\{y_j\}, t) = \prod_j \psi^{(j)}(y_j, t) \neq \prod_j \phi^{(j)}(x_j, t) \Rightarrow \text{entangled in physical coordinates} \quad (6)$$

Stability of scaling parameters

- The Ermakov-Pinney equation is highly non-linear:

$$\ddot{b}_j(t) + \omega_j^2(t) b_j(t) = \frac{\omega_j^2(0)}{b_j^3(t)} \quad ; \quad j = 1, ..., N \tag{7}$$

- Three distinct stability regimes:

$$b_j(t) \propto \begin{cases} \text{oscillatory} & \omega_j^2(t) > 0 \text{ (stable)} \\ \omega_j(0)t & \omega_j^2(t) = 0 \text{ (metastable/zero-mode)} \\ \exp\{|\nu_j|t\} & \omega_j^2(t) < 0 \text{ (unstable/inverted mode)} \quad ; \quad \nu_j = i\omega_j \end{cases} \tag{8}$$

- During inflation, modes that remain within the Hubble radius are stable/oscillatory and those that cross the radius get squeezed/inverted.

Loss of “quantumness” via Decoherence

- The density matrix can be viewed as an ensemble of states:

$$\hat{\rho} = \begin{cases} \begin{bmatrix} \langle\psi_1|\,\hat{\rho}\,|\psi_1\rangle & \langle\psi_1|\,\hat{\rho}\,|\psi_2\rangle & \cdots \\ \langle\psi_2|\,\hat{\rho}\,|\psi_1\rangle & \langle\psi_2|\,\hat{\rho}\,|\psi_2\rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} & \text{Quantum ensemble} \\ \\ \begin{bmatrix} \langle\psi_1|\,\hat{\rho}\,|\psi_1\rangle & 0 & \cdots \\ 0 & \langle\psi_2|\,\hat{\rho}\,|\psi_2\rangle & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} & \text{Classical ensemble} \end{cases} \tag{9}$$

- **Degree of quantum decoherence** : Purity of a given density matrix

$$\delta_{QD} = \mathrm{Tr}\,\rho^2 \rightarrow \begin{cases} 1 & \text{Pure} \\ 0 & \text{Decohered} \end{cases} \tag{10}$$

Quantifying instabilities

- **Entanglement entropy** : Measures decoherence of A due to B:

$$S = -Tr\,\rho_{red} \ln \rho_{red} \quad ; \quad \rho_{red} = \mathrm{Tr}_A \, |\Psi\rangle\langle\Psi| \; or \; \mathrm{Tr}_B \, |\Psi\rangle\langle\Psi|. \tag{11}$$

- **Loschmidt echo** : Deviation of the state from slightly different $H(t)$ evolutions:

$$\mathscr{M}(t) = \left| \langle\Psi_0| e^{i\int H' dt} e^{-i\int H dt} |\Psi_0\rangle \right| = |\langle\Psi_0|\Psi_2\rangle| \tag{12}$$

- **Quantum Lyapunov exponents**: Exponential decay rate of Loschmidt Echo

$$\lambda_L^{(k)} = v_k \tag{13}$$

General behavior of EE for sub-system size n

$$S_n \sim \underbrace{\left(\sum_{j=1}^{2n} \lambda_L^{(j)} \right) t}_{\text{inverted mode}} + \underbrace{\log(t)}_{\text{zero-mode}} + \underbrace{S_0(t)}_{\text{stable mode}}$$

Classicality of EE, Asymptotic convergence

- **Classicality of EE** : When subsystem size $n \geq \frac{m}{2}$ where $m = dim\{\lambda_L^{(j)}\}$, EE growth saturates:

[L. Hackl et. al. '18]

$$S_n \sim h_{KS}\, t \quad ; \quad h_{KS} = \sum_{j=1}^{m} \lambda_L^{(j)} \tag{14}$$

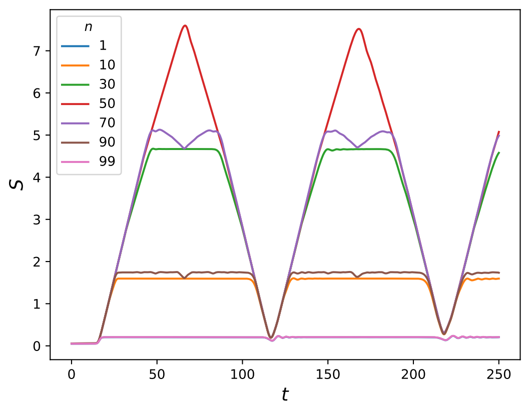
where h_{KS} is the Kolmogorov-Sinai rate, associated with a classically chaotic system.

