Partition function for a volume of space

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Horizon entropy as a probe of QG

- Bekenstein-Hawking entropy provides a low-energy window into quantum gravity

$$S = k_B \frac{Ac^3}{4\hbar G}$$

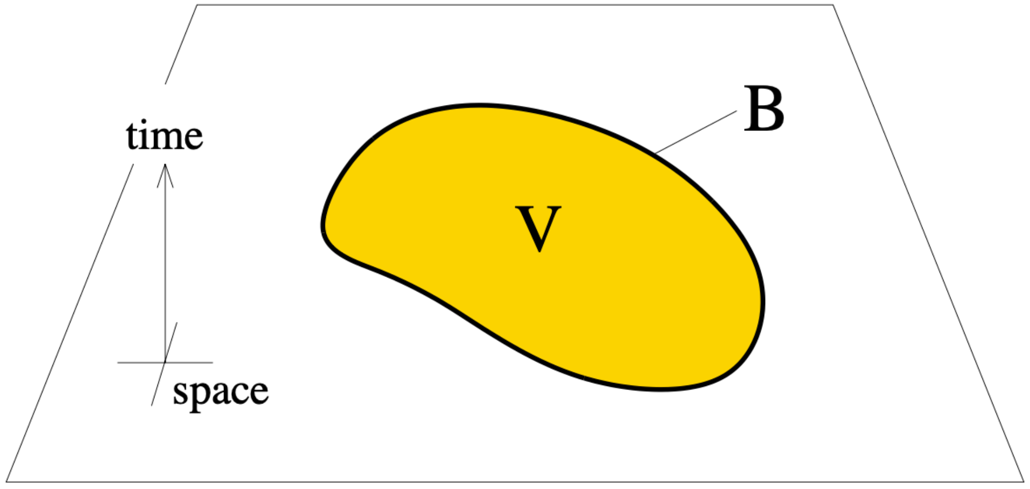
A = horizon area

- BH entropy is a universal formula: applies not only to black hole horizon, but also to cosmological and acceleration horizons.

“Entropy = area” for any volume of space

- It is expected that gravitational entropy is not only associated to the area of black hole or de Sitter horizon, but to the area of any boundary separating a region of space.

Banks-Fischler, Bousso, Bianchi-Myers, Jacobson-Parentani, …



How to justify this?

Gibbons-Hawking partition function

- Gibbons and Hawking (1977) derived the entropy of black hole and de Sitter horizons from a Euclidean saddle approximation of the gravitational partition function.

Can the entropy of a volume of space be derived from a saddle approximation to a partition function?

See also Banks-Draper-Farkas ’20
and our recent statistical interpretation (Jacobson-MV ’22)

Gibbons-Hawking partition function

- Gibbons and Hawking (1977) derived the entropy of black hole and de Sitter horizons from a Euclidean saddle approximation of the gravitational partition function.

Can the entropy of a volume of space be derived from a saddle approximation to a partition function?

Yes! Using the method of constrained instantons

Gibbons-Hawking partition function

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Action integrals and partition functions in quantum gravity

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(Received 4 October 1976)

One can evaluate the action for a gravitational field on a section of the complexified spacetime which avoids the singularities. In this manner we obtain finite, purely imaginary values for the actions of the Kerr-Newman solutions and de Sitter space. One interpretation of these values is that they give the probabilities for finding such metrics in the vacuum state. Another interpretation is that they give the contribution of that metric to the partition function for a grand canonical ensemble at a certain temperature, angular momentum, and charge. We use this approach to evaluate the entropy of these metrics and find that it is always equal to one quarter the area of the event horizon in fundamental units. This agrees with previous derivations by completely different methods. In the case of a stationary system such as a star with no event horizon, the gravitational field has no entropy.

