The Noise of Gravitons

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Summary

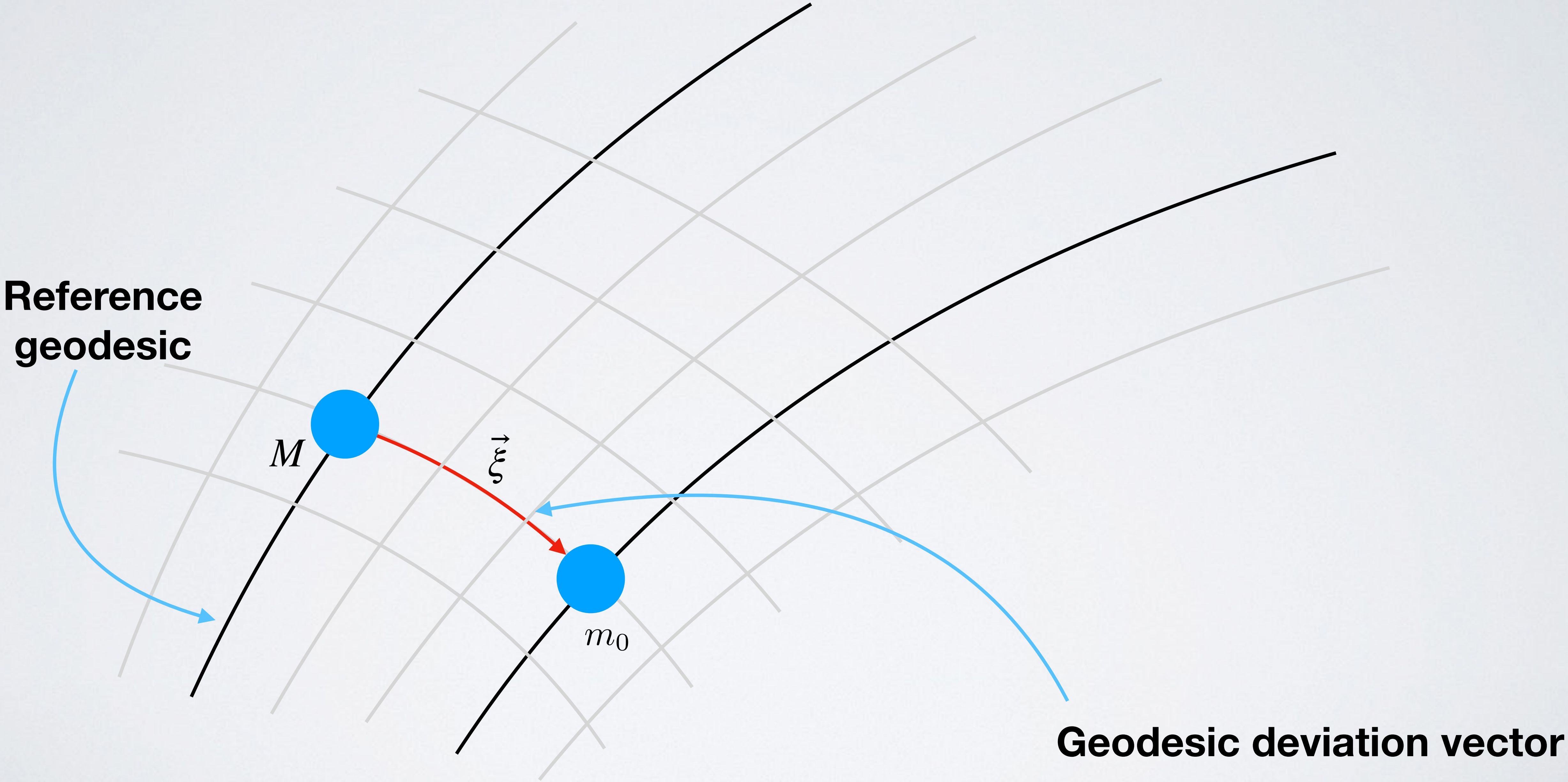
Quantum fluctuations of spacetime induce noise in the deviation of particles

