Covariance in effective models of quantum black holes

David Brizuela

Department of Physics
University of the Basque Country UPV/EHU, Bilbao (Spain)

In collaboration with A. Alonso-Bardaji and R. Vera

eman ta zabal zazu

Universidad del País Vasco   Euskal Herriko Unibertsitatea

Nijmegen, July 2023

Motivation

- Loop quantum gravity is formulated in terms of triads and connections.
- The operator associated to the connection A_a^j is not well defined, one quantizes the holonomies $(\exp[i(\int \sigma_j \dot{\gamma}^a A_a^j)])$.
- Effective theories are supposed to encode the main quantum effects. Polymerization: $A \rightarrow \frac{\sin(\lambda A)}{\lambda}$
- Singularity resolution in homogeneous models.

Objetives

- Construct an effective theory of a spherical quantum black hole in the context of loop quantum gravity.
- Modify the General-Relativity (GR) Hamiltonian constraint, so that the deformed Hamiltonian covariantly defines a spacetime metric.
- Analyze the singularity resolution for black holes with Q and Λ.

Polymerization of homogeneous cosmological models

Let us assume a homogeneous and isotropic cosmology with a scalar matter field ϕ. Two couple of conjugate variables:

$$\{b, v\} = 1, \qquad \{\phi, p_\phi\} = 1.$$

- **Classical dynamics**

  The GR Hamiltonian constraint: $\mathcal{H}_{GR} = -vb^2 + \frac{p_\phi^2}{v} = 0$

  The energy density: $\rho \equiv \frac{p_\phi^2}{v^2} = b^2$

  The Hubble rate: $\left(\frac{\dot{a}}{a}\right)^2 \propto \left(\frac{\dot{v}}{v}\right)^2 = \rho \Longrightarrow$ Singularity

- **Polymerized effective theory**

  The polymerized Hamiltonian constraint: $\mathcal{H} = -v\frac{\sin^2(\lambda b)}{\lambda^2} + \frac{p_\phi^2}{v} = 0$

  The energy density: $\rho \equiv \frac{p_\phi^2}{v^2} = \frac{\sin^2(\lambda b)}{\lambda^2}$

  The Hubble rate: $\left(\frac{\dot{a}}{a}\right)^2 \propto \left(\frac{\dot{v}}{v}\right)^2 = \rho\left(1 - \frac{\rho}{\rho_{\max}}\right) \Longrightarrow$ Bounce

Spherical vacuum in General Relativity

Two conjugate couples: $\{E^x(x_1), K_x(x_2)\} = \{E^\varphi(x_1), K_\varphi(x_2)\} = \delta(x_1, x_2)$.

The total Hamiltonian $H_T = H[N] + D[N^x]$ is a sum of constraints,

$$\mathcal{H} = \frac{E^\varphi}{2\sqrt{E^x}}(1 + K_\varphi^2) - 2\sqrt{E^x} K_x K_\varphi + \frac{(E^{x\prime})^2}{8\sqrt{E^x}E^\varphi} - \frac{\sqrt{E^x}}{2(E^\varphi)^2} E^{x\prime} E^{\varphi\prime} + \frac{\sqrt{E^x}}{2E^\varphi} E^{x\prime\prime},$$
$$\mathcal{D} = -E^{x\prime} K_x + E^\varphi K_\varphi'.$$

The hypersurface deformation algebra:

$$\{D[f_1], D[f_2]\} = D\left[f_1 f_2' - f_1' f_2\right],$$
$$\{D[f_1], H[f_2]\} = H\left[f_1 f_2'\right],$$
$$\{H[f_1], H[f_2]\} = D\left[\frac{1}{q_{xx}}(f_1 f_2' - f_1' f_2)\right].$$

The metric:

$$ds^2 = -N^2 dt^2 + q_{xx}(dx + N^x dt)^2 + r^2 d\Omega^2,$$

with $q_{xx} = (E^\varphi)^2/E^x$ and $r = \sqrt{E^x}$.

