Algebras and states for quantum fields on causal sets

Christoph Minz[1]

[1]as part of my PhD research, supervised by Eli Hawkins and Kasia Rejzner
at the Department of Mathematics, University of York

Radboud University Nijmegen, Quantum Gravity, 11 July 2023

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1. Discreteness: What are causal sets and what is the classical algebra?
2. Quantization: Construction of quantum algebras
3. Dequantization: A tool to define states
4. Generalization: Summary with a view on interacting field theory

Note that thoughout the talk, we take the free, scalar field theory as the main example.

The main part of this talk is based on the publication [Hawkins-Minz-Rejzner] (currently under review).

Definition (Causal set)

A *causal set* (causet) is a partially ordered set $(S, \preceq)$ that is locally finite, i.e. the cardinality of the interval between any two elements (events) $x, y \in S$,

$$[x, y] := \{z \in S \mid x \preceq z \preceq y\},$$

is finite.

For an overview, see the living review [Surya 2019].

Similarities with spacetime manifolds

- partial order describes causal structure (definitions of past and future subsets $J^{\mp}(x)$ are identical)
- local compactness of a spacetime manifold ⇒ local finiteness of a causet model of this manifold

Hasse diagram of a finite causet (3-simplex) that embeds in d-dimensional Minkowski spacetime with $d \geq 1 + 3$.

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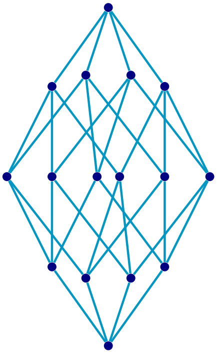

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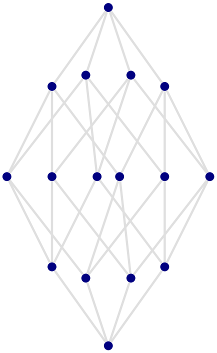
Definition (Scalar field configuration space)

For (a local region of) a causet C, the configuration space is given by all functions $\mathcal{E}(C) := \{f : C \to \mathbb{R}\}$.

For a spacetime manifold M, the space of real scalar fields is the space of smooth functions, $\mathcal{E}(M) := \mathrm{C}^\infty(M, \mathbb{R})$.



Definition (Algebra of classical observables)

The algebra of classical observables on $\mathcal{X} = C$ or $\mathcal{X} = M$ is the space of smooth, complex-valued functionals over the configuration space $\mathcal{F}(\mathcal{X}) = \mathrm{C}^\infty(\mathcal{E}(\mathcal{X}), \mathbb{C})$ with pointwise addition and multiplication.

This can be equipped with an (off-shell) Poisson bracket [DableHeath-Fewster-Rejzner-Woods 2020]

$$\{F_1, F_2\}(\varphi) = \pi_{\mathrm{off}}\Big(F_1'(\varphi), F_2'(\varphi)\Big).$$

For a finite causet C with cardinality n, $\mathcal{E}(C) \cong \mathbb{R}^n$, write $\vec{f} = (f_1, f_2, \ldots, f_n)^\mathsf{T}$.

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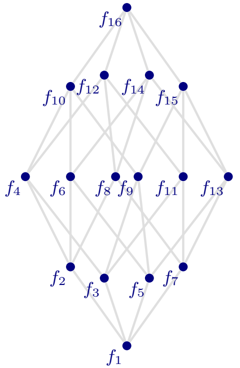

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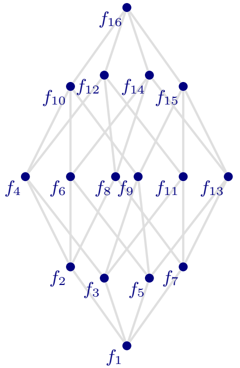

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Imposing field equations

To find the on-shell algebra, we have to discretize the Klein-Gordon equations ($P\varphi = 0$)

$$(P\varphi)(x) = c_0\varphi(x) + c_1 \sum_{z \in L_1(x)} \varphi(z) + c_2 \sum_{z \in L_2(x)} \varphi(z) + \dots$$

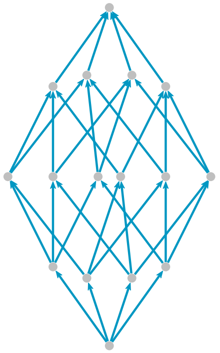
where $L_i(x)$ are certain subsets of events in the past of x. Define

retarded Green operator: $E^+ := P^{-1}$,

advanced Green operator: $E^- := (E^+)^* = (E^+)^\intercal$,

Pauli-Jordan operator: $E := E^+ - E^-$.