The noise adds a quantum term to the Raychaudhuri equation

The characteristics of the noise depend on the quantum state of gravity

For a squeezed state, the enhanced noise might even be observable

Geodesic Deviation



Geodesic Deviation Equation

$$\ddot{\xi}^\mu = -R^\mu_{0\nu0}\xi^\nu$$

geodesic separation

Riemann tensor

$$R_{i0j0}(t,0) = -\frac{1}{2}\ddot{h}_{ij}(t,0)$$

gravitational wave

$$\ddot{\xi} = \frac{1}{2}\ddot{h}\xi$$

This is the geodesic deviation equation in the presence of a classical gravitational wave

Quantum Geodesic Deviation Equation?

$$\ddot{\xi} = \frac{1}{2}\ddot{h}\xi$$

What is the generalization of this equation when the spacetime metric is treated as a quantum field?

"The Noise of Gravitons," arXiv:2005.07211

The Noise of Gravitons

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Abstract

We show that when the gravitational field is treated quantum-mechanically, it induces fluctuations – noise – in the lengths of the arms of gravitational wave detectors. The characteristics of the noise depend on the quantum state of the gravitational field, and can be calculated exactly in several interesting cases. For coherent states the noise is very small, but it can be greatly enhanced in thermal and (especially) squeezed states. Detection of this fundamental noise would constitute direct evidence for the quantization of gravity and the existence of gravitons.

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"Quantum Mechanics of Gravitational Waves," PRL, arXiv:2010.08205

"Signatures of the Quantization of Gravity at Gravitational Wave Detectors," PRD, arXiv:2010.08208

Frank Wilczek

George Zahariade

Action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R - M\int d\lambda\sqrt{-g_{\mu\nu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda}} - m_0\int d\tau\sqrt{-g_{\mu\nu}\frac{dY^\mu}{d\tau}\frac{dY^\nu}{d\tau}}$$

Einstein-Hilbert action + action for two free-falling particles

Use Fermi normal coordinates, putting mass M on classical trajectory

$$X^\mu = (t, \vec{0})$$

Let the other particle be at

$$Y^\mu = (t, \vec{\xi})$$

Action

Next, insert metric in Fermi normal coordinates into particle action:

$$
\begin{aligned}
g_{00}(t,\xi) &= -1 - R_{i0j0}(t,0)\xi^i\xi^j + O(\xi^3)\\
g_{0i}(t,\xi) &= -\frac{2}{3}R_{0jik}(t,0)\xi^j\xi^k + O(\xi^3)\\
g_{ij}(t,\xi) &= \delta_{ij} - \frac{1}{3}R_{ikjl}(t,0)\xi^k\xi^l + O(\xi^3)\ .
\end{aligned}
$$

Inserting this into the particle action gives

$$
-m_0\int d\tau\sqrt{-g_{\mu\nu}\frac{dY^\mu}{d\tau}\frac{dY^\nu}{d\tau}} \approx -m_0\int dt\left(\frac{1}{2}R_{i0j0}(t,0)\xi^i\xi^j - \frac{1}{2}\delta_{ij}\dot{\xi}^i\dot{\xi}^j\right)
$$

Action

Expanding action to lowest order in metric perturbation, we have:

$$S = -\frac{1}{64\pi G}\int d^4x \ \partial_\mu h_{ij}\partial^\mu h^{ij} + \int dt \frac{1}{2} m_0 \left(\delta_{ij}\dot{\xi}^i\dot{\xi}^j - \dot{h}_{ij}\dot{\xi}^i\xi^j\right)$$

Mode Decomposition

$$S = -\frac{1}{64\pi G}\int d^4x\ \partial_\mu h_{ij}\partial^\mu h^{ij} + \int dt \frac{1}{2} m_0 \left(\delta_{ij}\dot{\xi^i}\dot{\xi^j} - \dot{h}_{ij}\dot{\xi^i}\xi^j\right)$$