Polymerized vacuum model with a closed algebra

"Carefully" polymerizing the Hamiltonian

$$\mathcal{H} = \frac{E^\varphi}{2\sqrt{E^x}}\left(1+\frac{\sin^2(\lambda K_\varphi)}{\lambda^2}\right) - \sqrt{E^x}K_x\frac{\sin(2\lambda K_\varphi)}{\lambda} + \frac{(E^{x\prime})^2}{8\sqrt{E^x}E^\varphi} - \frac{\sqrt{E^x}}{2(E^\varphi)^2}E^{x\prime}E^{\varphi\prime} + \frac{\sqrt{E^x}}{2E^\varphi}E^{x\prime\prime},$$

one obtains the closed (anomaly-free) algebra,

$$\begin{aligned}
\{D[f_1], D[f_2]\} &= D\left[f_1 f_2' - f_1' f_2\right],\\
\{D[f_1], H[f_2]\} &= H\left[f_1 f_2'\right],\\
\{H[f_1], H[f_2]\} &= D\left[F(f_1 f_2' - f_1' f_2)\right],
\end{aligned}$$

with $F = E^x \cos(2\lambda K_\varphi)/(E^\varphi)^2$. However,

- $1/F$ does not have the correct transformation properties to be interpreted as q_{xx}.
- No known way to couple matter while keeping a closed algebra.

Covariance

- In phase space, with Hamiltonian $H_T = H[N] + D[N^x]$, the first-class constraints

$$\begin{aligned} \{D[f_1], D[f_2]\} &= D\left[f_1 f_2' - f_1' f_2\right], \\ \{D[f_1], H[f_2]\} &= H\left[f_1 f_2'\right], \\ \{H[f_1], H[f_2]\} &= D\left[F(f_1 f_2' - f_1' f_2)\right], \end{aligned}$$

  are generators of gauge transformations $\delta_\epsilon \Phi = \{\Phi, H[\epsilon] + D[\epsilon^x]\}$.
- Under a coordinate transformation in spacetime, the metric g_{ab} changes as $\mathcal{L}_\xi g_{ab}$, with $\xi^\mu \partial_\mu = \xi^t \partial_t + \xi^x \partial_x$.
- Both transformations must coincide if the gauge parameters are the components of ξ^μ in the normal-tangential basis: $\xi^\mu = \epsilon \partial_n + \epsilon^x \partial_x$.
- In summary, one can covariantly define the metric

$$ds^2 = -N^2 dt^2 + q_{xx}(dx + N^x dt)^2 + r^2 d\Omega^2,$$

  with $q_{xx} := 1/F$, as long as $\delta_\epsilon(1/F) = \mathcal{L}_\xi q_{xx}$ and r is a scalar.

Set up to construct a deformed covariant Hamiltonian

The classical Hamiltonian:

$$\mathcal{H}_{GR} = -\frac{E^\varphi}{2\sqrt{E^x}}(1+K_\varphi^2) - 2\sqrt{E^x}K_x K_\varphi + \frac{(E^{x\prime})^2}{8\sqrt{E^x}E^\varphi} - \frac{\sqrt{E^x}}{2(E^\varphi)^2}E^{x\prime}E^{\varphi\prime} + \frac{\sqrt{E^x}}{2E^\varphi}E^{x\prime\prime}$$

Ansatz: the most general Hamiltonian constraint quadratic in derivatives of E^x and E^φ:

$$\mathcal{H} = a_0 + a_{xx}(E^{x\prime})^2 + a_{\varphi\varphi}(E^{\varphi\prime})^2 + a_{x\varphi}E^{x\prime}E^{\varphi\prime} + a_1 E^{x\prime\prime} + a_2 E^{\varphi\prime\prime},$$

with all a_I and a_{ij} free functions of $(E^x, K_x, E^\varphi, K_\varphi)$

Requirements:

- Anomaly freedom: $\mathcal{H}$ forms a closed algebra with the diff constraint $\mathcal{D}$.
- Spacetime embeddability: $\delta_\epsilon(1/F) = \mathcal{L}_\xi q_{xx}$.
- The classical Hamiltonian $\mathcal{H}_{GR}$ is recovered in a continuous limit.