So the solution space is the vector space $\mathcal{S} = \mathrm{img}(E) = \mathrm{img}(\pi^\sharp_{\mathrm{off}})$.



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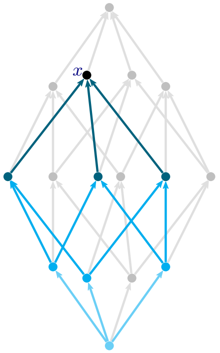

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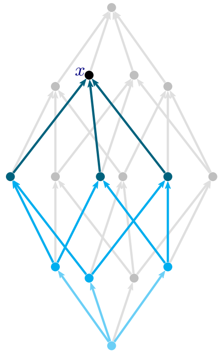

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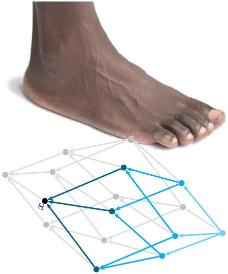

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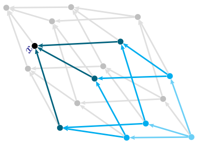

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How to quantize the classical algebra?

- Formal deformation quantization – via star products (for causal sets, see [DableHeath-Fewster-Rejzner-Woods 2020])
- Strict deformation quantization – via a field of C*-algebras
- ...
- Geometric quantization
  1. via a quantization line bundle
  2. the Bochner Laplacian and
  3. the Toeplitz quantization map

Input: Starting structure for geometric quantization

- a $2N$-dimensional vector space $\mathcal{S}$ with
- an inner product $\langle \cdot, \cdot \rangle$, and
- a symplectic form ω as inverse of the non-degenerate, on-shell Poisson bracket

such that

$$\forall v_1, v_2 \in \mathcal{S} : \qquad \omega(v_1, v_2) = \langle v_1, E^{-1} v_2 \rangle ,$$

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Let $(\mathcal{M}, \omega)$ be a real, symplectic manifold. A *quantization bundle* is a Hermitian line bundle $\mathcal{L}_\hbar \to \mathcal{M}$ with connection $\nabla_\hbar$ such that its curvature $R^{\mathcal{L}_\hbar}$ is proportional to the symplectic form,

$$R^{\mathcal{L}_\hbar} = -\frac{\mathrm{i}}{\hbar}\omega,$$

parametrized by the quantization parameter, $\hbar > 0$. The connection respects the Hermitian inner product on $\mathcal{L}_\hbar$.

Physical Hilbert space = subbundle

A *physical Hilbert space* $\mathcal{H}_\hbar$ is constructed from a subbundle (polarized sections).
As subbundle, consider the eigensections corresponding to the lowest part of the spectrum of the *Bochner Laplacian*

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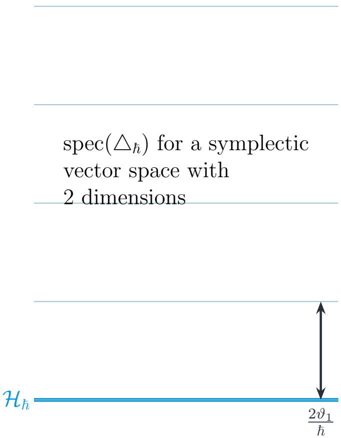


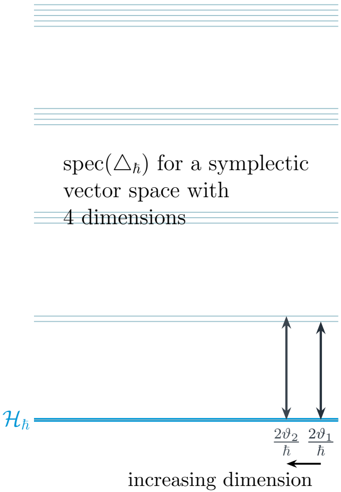
Theorem (Spectrum of the Bochner Laplacian)