Write metric perturbation in Fourier modes:

$$h_{ij}(t,\vec{x}) = \frac{1}{\sqrt{\hbar G}}\sum_{\vec{k},s} q_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}} \epsilon^s_{ij}(\vec{k})$$

$$S = \int dt \frac{1}{2} m_0 \dot{\xi}^2 + \int dt \sum_{\vec{k},s} \frac{1}{2} m \left(\dot{q}^2_{\vec{k},s} - \omega^2_{\vec{k}} q^2_{\vec{k}}\right) - g\int dt \sum_{\vec{k},s} \dot{q}_{\vec{k},s}\epsilon^s_{ij}(\vec{k})\dot{\xi}^i\xi^j$$

where

$$m \equiv \frac{L^3}{16\pi\hbar G^2} \qquad g \equiv \frac{m_0}{2\sqrt{\hbar G}}$$

Geodesic Deviation Interacting with Graviton Mode

free particle

$$S_\omega = \int dt \left(\frac{1}{2} m(\dot{q}^2 - \omega^2 q^2) + \frac{1}{2} m_0 \dot{\xi}^2 - g \dot{q} \dot{\xi} \xi \right)$$

simple harmonic
oscillator

cubic interaction term

Quantization Strategy

We will treat both the deviation and gravity quantum mechanically.

We will then integrate out gravity and take a saddle point to obtain the geodesic deviation in the presence of a quantized metric.

Quantum Mechanics

Suppose the gravitational field is initially in state $|\Psi\rangle$

We don’t know what the final state of the field is.

Formally, we calculate the transition probability of the deviation to go from state A to state B in time T in the presence of a gravitational field that is initially in state $|\Psi\rangle$

$$P_\Psi(A \to B) = \sum_{|f\rangle} |\langle f, B|\hat{U}(T)|\Psi, A\rangle|^2$$

The relatively simple form of the action allows the calculation to be performed exactly

Aside: Influence Functionals

Feynman and Vernon (1963) considered a very general problem in quantum mechanics

Suppose we have two interacting systems but we only have access to or interest in one of them

Then the effect of the system we are not interested in on the system we are interested in is completely encoded by the influence functional

The influence functional is a double path integral

Integrating Out Gravity

We wish to integrate out gravity to see the effect on the deviation

encodes all effects of gravity on deviation

$$P_\Psi(A \to B) \sim \int D\xi D\xi' e^{\frac{i}{\hbar}\int dt \frac{1}{2} m_0 (\dot{\xi}^2 - \dot{\xi'}^2)} F_\Psi[\xi, \xi']$$

Here the influence functional F is

$$F_\Psi[\xi, \xi'] = \sum_{|f\rangle} \int \mathcal{D}h \, \mathcal{D}h' \, e^{\frac{i}{\hbar}(S_{h,\xi} - S_{h',\xi'})}$$

boundary conditions depend on initial and final state of gravitational field

gravity-dependent part of action

Influence Functional for Gravity

A calculation in ordinary quantum mechanics
yields the influence functional for gravity

$$|F_\Psi| = \exp\left[-\frac{m_0^2}{32\hbar^2}\int_0^T\int_0^T dt\,dt'\,A_\Psi(t-t')\,(X(t)-X'(t))\,(X(t')-X'(t'))\right]$$

where

$$X(t) = \frac{d^2}{dt^2}(\xi^2)$$

"Signatures of the Quantization of Gravity at
Gravitational Wave Detectors," arXiv:2010.08208

A Mathematical Trick

We now exploit the fact that exponentials can be written as Gaussian integrals:

$$e^{\frac{b^2}{4a}} \sim \int dy \, e^{-ay^2+by}$$

The infinite-dimensional generalization of this is a path integral:

$$\exp\left[-\frac{m_0^2}{32\hbar^2}\int_0^T\int_0^T dt\,dt'\;A(t,t')\,(X(t)-X'(t))\,(X(t')-X'(t'))\right]$$
$$=\int \mathcal{D}N\exp\left[-\frac{1}{2}\int_0^T\int_0^T dt\,dt'\;A^{-1}(t,t')N(t)N(t')+\frac{i}{\hbar}\int_0^T dt\frac{m_0}{4}N(t)\,(X(t)-X'(t))\right]$$

N(t) is a zero-mean Gaussian stochastic function with auto-correlation A(t-t'): **noise!**