The deformed covariant Hamiltonian constraint

$$\mathcal{H} = -\mathfrak{g}\Bigg(\frac{E^\varphi}{2E^x}\left(1 - E^x V + A\frac{\sin^2(\omega K_\varphi)}{\omega^2}\right) + K_x\left(A\frac{\sin(2\omega K_\varphi)}{\omega} - \left(\frac{E^{x\prime}}{2E^\varphi}\right)^2 \omega\sin\left(2(\omega K_\varphi + \phi)\right)\right) + E^\varphi\frac{\partial}{\partial E^x}\left[A\frac{\sin^2(\omega K_\varphi)}{\omega^2} - \left(\frac{E^{x\prime}}{2E^\varphi}\right)^2\cos^2(\omega K_\varphi + \phi)\right] - \frac{1}{2}\left(\frac{E^{x\prime\prime}}{E^\varphi} - \frac{E^{x\prime}E^{\varphi\prime}}{E^{\varphi 2}} + \frac{(E^{x\prime})^2}{4E^x E^\varphi}\right)\cos^2(\omega K_\varphi + \phi)\Bigg),$$

with $\mathfrak{g}$, A, ω, ϕ, and V free functions of the scalar E^x only.

Vacuum GR corresponds to $\omega \to 0$, $\phi \to 0$, $V \to 0$, $A \to 1$, and $\mathfrak{g} \to \sqrt{E^x}$.

Defomed covariant Hamiltonian constraint: properties

- The complexity of the initial ansatz is radically reduced by the covariance requirement.
- Trigonometric functions have not been chosen by hand, rather a consequence of covariance.
- Using the equations of motion, one can show that the function $m := \sqrt{E^x}(1 + A\frac{\sin^2(\omega K_\varphi)}{\omega^2} - \left(\frac{E^{x\prime}}{2E^\varphi}\right)^2 \cos^2(\omega K_\varphi + \phi))$ is given on-shell by

$$m \approx M + \int V(E^x) dE^x.$$

- In particular, m is a constant of motion if $V = 0$.
- The potential V can reproduce a cosmological-constant and charge.
- The associated metric of the deformed theory is given by,

$$ds^2 = -N^2 dt^2 + \frac{1}{F}(dx + N^x dt)^2 + r^2 d\Omega^2,$$

with $F = \frac{\mathfrak{g}^2}{E^\varphi}\left(A\cos^2(\phi) + \omega^2\left(1 - \frac{2m}{\sqrt{|E^x|}}\right)\right)$ and $r = r(E^x)$.

Defomed Hamiltonian constraint: spacetime structure

- There exists a Killing vector field $\xi = \xi_\mu dx^\mu$, with $\mu = 0, 1$.
- ξ_μ is everywhere orthogonal to $\nabla_\mu r$, that is, $\xi^\mu \nabla_u r = 0$
- $G := \xi^\mu \xi_\mu$, $H := \nabla_\mu r \nabla^\mu r$. Wherever $\nabla_\mu r \neq 0$, then $\mathrm{sign}(G) = -\mathrm{sign}(H)$ as long as .
  Four different regions of the spacetime:
  - $G < 0$ and $H > 0$: static nontrapped regions with $\nabla_\mu r$ spacelike and ξ_μ timelike.
  - $G > 0$ and $H < 0$: trapped homogeneous regions with $\nabla_\mu r$ timelike and ξ_μ spacelike.
  - $G = 0$ and $H = 0$ (with $\nabla_\mu r \neq 0$): Killing horizons, which separate trapped and nontrapped regions, where both $\nabla_\mu r$ and ξ_μ are lightlike.
  - $\nabla_\mu r = 0$: critical points. (For $\phi = 0$, $\nabla_\mu r = 0 \iff F = 0$).

Other specific properties of the spacetime under consideration depend on the chosen free functions.