The spectrum of the Laplacian for the quantization bundle $\mathcal{L}_\hbar \to \mathcal{S}$ over $(\mathcal{S}, \omega, \langle\cdot,\cdot\rangle)$ is determined by a set of strictly-positive numbers $\vartheta_i \in \mathbb{R}$ such that

$$\mathrm{spec}(\triangle_\hbar) = \left\{ \frac{1}{\hbar} \sum_{i=1}^{N} (2n_i + 1)\vartheta_i \;\middle|\; n_i \in \mathbb{N} \right\}.$$

The illustration shows the spectrum for a $2N$-dimensional space (with $N \in [1, 9]$) and bounds in the more general case of a symplectic manifold [Ma-Marinescu 2002, 2008].

Physical Hilbert space $\mathcal{H}_\hbar$ from sections of the lowest part of the spectrum (here, a single eigenvalue)



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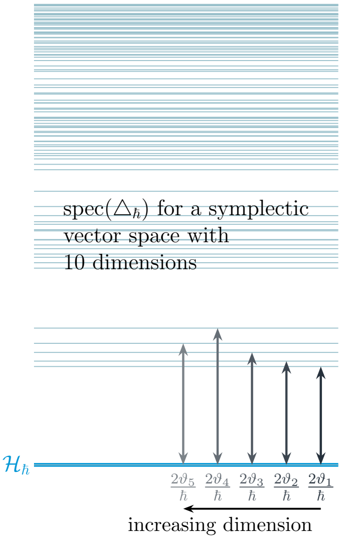

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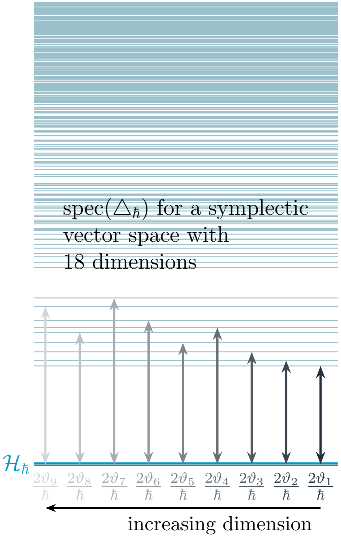

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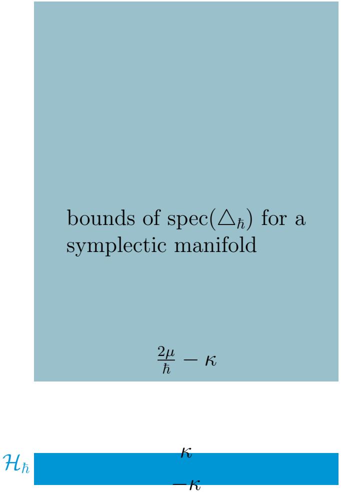

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For the finite-dimensional vector space, use complex coordiantes $(z^i, \overline{z}^{\overline{\imath}})$.

- The lowest eigensections are the holomorphic sections

$$\psi(z) = \frac{\alpha(z)}{\sqrt{2\pi\hbar}^N} \exp\left(-\frac{1}{2\hbar}|z|^2\right)$$

 with some holomorphic function α
- Choose Hilbert space basis with $\langle m_1, \ldots, m_N | n_1, \ldots, n_N \rangle_\hbar = \prod_{i=1}^N \delta_{m_i n_i}$ (for any $n_1, \ldots, n_N \in \mathbb{N}$), and
- ladder operators $a_{\overline{\imath}}^+ := \frac{1}{\sqrt{\hbar}} \delta_{\overline{\imath} i} z^i - \sqrt{\hbar} \nabla_{\overline{\imath}}$

Definition (Toeplitz quantization map)