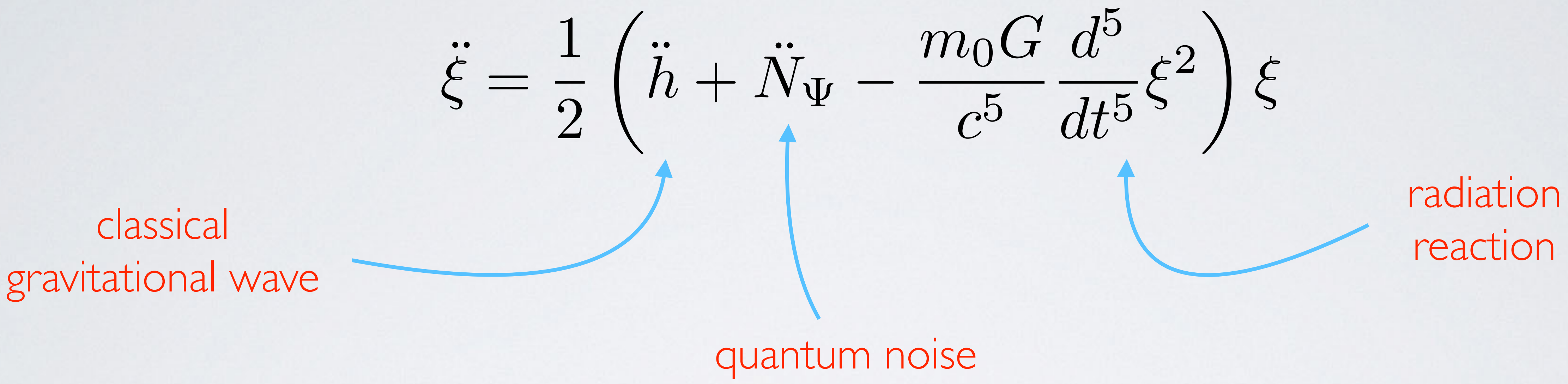
Transition Probability

Putting everything together we obtain the
deviation transition probability

$$P_\Psi(A \to B) \sim \int D\xi D\xi' e^{\frac{i}{\hbar}\int dt \frac{1}{2} m_0 (\dot{\xi}^2 - \dot{\xi}'^2)} \left\langle e^{\left(\frac{i}{\hbar}\int_0^T dt \frac{m_0}{4} N(t)\left(X(t) - X'(t)\right)\right)} \right\rangle_N$$

We can now take a saddle point to obtain the geodesic deviation
equation in the presence of a quantized metric

Langevin-Like Quantum Geodesic Deviation Equation



$$\ddot{\xi} = \frac{1}{2}\left(\ddot{h} + \ddot{N}_{\Psi} - \frac{m_0 G}{c^5}\frac{d^5}{dt^5}\xi^2\right)\xi$$