Particular case

- Let us consider a constant value for $\omega = \lambda$ and the GR values for $\phi = 0$, $A = 1$, and $\mathfrak{g} = \sqrt{E^x}$.

$$\mathcal{H} = -\frac{E^\varphi}{2\sqrt{E^x}}\left(1 + \frac{\sin^2(\lambda K_\varphi)}{\lambda^2}\right) - \sqrt{E^x} K_x \frac{\sin(2\lambda K_\varphi)}{\lambda}\left(1 + \left(\frac{\lambda E^{x\prime}}{2E^\varphi}\right)^2\right)$$
$$+ \frac{\cos^2(\lambda K_\varphi)}{2}\left(\frac{E^{x\prime}}{2E^\varphi}\left(\sqrt{E^x}\right)' + \sqrt{E^x}\left(\frac{E^{x\prime}}{E^\varphi}\right)'\right) + \frac{1}{2}\sqrt{E^x} E^\varphi V(E^x)$$

- The potential $V(E^x)$ will be chosen below to describe Λ and Q.
- This Hamiltonian can be obtained from a canonical transformation plus a linear combination of the GR constraints:

$$E^x_{(GR)} = E^x\,, \quad K_x^{(GR)} = K_x\,, \quad E^\varphi_{(GR)} = \frac{E^\varphi}{\cos(\lambda K_\varphi)}\,, \quad K_\varphi^{(GR)} = \frac{\sin(\lambda K_\varphi)}{\lambda}\,.$$

$$\left(\mathcal{H}_{GR} + \lambda \sin(\lambda K_\varphi)\frac{\sqrt{E^x} E^{x\prime}}{2{E^\varphi}^2}\mathcal{D}\right)\cos(\lambda K_\varphi) = \mathcal{H}$$

- This motivates the choice $V(E^x) = \left(\Lambda + \left(\frac{Q}{E^x}\right)^2\right)$ and $r = \sqrt{E^x}$.

Particular case: the structure and the mass functions

- The structure function in $\{H[f_1], H[f_2]\} = D[F(f_1 f_2' - f_1' f_2)]$ reads

$$F = \frac{\cos^2(\lambda K_\varphi)}{1+\lambda^2}\left(1 + \left(\frac{\lambda E^{x\prime}}{2E^\varphi}\right)^2\right)\frac{E^x}{(E^\varphi)^2} \geq 0$$

- In terms of the mass function $m(r) = M - \frac{Q^2}{2r} + \frac{\Lambda}{6} r^3$,

$$F = \left(1 - \frac{2\bar{\lambda} m(r)}{r}\right)\frac{r^2}{(E^\varphi)^2} \geq 0 \Longrightarrow 2\bar{\lambda} m(r) \leq r,$$

  with $\bar{\lambda} := \frac{\lambda^2}{1+\lambda^2} \in [0,1]$.
- For vacuum (with $Q = 0 = \Lambda$), $m = M$ constant and $r_0 := 2\bar{\lambda} M \leq r$ is a minimum for r. If $M > 0$ this will lead to the singularity resolution.
- For nonvacuum $2\bar{\lambda} m \leq r$ applies, but the possible ranges of definition of r depends on the specific values of the parameters (M, Q, Λ).

Particular case: metric and curvature

- By solving the equations of motion

$$\dot{E}^x = \{E^x, \mathcal{H}\}, \quad \dot{K}_x = \{K_x, \mathcal{H}\}, \quad \dot{E}^\varphi = \{E^\varphi, \mathcal{H}\}, \quad \dot{K}_\varphi = \{K_\varphi, \mathcal{H}\}$$

in certain gauge, we obtain the metric in diagonal form

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right) dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dx^2 + r^2 d\Omega^2,$$

valid for $r \neq 2m(r) =$ and with $r = r(x)$ defined by

$$\left(\frac{dr(x)}{dx}\right)^2 = 1 - \frac{2\lambda m(r(x))}{r(x)}$$

- Here appears again the condition $2\lambda m(r) \leq r$.
- The norm of the Killing $\xi^\mu \xi_\mu = (1 - 2m(r)/r) \Longrightarrow$ same horizon structure as the corresponding GR solution.