GH represented the canonical partition function in gravity as a Euclidean path integral over metrics

$$Z = \mathrm{Tr}\, e^{-\beta H} \quad \longleftrightarrow \quad Z = \int \mathcal{D}g\, e^{-I_E[g]/\hbar}$$

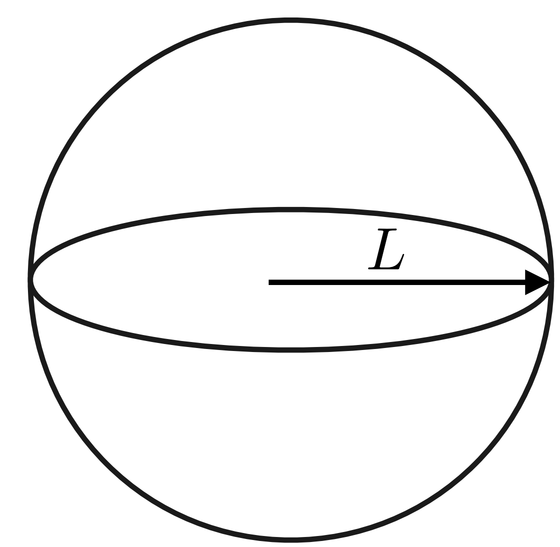
Entropy from the partition function

- In a saddle approximation the path integral can be estimated as:

$$Z \sim \exp\left(-I_E^{\text{saddle}}/\hbar\right)$$

- If the saddle geometry is Euclidean de Sitter space (a round sphere with radius equal to the dS curvature scale *L*), then

$$I_E^{\text{saddle}}/\hbar = -\frac{A(L)}{4\hbar G} = -S_{dS}$$



S^D

Gibbons-Hawking partition function

- Since the energy vanishes for Euclidean de Sitter space, the partition function counts the dimension of the quantum gravity Hilbert space

Banks, Fischler

$$Z = \mathrm{Tr}\, e^{-\beta H} \qquad \& \qquad H = 0$$

Jacobson, Banihashemi '22

$\longrightarrow$

$$Z \to \mathrm{Tr}_{\mathcal{H}}\, 1 = e^{S_{dS}}$$

= dimension of Hilbert space
of states surrounded by a horizon,
i.e. states of a ball

Partition function for a volume of space

- Should not "area = entropy" apply to *any* volume of space (topological ball)?
- To specify a region of space, one must somehow fix its size.
- We fix the spatial volume, by adding a constraint in the path integral

$$Z[V] = \mathrm{Tr}_{\mathcal{H}} 1 \quad \longleftrightarrow \quad Z[V] = \int Dg \, \delta(\mathcal{C}[g] - V) \, e^{-\frac{1}{\hbar} I_E[g]}$$

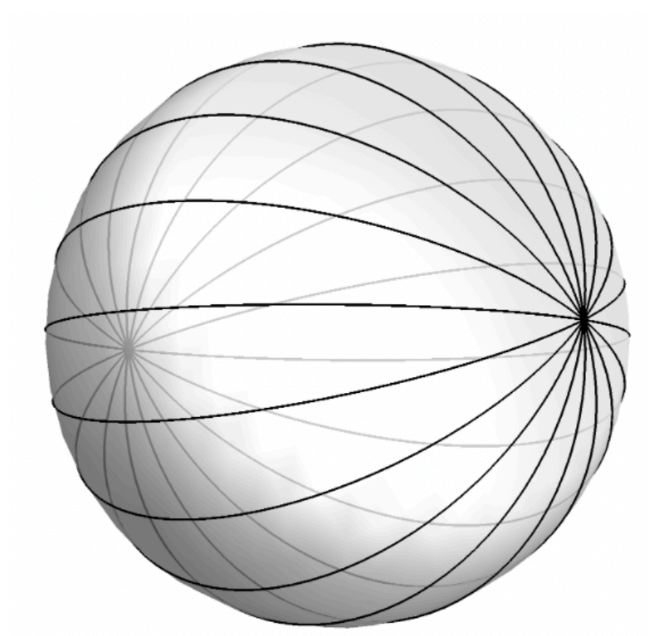
- Introduce a Lagrange multiplier to impose the constraint

$$Z[V] = \int \mathcal{D}\lambda \, \mathcal{D}g \, e^{-\frac{1}{\hbar} I_E[g] + \frac{1}{\hbar} \lambda(\mathcal{C}[g] - V)}$$

