Quantum Gravitational Dynamics: An Electric Shift in Perspective

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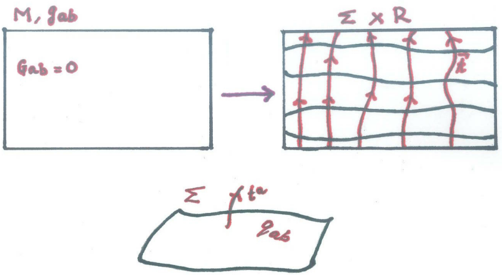
Introductory remarks:



- Phase space variables: $q_{ab}(x)$, $p^{ab}(x)$. $\{q, p\} \sim \delta$.
- Dynamics along t^a generated by Hamiltonian H.
  Solution to Hamilton's eqns $= (q(x,t), p(x,t))$. Can re-construct $g_{ab}(x,t)$ from this soln.
- Useful to decompose time flow into components normal, tangential to slice. $t^a = Nn^a + N^a$ $N\sim$ Lapse $N^a\sim$ Shift
- $H = H(N) + D(N^a) =$ Ham constraint + Diffeo constraint.

- Non trivial classical evolution is generated by the Hamiltonian constraint. In quantum theory, the corresponding operator controls the dynamics and, hence, lies at the very heart of canonical quantum gravity.
- The classical constraint is a very complicated function of q, p both for Lorentzian and Euclidean gravity. Difficult to quantize. Let us focus on the Euclidean constraint.
- The Euclidean constraint acquires a simple form in Ashtekar-Barbero variables which makes it more amenable to quantization.
- These variables are an $su(2)$ electric field E^a_i and conjugate connection A^i_a.
  $E^a_i, i = 1, 2, 3$ define a spatial triad, A^i_a has extr curv info.
  $H(N)$ takes the form $\approx \int_\Sigma N(EEF)(\frac{1}{\sqrt{q}}), \quad F = dA + [A, A]$
- Our aim is to construct the corresponding operator $\hat{H}(N)$ in the LQG repn which Hal introduced.

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- Since spacetime is itself dynamical, LQG aims to construct this Hamiltonian constraint operator without relying on any fixed background spacetime. This requires new ideas and techniques beyond those of qft in fixed, flat spacetime
- These were developed through early pioneering contributions by Jacobson, Smolin, Rovelli, Gambini, later by Blencowe, Pullin, Bruegmann, Borissov, Ashtekar, Lewandowski, Loll (partial list!) culminating in Thiemann's construction of the Euclidean constraint operator in his QSD work.
- Thiemann also showed how to construct the Lorentzian operator from the Euclidean one and the Volume operator.
- **It is for this reason that we restrict attention to the Euclidean Hamiltonian constraint operator in this talk.**

- The QSD construction of the Euclidean constraint operator is a spectacular achievement.
- However, some open problems remain:
  1. Many ambiguities in final operator action.
  2. Constraint commutator $[\hat{H}(M), \hat{H}(N)]$ does not reproduce correct lapse dependence of $\widehat{\{H(M), H(N)\}}$.

  (From work of H-K-T, this dependence turns out to be connected with implementation of spacetime covariance in quantum theory.)
- In our view, further progress on these issues was handicapped by absence of any intrinsically 3d interpretation of classical evolution generated by Ham constraint...

  (Existence of such an interpretation for spatial diffeo constraints, namely that they generate 3d diffeos, is what ensures their correct quantum implementation in LQG.)

Main Take Home Message:

- Classical evolution in **time** generated by the Hamiltonian constraint can now be thought of in terms of certain **spatial** diffeomorphism like transformations generated by a triple of 'shift' vector fields constructed from the Electric field. (A.Ashtekar, MV).
- This classical insight dovetails naturally into a new construction of the Hamiltonian constraint operator for Euclidean gravity in which diffeomorphism like transformations of quantum states play a key role. The new operator has improved properties with regard to ambiguities and spacetime covariance.

We turn now to an account of the new classical insight.

A Digression on Gauge Covariant Lie Derivatives:

Lie derivative of w_i^a wrto v^b: $L_{\vec{v}} w_i^a = v^b \partial_b w_i^a - w_i^b \partial_b v^a$

Gauge covariant Lie derivative: $\mathcal{L}_{\vec{v}} w_i^a = v^b \mathcal{D}_b w_i^a - w_i^b \partial_b v^a$

Geometric interpretation in terms of diffeos along horizontal lift of v.