Particular case: covering domain $\mathcal{U}$

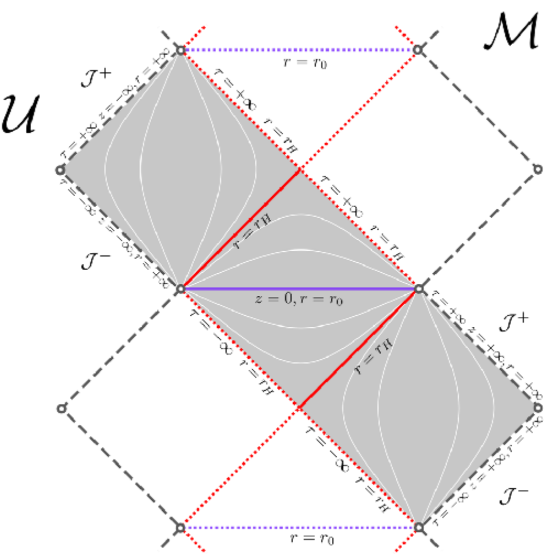
- Another gauge choice leads to

$$ds^2 = -\left(1 - \frac{2m\big(r(z)\big)}{r(z)}\right)d\tau^2 + 2\sqrt{\frac{2m\big(r(z)\big)}{r(z)}}d\tau dz + dz^2 + r(z)^2 d\Omega^2,$$

  where (t, x) have been renamed as (τ, z).
- (τ, z) are horizon-crossing coordinates and their domain of definition is named $\mathcal{U}$ (covering domain).
- $\tau \in (-\infty, \infty)$ and z restricted by $m(r(z)) \geq 0$.
- By transforming to null coordinates in different regions, and extending then the domains, one can construct the conformal diagram and obtain the maximal analytic extension of the spacetime $\mathcal{M}$.

arXiv:2205.02098[gr-qc]

Conformal diagram for vacuum ($Q = 0 = \Lambda$) with $M > 0$



- Same horizon as in GR: $r = r_H \equiv 2M$.
- The critical surface $r = r_0 \equiv 2\lambda M$ replaces the classical singularity and separates a trapped and antitrapped region.
- A perfectly regular and geodesically-complete spacetime.
- However, for $M < 0$ the singularity is not resolved and the conformal diagram coincides with the classical one.

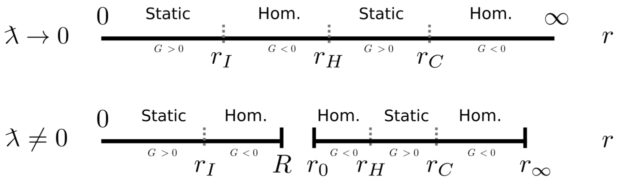
Nonvacuum cases ($Q \neq 0$ or $\Lambda \neq 0$)

- There are a lot of possible different cases depending on the specific values of M, Q, Λ,
- Not all the singularities are resolved.
- The Ricci scalar diverges at $r = 0$ (except for $M = Q = \Lambda = 0$) and at $r \to \infty$ (for $\Lambda \neq 0$):

$$\mathcal{R} = 4\Lambda\left(1 + \frac{\lambda}{2}\right) + 2\lambda\left(\frac{3M^2}{r^4} + \frac{Q^2}{r^4}\left(1 - \frac{4M}{r} + \frac{Q^2}{r^2}\right) - \Lambda\left(\frac{4M}{r} + \Lambda r^2\right) + \frac{4\Lambda Q^2}{3r^2}\right)$$

Nonvacuum cases ($Q \neq 0$ or $\Lambda \neq 0$)

- The range for r is defined by $r \geq 2\lambda\!\!\!^{-} m = 2\lambda\!\!\!^{-}\left(M - \frac{Q^2}{2r} + \frac{\Lambda}{6} r^3\right)$.
- The saturation of the above condition $r_{\rm crit} = 2\lambda\!\!\!^{-}\left(M - \frac{Q^2}{2r_{\rm crit}} + \frac{\Lambda}{6} r_{\rm crit}^3\right)$ is equivalent to a fourth-order polynomial equation and it defines at most three possible positive critical values $r_{\rm crit} = R, r_0, r_\infty$.
- The location of the horizons is defined by $G \equiv \xi^\mu \xi_\mu = 0 \iff r = 2m$. There are at most three: $r_{\rm hor} = r_I, r_H, r_C$.
- Schematically:



Nonvacuum cases ($Q \neq 0$ or $\Lambda \neq 0$)

We have classified all the possible singularity-free solutions of the theory:

- There exists a minimum value of $r = r_0$, so that $r = 0$ is not contained in the domain.
- If $\Lambda \neq 0$, there exists a maximum value of $r = r_\infty$, so that $r \to \infty$ is not contained in the domain.