Euclidean sphere geometry

- What are the topologies that we integrate over in the path integral?
- Consider a spatial topological *(D-1)*-ball whose boundary has topology S^{D-2}.
- The Euclidean manifold generated by rotating the ball through a complete circle about the ball boundary is a topological *D*-sphere

  e.g. *D=2* version:



S^D

Constrained sphere partition function

Method of constrained instantons Affleck, Cotler-Jensen

$$Z[V,\Lambda] = \int \mathcal{D}\lambda\, \mathcal{D}g\, \exp\left[\frac{1}{16\pi\hbar G}\int d^D x\sqrt{g}(R-2\Lambda) + \frac{1}{\hbar}\int d\phi\, \lambda(\phi)\left(\int d^{D-1}x\sqrt{\gamma} - V\right)\right]$$

- Foliate S^D by *(D-1)*-balls at constant ϕ with induced metric $\gamma_{ab} = g_{ab} - N^2 \phi_{,a}\phi_{,b}$

$$N \equiv (g^{ab}\phi_{,a}\phi_{,b})^{-1/2}$$

- The saddle point equations are the Einstein equations sourced by an effective perfect fluid with vanishing energy density,

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \quad \text{with} \quad T_{ab} = \frac{\lambda}{N}\gamma_{ab} \equiv P\gamma_{ab}$$

Static, spherically symmetric saddle

- $\Lambda = 0$ static, spherically symmetric solution:

$$ds^2 = \frac{1}{4R_V^2}(R_V^2 - r^2)^2 d\phi^2 + dr^2 + r^2 d\Omega_{D-2}^2$$

horizon radius $R_V = [(D-1)V/\Omega_{D-2}]^{1/(D-1)}$

- Euclidean constrained instanton has topology S^D, is conformally flat, and has a curvature singularity at the horizon.

Euclidean action

- Even though the saddle has a 1/(r-R_V) curvature singularity at the horizon, the action is finite.

- For $\Lambda = 0$ the on-shell Euclidean action is

$$I_{\text{saddle}} = -\frac{1}{16\pi G}\int d^D x\sqrt{g}R = -\frac{A_V}{4G}$$

- Hence, in the zero-loop saddle-point approximation:

$$\boxed{Z[V] \approx \exp(A_V/4\hbar G)}$$

This generalizes the Gibbons-Hawking partition function to a finite volume of space!

Conclusions

- *Partition function of a volume of space* = dimension of the quantum gravity Hilbert space of a topological ball with fixed proper volume.

- The Hilbert space dimension matches with the semiclassical *BH entropy* attributed to spatial regions