classical gravitational wave

quantum noise

radiation reaction

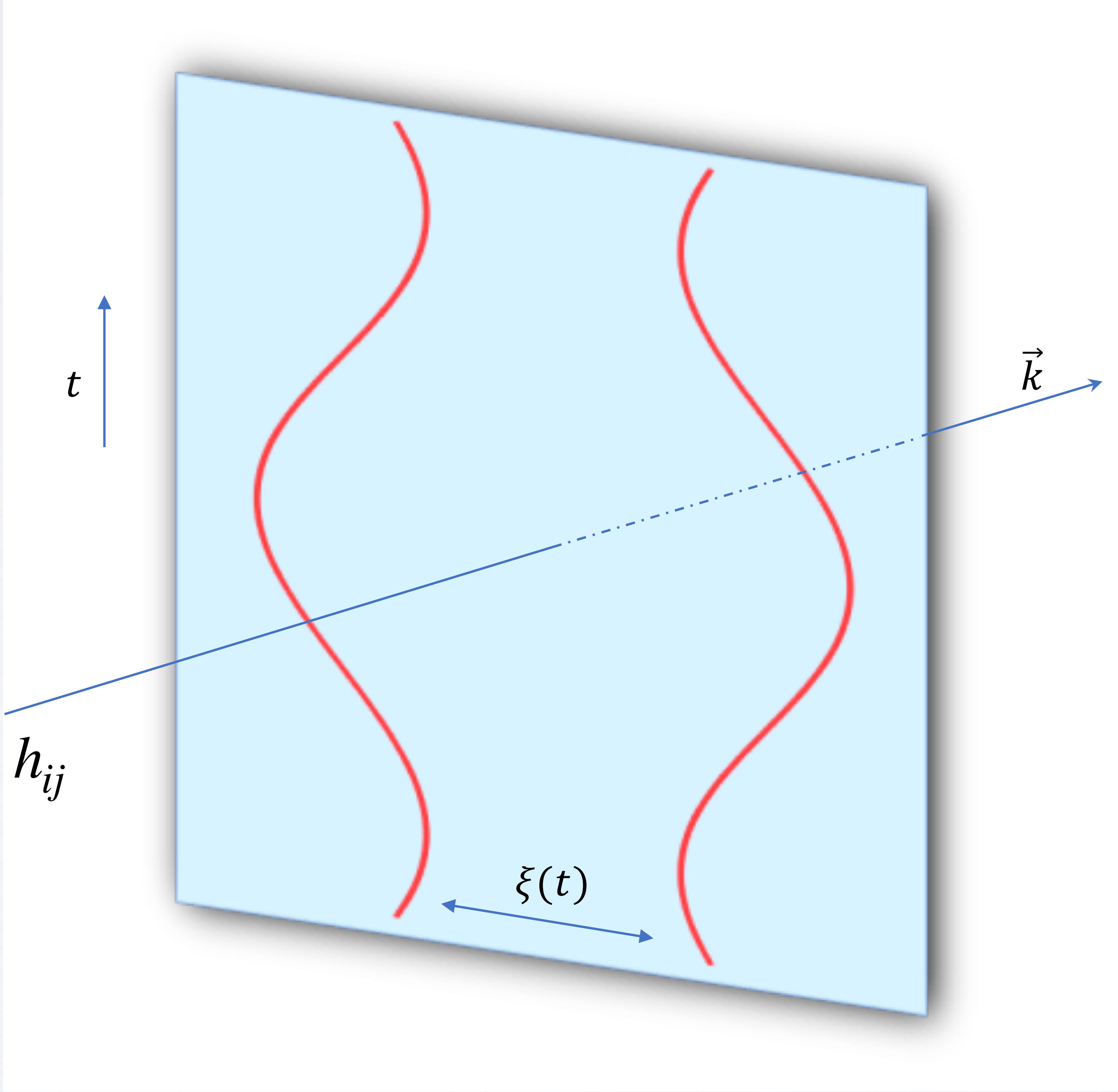
This is effectively the generalization of the classical geodesic deviation equation when the spacetime metric is a *quantum* field