- Other diagnostic measures : Ground state fidelity $\mathscr{F}_0$, Circuit complexity C_{CM}.

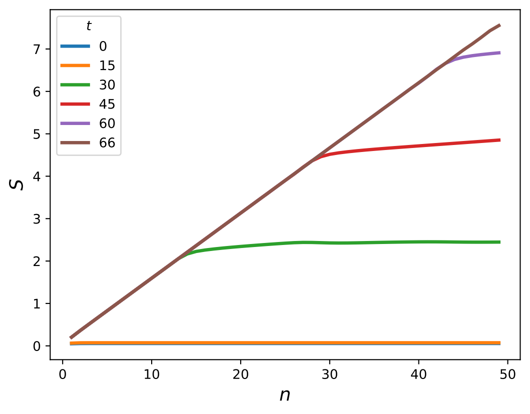
- **Asymptotic convergence** : Subsystem measure EE converges with full-system measures:

$$S^{\rm inv}_{n\geq\frac{m}{2}} \sim -\log \mathscr{F}_0^2 \sim -\log \mathscr{M} \sim C_{\rm CM} \sim h_{\rm KS}\, t \tag{15}$$

Thermality of EE : Subsystem scaling & Particle creation



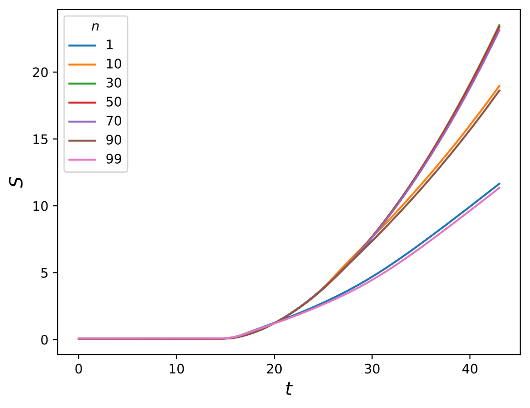
(a)



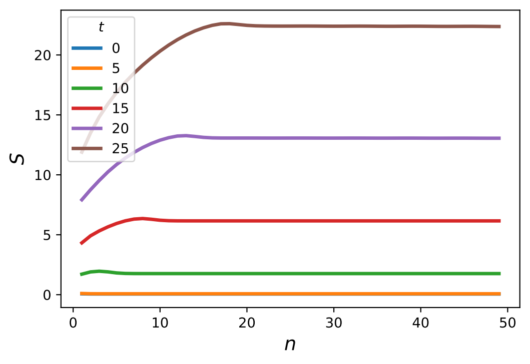
(b)

Thermal signatures of EE for stable modes — (a) Dynamics for various subsystem sizes, and (b) Area-law to volume-law transition.

Subsystem Scaling : Inverted modes



(a)



(b)

Thermal signatures of EE for inverted modes — (a) Dynamics for various subsystem sizes, and (b) Area-law to volume-law transition.

Squeezing and Classicality Criteria

- **Classicality parameter** : Squeezing of the Wigner fn about classical trajectories

$$\mathscr{C} = \frac{\langle xp \rangle_W}{\sqrt{\langle p^2 \rangle_W \langle x^2 \rangle_W}} = \sqrt{1 - \frac{\det \Sigma}{\sigma_{XX}\sigma_{PP}}} \to \begin{cases} 0 & \text{“Quantum”} \\ 1 & \text{“Classical”} \end{cases} \tag{16}$$

[G. Mahajan, T. Padmanabhan '08]

- "Quantum" limit : max. uncertainty $\Longrightarrow$ Wigner function separable in x and p.
- Decoherence and squeezing can be measured from the Covariance matrix:

$$\Sigma = \begin{bmatrix} \sigma_{XX} & \sigma_{XP} \\ \sigma_{XP}^T & \sigma_{PP} \end{bmatrix} ; (\sigma_{XX})_{ij} = \langle \{x_i, x_j\} \rangle ; (\sigma_{XP})_{ij} = \langle \{x_i, p_j\} \rangle ; (\sigma_{PP})_{ij} = \langle \{p_i, p_j\} \rangle \tag{17}$$

Classicality criteria for multi-mode Gaussian states

$$S \to \infty \quad ; \quad LC(t) = -\log \sqrt{1 - \det \mathscr{C}^2} \to \infty$$