What if $v \equiv v_j^a$ has internal indices? Define:

Generalised gge cov Lie deriv: $\mathbb{L}_{\vec{v}_j} w_i^a = v_j^b \mathcal{D}_b w_i^a - w_i^b \mathcal{D}_b v_j^a$

Depends on connection. Exact geom interpretation open.

- Back to GR. Define Electric Shift$\sim$ NE_i^a.

  Evolution Equations generated by Hamiltonian Constraint on the constraint surface:

  $\dot{E}^a{}_i = \frac{1}{2}\epsilon_i{}^{jk} \mathbb{L}_{\vec{N}_j} E_k^a$, $\qquad \dot{F_{ab}}{}^i = -\epsilon^{ij}{}_k \mathbb{L}_{\vec{N}_j} F_{ab}^k$

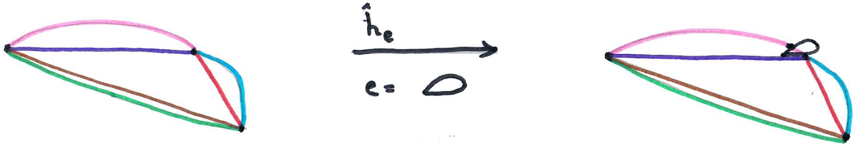
  $\mathbb{L}_{\vec{N}_j}$ is exactly $\mathcal{L}_{\vec{N}_j}$ with ordinary derivatives replaced by gauge covariant ones.

  The message for quantum theory is to try to encode the action of the Hamiltonian constraint operator in terms of transformations generated by the electric shift operator $\sim N\hat{E}$.

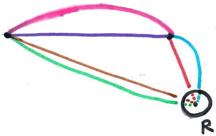
  On to quantum theory...

Quantum Kinematics: A ‘Connection’ Repn

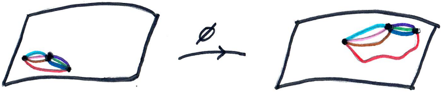
- Connection dep operators are holonomies along edges in the Cauchy slice $h_e = P \exp - \int_e ds \dot{e}^a A_a$.
  $\hat{h}_e(A)\psi(A) = h_e(A)\psi(A) \quad \hat{E}(x)\psi(A) = iG\hbar \frac{\delta}{\delta A(x)}\psi(A)$
- States are linear combinations of products of edge holonomies so live on colored graphs. Holonomy operator $\hat{h}_e$ typically adds edge e to the graph.



- Volume operator constructed from $\hat{E}$:



- Evoltn along slice is a diffeo. Due to background free quantization, diffeos are unitarily implemented.
- A Diffeo acts by moving the graph and preserving its colors:



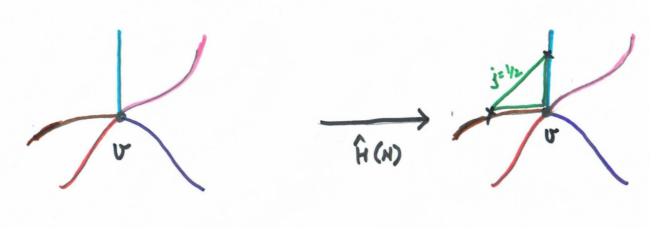
Quantum Dynamics Normal to the Slice

- Recall $H_E \sim \int NEEF \times (\sqrt{q}$ dep factor$)$.
- $V(R) = \int_R \sqrt{q}$. Just like $\hat{V}$, $\widehat{\sqrt{q}}$ also acts only at vertices of states. With appropriate operator ordering, Hamiltonian constraint action is a sum over vertex contributions.
- At each vertex, we need an action of the curvature operator $\hat{F}$. This presents us with following key problem.
- The Problem: $F(x)$ is a *local* function of the connection. Basic connection operators nonlocal holonomies. Classically: Extract F from δ size loop holonomy:

  $$\lim_{\delta\to 0} \frac{h_{\text{small loop}} - \mathbf{1}}{\delta^2}.$$

  QMly: We replace $\hat{F}$ in the constraint operator by $\frac{\hat{h}_{\text{small loop}} - \mathbf{1}}{\delta^2}$ and then take the $\delta \to 0$ limit of the resulting expression.
- Good News: Can define this limit and hence a constraint operator . Bad News: Final operator action depends on choice of small loops and repn of its holonomy: Ambiguities!.