Compare classical geodesic deviation: $\ddot{\xi} = \frac{1}{2}\ddot{h}\xi$

Because of the noise term, the new equation is a stochastic equation

Classical Geodesic Separation by Gravitational Waves

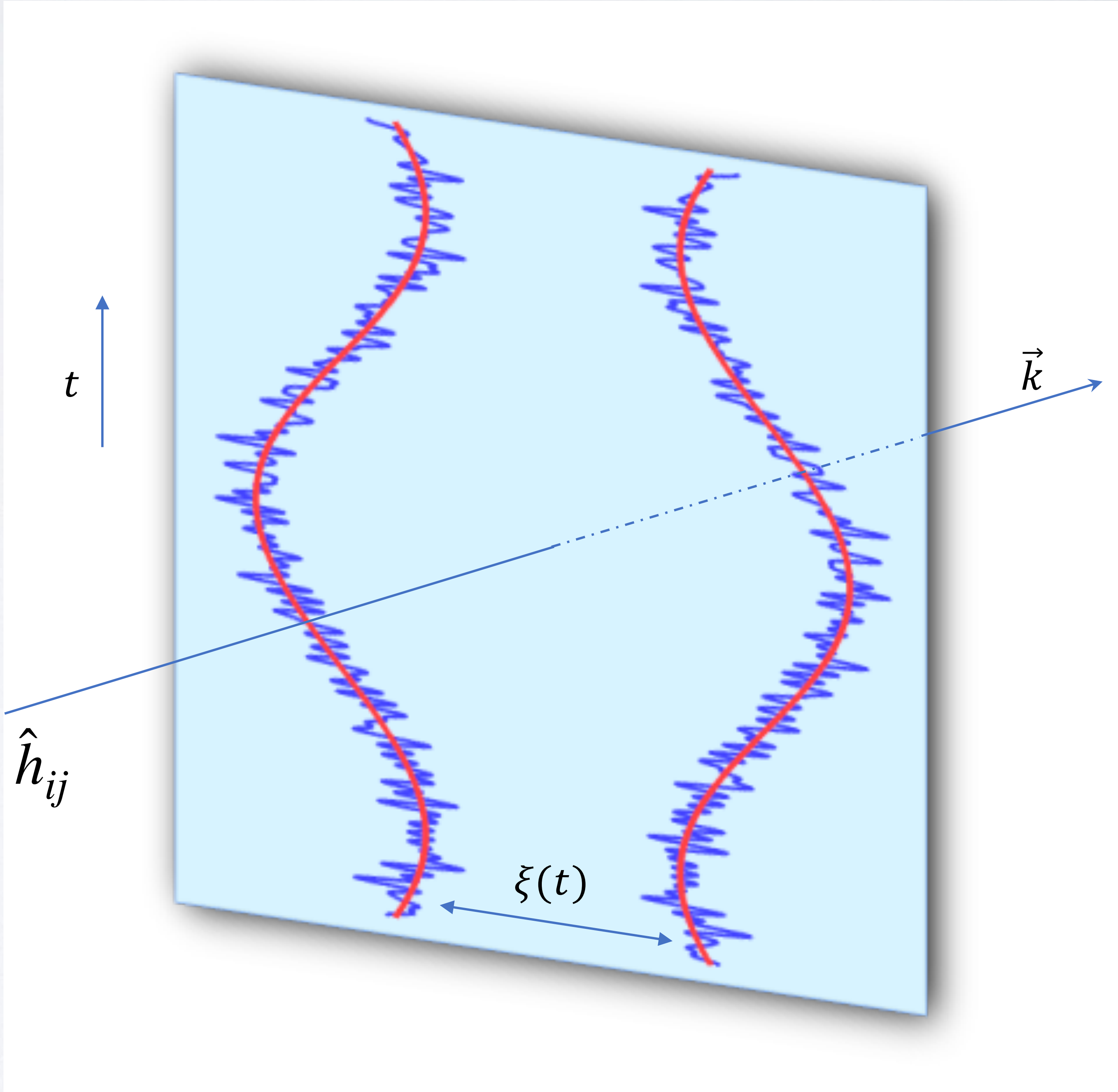
$$\ddot{\xi} = \frac{1}{2}\ddot{h}\xi$$



classical gravitational wave

The Noise of Gravitons

$$\ddot{\xi} \approx \frac{1}{2}\left(\ddot{h} + \ddot{N}_{\Psi}\right)\xi$$



quantized gravitational wave

Punchline

The signal of **quantum gravity** is in the **noise.**

G. Amelino-Camelia

MP, F. Wilczek, G. Zahariade

see also
S. Kanno, J. Soda, J. Tokuda
H-T. Cho, B-L. Hu
E. Verlinde, K. Zurek

Noise Spectrum

The power spectrum is exactly calculable for many classes of quantum states (the vacuum, thermal states, coherent states, squeezed states…)

$$S_{\text{vac}}(\omega) = 4G\hbar\omega$$

$$S_{\text{squeezed}}(\omega) \approx \cosh(2r)4G\hbar\omega$$

Noise for Quantum States of Gravitational Field

The magnitude of the noise depends on the quantum state as well as on the detector

detector sensitivity

variance of
fluctuation

$$\sigma^2 \sim \xi_0^2 \int_0^{\omega_{\max}} d\omega S(\omega)$$

noise power spectrum

arm length

For the vacuum state and for coherent states (classical gravitational waves from weak sources)

$$\sigma_0 \sim \ell_P \xi_0 \omega_{\max}/c \lesssim 10^{-36} - 10^{-38}\mathrm{m}$$