Fluctuations in $(3+1)-$dimensions

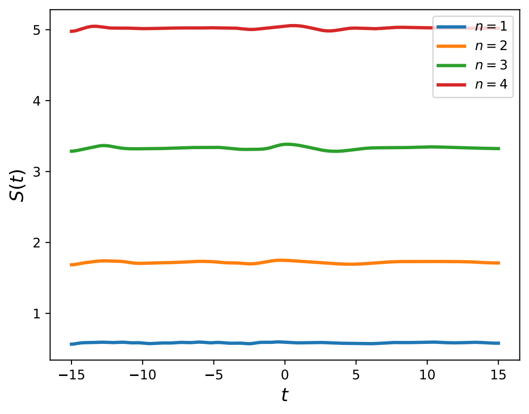
- Lattice regularized fluctuations:

$$\mathscr{H}_{lm}[t] = \frac{1}{2}\sum_{lmj}\left[\Pi_{lmj}^2 + \frac{1}{a^2(t)}\left(j+\frac{1}{2}\right)^2\left\{\frac{\Phi_{lmj}}{j} - \frac{\Phi_{lm,j+1}}{j+1}\right\}^2 + \Omega_{lmj}^2(t)\Phi_{lmj}^2\right] \quad ; \quad t = \frac{\tilde{t}}{\tilde{d}}, \tag{18}$$

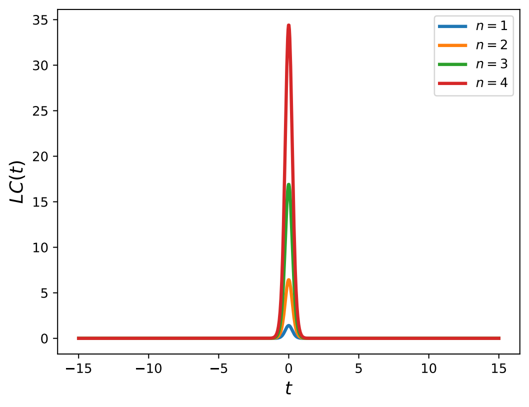
where,

$$\Omega_{lmj}^2(t) = \Lambda + \frac{l(l+1)}{j^2 a^2(t)} - \frac{3}{4}\left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \frac{3\ddot{a}(t)}{2a(t)} \tag{19}$$

Tanh Evolution



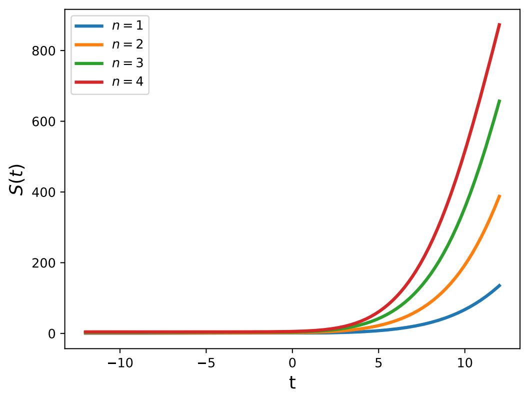
(a) Entanglement Entropy



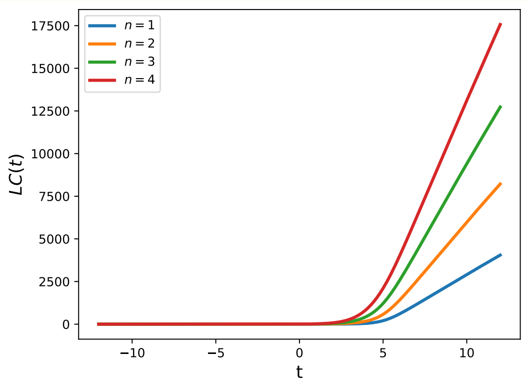
(b) Log Classicality

$S \to \infty$ ✗ ; $LC \to \infty$ ✗

de-Sitter expansion



(a) Entanglement Entropy



(b) Log Classicality

$$S \to \infty \quad ✓ \qquad ; \qquad LC \to \infty \quad ✓$$

Conclusions

- Classicality criteria is decided simultaneously by decoherence (entanglement entropy) and squeezing (log classicality).
- Inverted modes cause EE to classicalize for sufficiently large subsystem size $n \geq m/2$.
- For stable/zero modes, EE-scaling oscillates between area-law and volume-law. For inverted modes, there is a progressive deviation from area-law, asymptotically approaching extensive behaviour.
- In $(3+1)-$dimensions, de-Sitter expansion satisfies the criteria for quantum-classical transition of fluctuations.

Ongoing & Future Work

- Probing area-to-volume law transition in $(3+1)$-dimensions.
- Generalization to higher-spins.
- Black-hole evaporation : Page curve.

Thank You!