- Typically, the choice of loop is dictated by simplicity and the repn is chosen to be $j = \frac{1}{2}$. Skipping over some technicalities related to taking $\delta \to 0$, an accurate pictorial representation of the essential features of the operator action with these choices is:

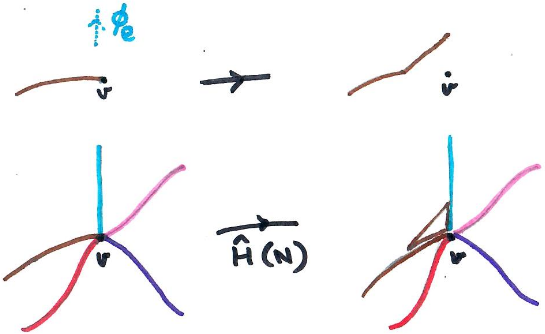


- We now move on to the new results.

New Construction:

- Recall that classical theory asks us to focus on Electric Shift. Lets do this in quantum theory and see what happens.
- Recall $H_E \sim \int NEEF \times (\sqrt{q}$ dep factor). $NE \sim$ Electric shift. Becomes an operator $\sim N\hat{E}^a_i$
  $\widehat{\sqrt{q}}$ dep factor yields sum over vertex contributions.
- $N\hat{E}^a_i$ acts at each vertex. The simplest objects at vertex with ^{a} index are edge tangents.
- Constraint action naturally implemented in terms of diffeo like translations along edges at vertex.
  Denote such a diffeo like translation along an edge e by ϕ_e.

- ϕ_e is called an Electric Diffeomorphism. Small loops which encode $\hat{F}$ are now built from the action of ϕ_e on remaining edges at **v**.



Since ϕ_e move colored edges without changing their spin labels, the choice of spin repn of small loop holonomies is uniquely determined and in correspondence with edge colors at the vertex.
Shape of small loop determined by the way electric diffeos act.

- Color ambiguity is fixed. ‘Geometric’ understanding of shape.
- Also get Correct Lapse Behavior.
  Can show on a suitable space of states that:
  $[\hat{H}(M), \hat{H}(N)] = i\hbar\{H(\widehat{M), H}(N)\}$
  This was one of the ingredients for emergence of Spacetime Covariance from the canonical theory.

In Summary:There are still ambiguities which remain but the Hamiltonian constraint operator action is better determined, beautifully implements our classical intuition, seems closer to its detailed, physically appropriate implementation in minisuperspace quantization (‘improved LQC’), and constitutes exciting progress!

Key issues for the immediate future

- Application of this new dynamics to mini/midisuperspaces: Improved LQC from improved LQG?New constraint treatment for plane waves (S. Major), sph. symmetry, cylindrical waves?
- Transition to Lorentzian Theory via Phase Space Wick Rotation: Thiemann constructed complex canonical transformation to Ashtekar's complex variables and suggested its implementation in quantum theory via the action of a certain quantum 'Wick rotator'. Quantum solutions to the Lorentzian constraint operator can then be obtained by action of this rotator of their Euclidean counterparts. With certain further improvements to the Euclidean constraint action (in progress), it may be possible to show the existence of 'Wick Rotated' solutions to Lorentzian Quantum Gravity!

3 key issues for the future :

- The issue of spacetime covariance: Is it enough to represent commutator of constraints with phase space indep multipliers in anomaly free manner (issue of the Bergmann-Komar group)? What about matter couplings?
- LQG for Asymptotically Flat spacetimes:
  Kinematics: How to implement bdry conditions at quantum level?
  Dynamics: Are Asymptotic Poincare Transformations unitarily implemented? Is there a quantum positive energy theorem?
- How do we extract physics from quantum states in canonical LQG:
  Problem: (Almost) No Explicit Dirac Observables
  How to construct the physical Hilbert Space inner product? Can we feed the new constraint action into Thiemann's Master Constraint Implementation? Can we develop the quantum theory in the context of finite boundaries (see Freidel's talk..) Can we interpret individual quantum spin network states (Speziale, Livine...) Use matter to define 'clocks and rods'? (Lewandowski, Giesel, Thiemann....) Coarse graining? (Dittrich, Livine..) Contact with Spin Foams?