For thermal states (cosmic gravitational background, evaporating black holes)

$$\sigma \sim \sigma_0 \sqrt{k_B T/\hbar\omega_{\max}} \lesssim 10^{-30} - 10^{-34}\mathrm{m}$$

For squeezed states (cosmology, non-linear gravitational waves)

$$\sigma \sim e^{r/2}\sigma_0$$

exponential enhancement in
squeezing parameter

Discussion

Lessons from EM: Why Quantum Gravity could be Observable

Many observed phenomena show that the electromagnetic field is quantized:

Photon anti-bunching, entangled photons, sub-Poissonian statistics,
Compton effect, Lamb shift, …

Most of these are *tree-level* effects in a state that has no classical counterpart

The same is true for gravity: there can be potentially observable effects if the quantum state of the gravitational field is not a coherent state

How Gravity Naturally Produces Squeezed States

If we have a field coupled to a classical source $\mathcal{L} = -\frac{1}{2}(\partial\varphi)^2 + J\varphi$

where $\varphi \sim a + a^\dagger$

then turning on the coupling naturally produces a **coherent state**:

$$|0\rangle \to e^{Ja^\dagger}|0\rangle$$

However if the coupling is **non-linear** $J\varphi^2$

we produce roughly a **squeezed state**: $|0\rangle \to e^{Ja^\dagger a^\dagger}|0\rangle$

Gravity naturally has such nonlinear couplings which can be expected to produce squeezed states during the merger phase of black hole collisions