Let $\mathcal{A}_0$ be the subspace of Schwarz functions in the classical algebra and $\mathcal{K}(\mathcal{H}_\hbar) \subseteq \mathcal{B}(\mathcal{H}_\hbar)$ be the algebra of compact operators. The *(Berezin-)Toeplitz quantization* map

$$T_\hbar : \mathcal{A}_0 \to \mathcal{K}(\mathcal{H}_\hbar)$$

is given by the projector $\Pi_\hbar : \mathrm{L}^2(\mathcal{S}, \mathcal{L}_\hbar) \to \mathcal{H}_\hbar$ as

$$\forall \psi \in \mathcal{H}_\hbar : \qquad T_\hbar(f)\psi = \Pi_\hbar(f\psi).$$

Toeplitz quantization extends to the bounded operators $\mathcal{B}(\mathcal{H}_\hbar)$.

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Let $\mu_\hbar$ be a measure such that for compactly supported functions

$$\mathrm{Tr}\big(T_\hbar(f)\big) = \int_{\mathcal{S}} f \, \mathrm{d}\mu_\hbar.$$

Definition

The *(Berezin)-Toeplitz dequantization* is a family of linear maps $\Xi_\hbar : \mathfrak{A}_\hbar \to \mathfrak{A}_0$ such that for all complex-valued, compactly supported functions $f \in \mathrm{C}_\mathrm{c}(\mathcal{S}, \mathbb{C})$ and all operators $A_\hbar \in \mathfrak{A}_\hbar$

$$\mathrm{Tr}\big(A_\hbar T_\hbar(f)\big) = \frac{1}{(2\pi\hbar)^N} \int_{\mathcal{S}} \Xi_\hbar(A_\hbar) f \, \mathrm{dvol}.$$

By construction, this map respects the involution, $\Xi_\hbar(A^*) = \overline{\Xi_\hbar(A)}$, and is normalized.

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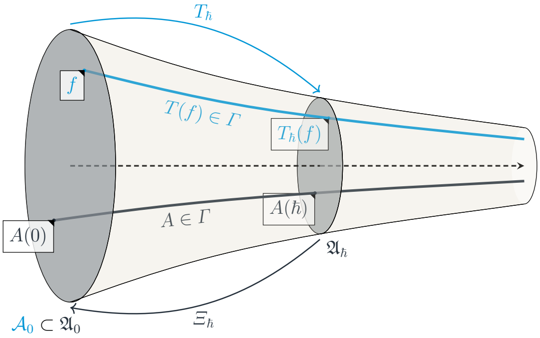

Definition (Algebraic state)

A state is a linear functional $\sigma : \mathfrak{A}_\hbar \to \mathbb{C}$ that is positive ($\forall A \in \mathfrak{A}_\hbar : \sigma(A^* A) \geq 0$) and normalized.

Theorem (Sorkin-Johnston state from dequantization)

The linear map $\sigma_\hbar : \mathfrak{A}_\hbar \to \mathbb{C}$ given by

$$\sigma_\hbar(A) := \Xi_\hbar(A)(0)$$

is the Sorkin-Johnston state.

For any Toeplitz operator $T_\hbar(f) \in \mathfrak{A}_\hbar$ ($f \in \mathcal{A}_0$):

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Originally [Johnston 2010, Sorkin 2011, 2017] derived it from a set of axioms on an operator A_{SJ} (defining a two-point function)

$$\begin{aligned} \text{positivity:} &\quad A_{\mathrm{SJ}} \geq 0, \\ \text{commutator:} &\quad A_{\mathrm{SJ}} - \overline{A}_{\mathrm{SJ}} = \mathrm{i}E, \\ \text{purity:} &\quad A_{\mathrm{SJ}} \overline{A}_{\mathrm{SJ}} = 0. \end{aligned}$$

- Note: In quantum field theory on curved spacetimes, this state is uniquely defined but not Hadamard [Fewster-Verch 2012, 2013], though
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Even though there are simpler ways to quantize the classical algebra for free field theory, the geometric quantization method directly generalizes to more involved settings, like interacting field theory. We do the same steps again, just with some modifications

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2. Take the image as solution space and construct a symplectic space, in general, a symplectic manifold with Riemannian metric
3. Consider the gap in the spectrum of the Bochner Laplacian to identify the physical Hilbert space
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