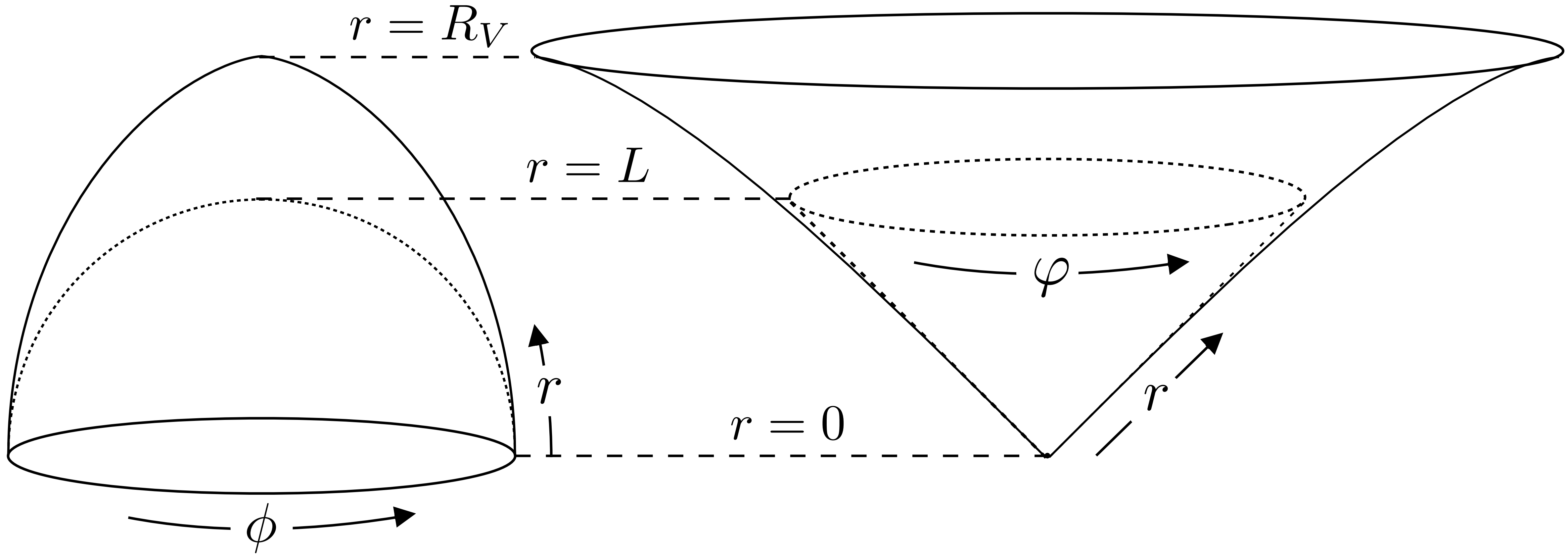
$$Z[V] = \dim \mathcal{H} \approx \exp(A_V / 4\hbar G)$$

Future direction:

- Determine whether *higher curvature corrections* regularise the curvature singularity at the horizon.

Euclidean saddle

- Comparison between diamond saddle and spherical dS saddle



$\Lambda > 0$ saddle

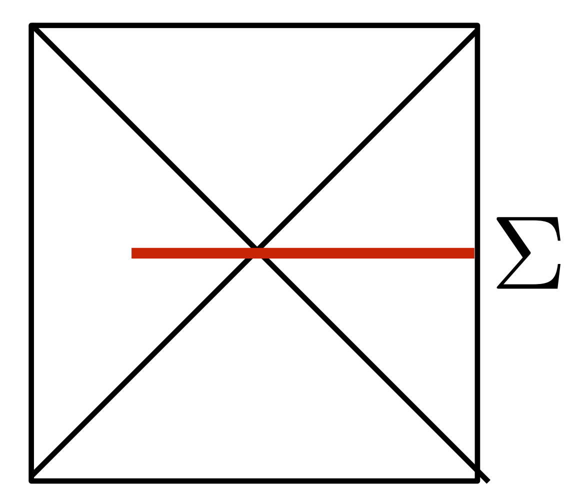
- The saddle solution with positive cosmological constant is similar, BUT:

  1) If $V = V_{dS}$ (static patch spatial volume), then the saddle is dS, which is smooth

  2) If V is larger than the dS spatial hemisphere, the entropy *decreases* as volume increases

  2) There is no saddle if V is larger than the full de Sitter spatial sphere.

  3) The integral over all V is dominated by the de Sitter saddle.



Regulation of the saddle singularity in EFT?

- Suppose the singularity is regulated by higher derivative terms in the action, governed by a UV length ℓ:

$$I = -\frac{1}{16\pi G}\int d^D x\sqrt{g}(R + \ell^2 R^2 + \cdots)$$

$$\rho \equiv r - R_V$$

- The field R^2 term contributes to the field equation $\sim \ell^2 \partial_r^2 R \sim \frac{\ell^2}{\rho^2} R$

  which is of the same order as the Einstein term when $\rho \sim \ell$, at which point $R \sim \frac{1}{R_V \ell}$

  If the curvature saturates at this value, then EFT remains effective, and the higher curvature

  corrections to the entropy are of order $\ell^2 R \sim \frac{\ell}{R_V} \ll 1$ relative to the BH entropy.

  We conjecture that this is what happens . . .