Inflation is also expected to produce squeezed gravitational states

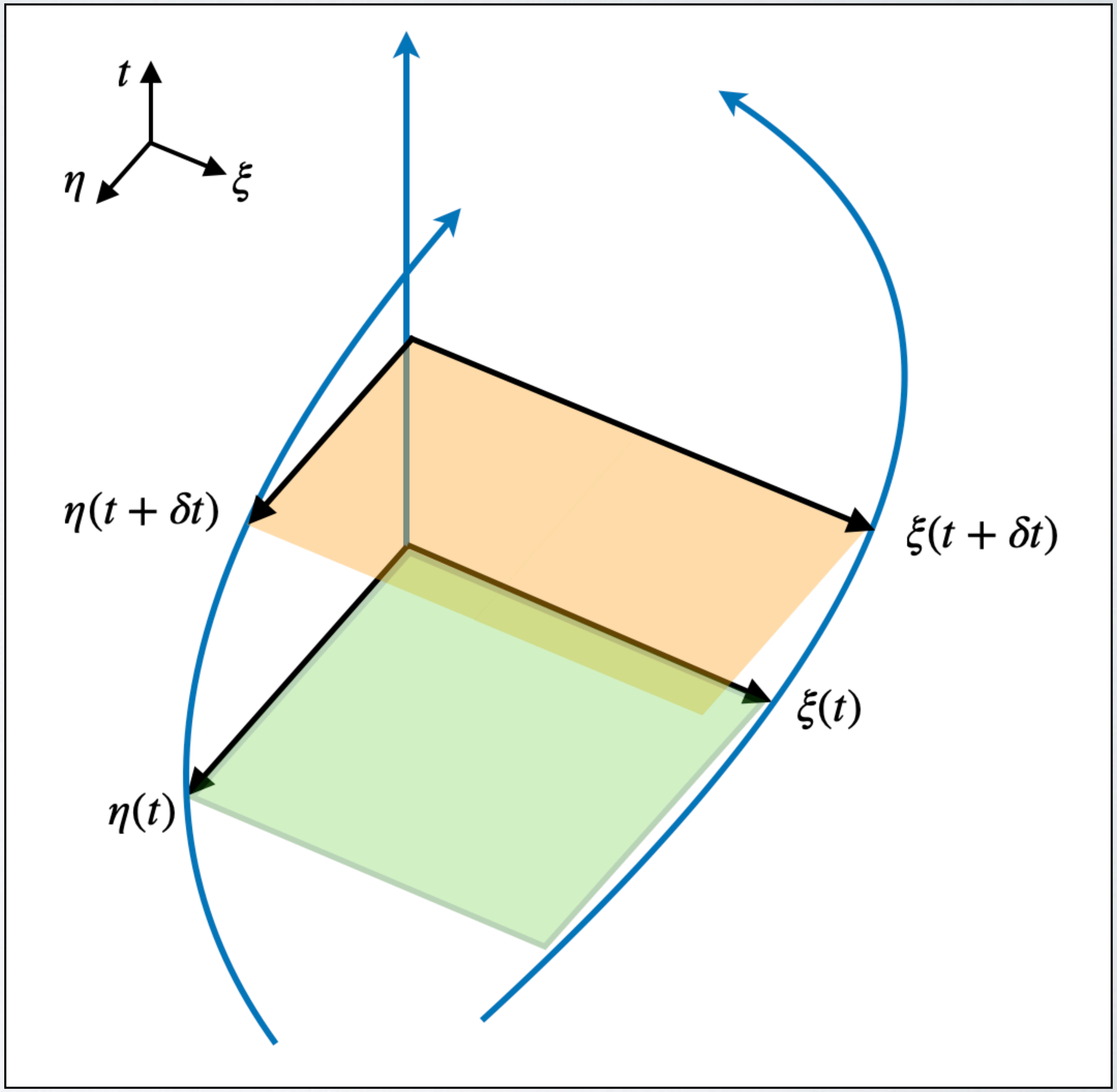
Congruences in Quantum Spacetime

Consider a cube-shaped congruence of timelike geodesics

The congruence volume is the product of the geodesic deviations of the origin to the corners:

$$V(t) = \xi(t)\eta(t)\zeta(t)$$

Sang-Eon Bak, MP, Sudipta Sarkar, Francesco Setti, 2212.14010



Quantum Volume of a Congruence

Each of $\xi(t)\ \eta(t)\ \zeta(t)$ obeys our Langevin equation:

$$\xi(t) = \xi_0(1 + N(t)), \eta(t) = \eta_0(1 + N(t)), \zeta(t) = \zeta_0(1 + N(t))$$

Then the *expectation* of the volume picks up the *variance* of the deviations:

$$\langle V \rangle = \langle \xi\eta\zeta \rangle \approx \xi_0\eta_0\zeta_0(1 + 3\langle N^2 \rangle) \equiv V_c + V_q$$

Quantum Raychaudhuri Equation

Define the congruence expansion

$$\theta \equiv \frac{\frac{d}{dt}\langle V(t)\rangle}{\langle V\rangle} \approx \frac{\dot{V}_c}{V_c} + \frac{\dot{V}_q}{V_c} \equiv \theta_c + \theta_q$$

The classical expansion obeys the classical Raychaudhuri equation

We therefore find a quantum Raychaudhuri equation

$$\dot{\theta} = -\frac{1}{3}\theta_c^2 - 2\sigma_c^2 + 2\omega_c^2 - R_{ab}u^a u^b + \dot{\theta}_q$$

quantum timelike Raychaudhuri equation

For the vacuum, we find $\dot{\theta}_{q_{\mathrm{vac}}} = -6\frac{l_P^2}{V^{2/3}}$

Summary

Quantum fluctuations of spacetime induce noise in the deviation of particles

$$\ddot{\xi} \approx \frac{1}{2}\left(\ddot{h} + \ddot{N}_{\Psi}\right)\xi$$

The noise adds a quantum term to the Raychaudhuri equation

$$\dot{\theta} = -\frac{1}{3}\theta_c^2 - 2\sigma_c^2 + 2\omega_c^2 - R_{ab}u^a u^b + \dot{\theta}_q$$

The characteristics of the noise depend on the quantum state of gravity

$$S_{\mathrm{vac}}(\omega) = 4G\hbar\omega$$

For a squeezed state, the enhanced noise might even be observable

$$S_{\mathrm{squeezed}}(\omega) \approx \cosh(2r)4G\hbar\omega$$