In a nutshell, the spacetime to be singularity-free it must have

- $M \geq 0$,
- $\Lambda \geq 0$, and
- Q is bounded.

arXiv:2302.10619 [gr-qc]

Nonvacuum cases ($Q \neq 0$ or $\Lambda \neq 0$)

More specifically, one gets the singularity-free

- Reissner-Nordström-de Sitter
  $C_1 := \left\{\Lambda > 0,\ Q \neq 0,\ M > 0,\ 8Q^2 < 9\lambda M^2, \text{ and } \Lambda \in (\Lambda_-, \Lambda_+) \cap (0, \Lambda_+)\right\}$,
- Schwarzschild-de Sitter,
  $C_2 := \left\{\Lambda > 0,\ Q = 0, \text{ and } \Lambda \in \left(0, \frac{1}{9\lambda^3 M^2}\right)\right\}$,
- Reissner-Nordström,
  $C_3 := \left\{\Lambda = 0 \text{ and } |Q| < \sqrt{\lambda} M\right\}$,

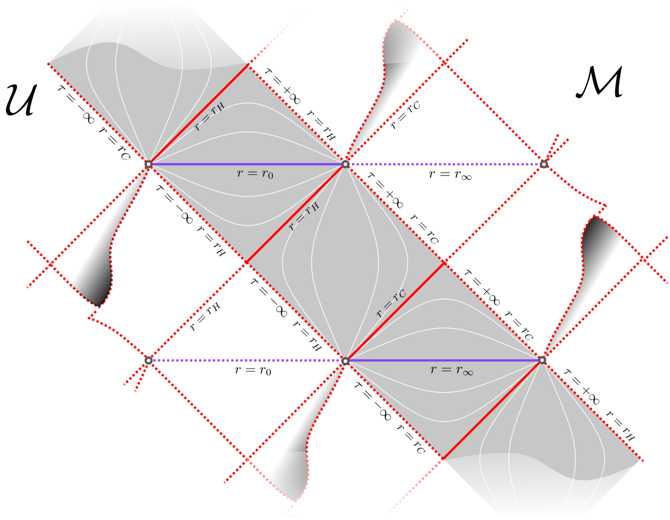
with $\Lambda_\pm := \frac{3}{32\lambda^4 Q^6}\left[36\lambda^3 M^2 Q^2 - 27\lambda^4 M^4 - 8\lambda^2 Q^4 \pm \sqrt{\lambda^5 M^2 \left(9\lambda M^2 - 8Q^2\right)^3}\right]$.
There are also the *degenerate* cases D_1 and D_3.

Nonvacuum cases ($Q \neq 0$ or $\Lambda \neq 0$)

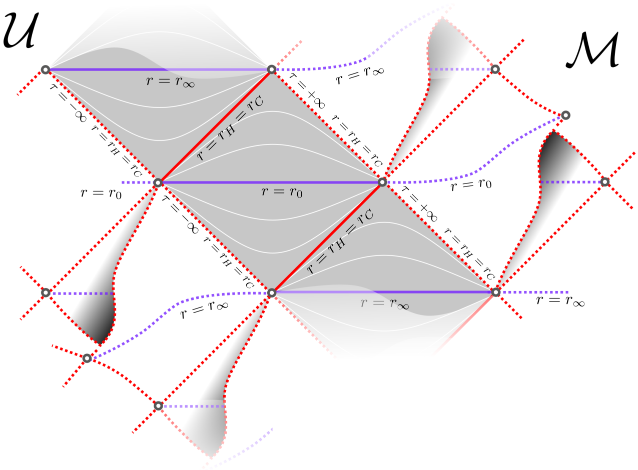
General properties of the conformal diagrams:

- Critical points r_0 and r_∞ always appear in homogeneous regions.
- Cases with $Q = 0$ and $Q \neq 0$ have the same conformal diagram. In particular no Cauchy horizon.
- r_0 replaces the classical singularity at $r = 0$ and, if exist, r_∞ replaces the $r \to \infty$ surface.
- $r = r_0$ is a spacelike (reachable) surfaces in "regular" cases C_1, C_2, and C_3; while it defines $\mathcal{J}^\pm$ in the degenerate cases D_1 and D_3.
- Cases C_1, C_2, and D_1 are subdivided in
  - Black-hole solution (if they present 2 horizons).
  - Extremal solution (if they present 1 degenerate horizon).
  - Cosmological solution (if they present no horizons).