A few useful references:

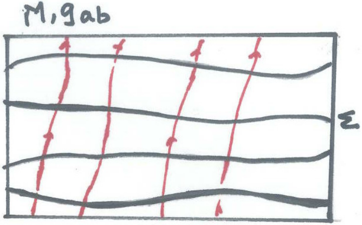
1. Classical Dynamics: Universe 7 (2021) 1, 13, e-Print: 2012.12094 [gr-qc] (Ashtekar, MV)
2. New action: Class.Quant.Grav. 38 (2021) 13, 135020, e-Print: 2101.03115 [gr-qc] (MV)
3. Non-trivial anomaly free commutators: e-Print: 2205.10779 [gr-qc] (MV)
4. Propagation in LQG e-Print: 2112.03992 [gr-qc] (Thiemann, MV).
5. Complexifier: Class.Quant.Grav. 13 (1996) 1383-1404 e-Print: gr-qc/9511057 [gr-qc] (T. Thiemann)
   Phys.Rev.D 53 (1996) 2865-2869, e-Print: gr-qc/9511083 [gr-qc] (A. Ashtekar)
   Class.Quant.Grav. 36 (2019) 1, 015016, e-Print: 1808.00673 [gr-qc] (MV).

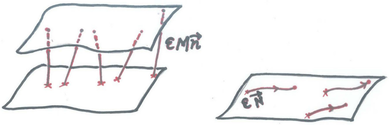
The Issue of Anomaly Free Constraint Commutators

- Classical 3+1 Hamiltonian formulation of a theory of spacetime butchers spacetime into space and time.
- Question: Is there a structure in the Hamiltonian formulation which encodes the 4d spacetime covariance of the system?
  Answer (H-K-T): YES! The constraint algebra of such a system has a certain characteristic structure. Thus 4d covariance of 3+1 Hamiltonian GR is encoded in the structure of its constraint algebra.
- The most non-trivial part of this in Euclidean case is:
  $\{H(M), H(N)\} = D(\vec{A})$ where $D(\vec{A})$ is the diffeo constraint smeared with shift $A^a = q^{ab}(N\nabla_b M - M\nabla_b N)$. In quantum theory, if the constraint commutators reflect this structure so that
  $[\hat{H}(M), \hat{H}(N)] = i\hbar\widehat{D(\vec{A})}$, we expect the emergent classical theory to be spacetime covariant.

What about higher order commutators?

- The algebra of gravitational constraints is not a Lie algebra : $\{H(M), H(N)\} = D(\vec{Q})$ where $D(\vec{Q})$ is the diffeo constraint D_a smeared with phase space dependent shift $A^a = q^{ab}(N\nabla_b M - M\nabla_b N)$. P.B. of $D(\vec{Q})$ with $H(N)$ generates constraints smeared with connection and electric field dependent lapses and shifts. Higher order PBs yield more and more complicated phase space dependent smearings.
- This higher order algebraic structure is not the same as that of hypersurface deformations. To see this, revisit H-K-T.





M= lapse, $\vec{N}$= shift

- Work out algebra of hypersurface deformations:
  $[\ \mathbf{Ta}\ (\vec{N}_1),\ \ \mathbf{Ta}\ (\vec{N}_2)] = \mathbf{Ta}(\mathcal{L}_{\vec{N}_2}\vec{N}_1),\ \ [\mathbf{No}(M),\mathbf{Ta}(\vec{N})] = \mathbf{No}(\mathcal{L}_{\vec{N}}M)$
  $[\mathbf{No}(M_1),\mathbf{No}(M_2)] = \mathbf{Ta}(\vec{N}_{M_1,M_2,q_{ab}})$
- Since deformations are geometrical, this structure holds even if lapses,shifts depend on q_{ab}! For e.g.:
  $[\mathbf{No}(M),\mathbf{Ta}(\vec{N}_{M_1,M_2,q_{ab}})] = \mathbf{No}(\mathcal{L}_{\vec{N}_{M_1,M_2,q_{ab}}}M)$
- In contrast, for gravity we have:
  $\{H(M), D(\vec{N}_{M_1,M_2,q_{ab}})\} = H(\mathcal{L}_{\vec{N}_{M_1,M_2,q_{ab}}}M)$
  $+ \int_\Sigma d^3x\ D_a\ \ \{H(M), N^a_{M_1,M_2,q_{ab}}(x)\}$

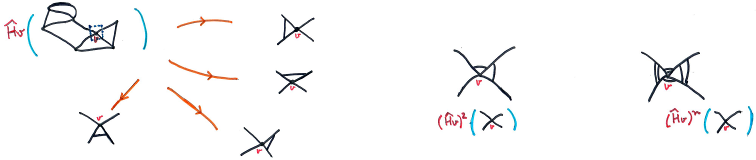
So the hypersurface deformation commutators disagree with the constraint algebra for higher order PBs. Hence, the relevant deep and universal off shell structure which encodes sptime cov is only the single PBs between constraints with phase space indep lapses,shifts and these must be represented without anomaly. This is fortunate because it is going to be impossible to represent the higher order PBs without anomalies and, indeed, doing that is probably asking for too much from the quantum theory.

Note: In simpler cases such as the $U(1)^3$ model, the connection dependence in the constraints is linear. This ensures that lapses and shifts in higher order commutators *only* depend on the Electric field. In a representation in which the Electric field is then diagonal, it is possible to represent the entire constraint algebra (Thiemann).

But for the gravitational case, this seems impossible due to the generation of lapses and shifts with arbitrarily complicated phase space dependence.

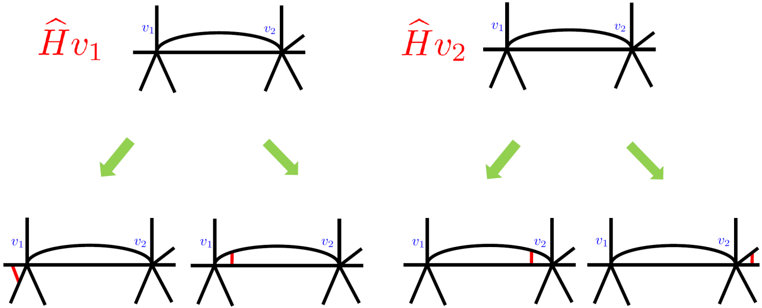
Propagation

The Hamiltonian constraint acts only at vertices of a spin net S.
At each vertex only a small neighborhood of v in S is affected.
Constraint action at one vertex indep of action at other.
Action is said to be ultralocal.

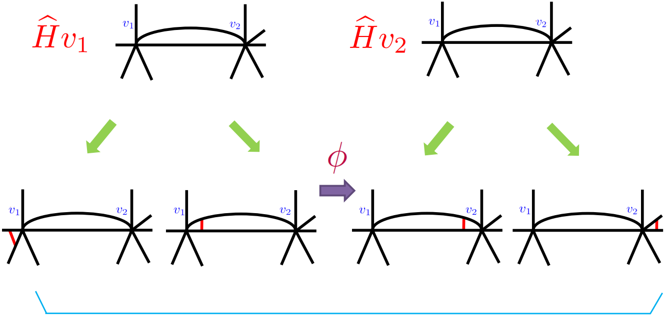


Clearly, repeated action of constraint only lead more ‘vertex embroidery’ but cannot propagate embroidery from one vertex of S to another. This lead to a folklore that ultra local action is incompatible with propagation and hence could not lead to the correct classical limit.

- Of course we do not have a true Hamiltonian, so propagation should be articulated in terms of properties of physical states.
  A physical state Ψ is a sum of bras:
  $\Psi = \sum_{\bar{S}} c_{\bar{S}} < \bar{S}|$ s.t. $\lim_{\epsilon\to 0} \Psi(\hat{H}_\epsilon(N)|S\rangle) = 0$ for all S
  Call states generated by action of $\hat{H}_\epsilon(N)$ on $|S\rangle$ as
  "Children of Parent $|S\rangle$".
  For $|S\rangle$ such that action of $\hat{H}_\epsilon(N)$ on $|S\rangle$ generates ket correspondents of these bras, nontrivial eqns result. Ψ will be linear combination of children of a set of parents.
- By considering lapses of support only around individual vertices of the parental graph, we get eqns at each vertex for coefficients $c_{\bar{S}}$.
- If these equations are such that the presence in Ψ of children with distortions at v_1 necessarily implies presence in Ψ of other children with distortions at v_2, then we say that distortions propagate from v_1 to v_2. Here by presence in Ψ I mean that coefficients of these children are non-vanishing in the sum representing Ψ.
- As we now illustrate, ultralocality implies no propagation.



Ultralocality implies no vertex coupled equations and no propagation.
This picture changes drastically with diffeo invariance!



VERTEX COUPLING OF EQUATIONS

Ultralocality is not a sharply defined notion in a diffeo inv setting!
States are diffeo inv and no-prop intuition **no longer holds**.