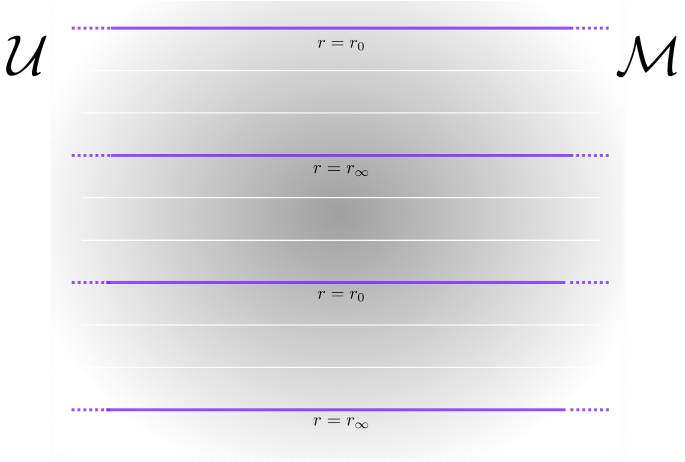
Conformal diagram for the black holes C_1 and C_2 ($\Lambda > 0$)



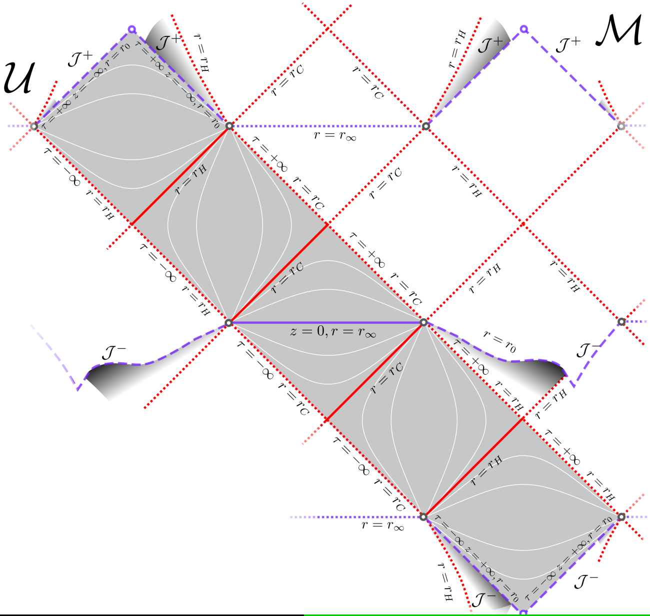
Conformal diagram for extremal C_1 and C_2 ($\Lambda > 0$)



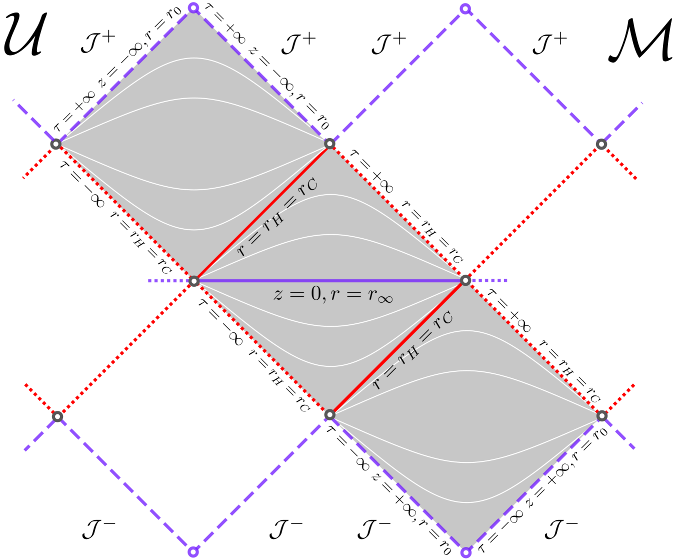
Conformal diagram for the cosmology C_1 and C_2 $(\Lambda > 0)$



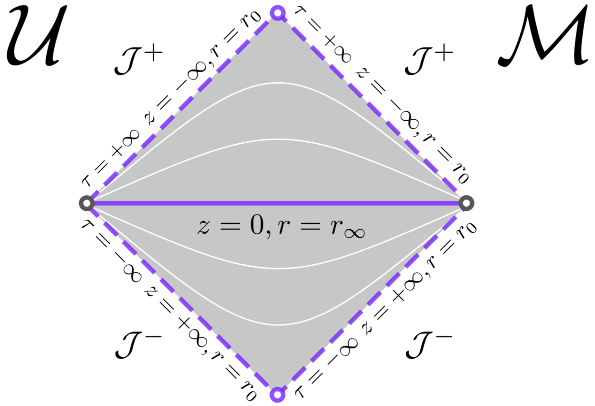
Conformal diagram for the black hole D_1 $(\Lambda > 0)$



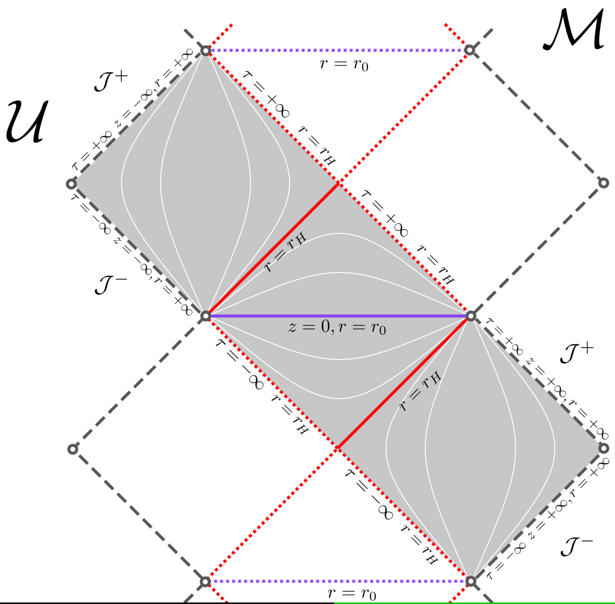
Conformal diagram for the extremal D_1 $(\Lambda > 0)$



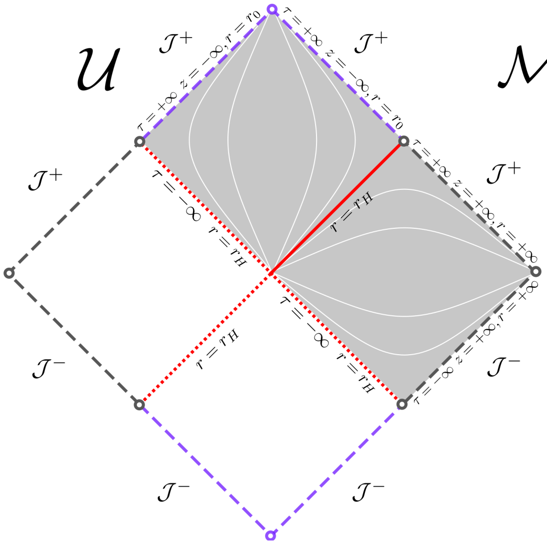
Conformal diagram for the cosmology D_1 $(\Lambda > 0)$



Conformal diagram for the cosmology C_3 $(\Lambda = 0)$



Conformal diagram for the cosmology D_3 $(\Lambda = 0)$



Conclusions

- The most general Hamiltonian constraint under the restrictions
  - Same derivative structure as GR.
  - Anomaly-free algebra.
  - Spacetime embedability.
  - Contains GR as a continuous limit.
- Analyzed in detail a ("minimally deformed") particular case.
  - Singularity resolution is not generic.
  - Singularity-free spacetimes
    - $M \geq 0$, $\Lambda \geq 0$, and bounded Q.
    - $r = r_0$ replaces the classical singularity at $r = 0$.
    - If $\Lambda \neq 0$, $r = r_\infty$ replaces the $r \to \infty$.