Exactly solvable lower-dimensional gravity models

Thomas Mertens

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Based on arXiv:2306.00941 with A. Blommaert and S. Yao
arXiv:2212.07696
and earlier work arXiv:2109.07770, arXiv:2006.07072, arXiv:1812.00918

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In this talk, I will discuss three interrelated 1+1d gravitational models, and in particular observe and unify the underlying structure of their gravitational amplitudes in terms of representation theory

This approach has been my main focus for several years, and has been pursued with several collaborators (A. Blommaert, Y. Fan, J. Simon, G.J. Turiaci, G. Wong, S. Yao)
+ WIP with F. Mariani, A. Belaey

Outline

Jackiw-Teitelboim gravity (1)

Dilaton gravity $= 1{+}1$ dimensional toy model of gravity
$S = \frac{1}{16\pi G}\int d^2x\sqrt{-g}\,(\Phi R + V(\Phi)) + S_{\text{bdy}}$
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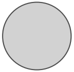
β  $= Z(\beta) = \text{Tr}\left[e^{-\beta H}\right] = \int_0^{+\infty} dk (k \sinh 2\pi k) e^{-\beta k^2}$

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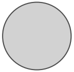
β  $= Z(\beta) = \text{Tr}\left[e^{-\beta H}\right] = \int_0^{+\infty} dk(k\sinh 2\pi k)e^{-\beta k^2}$

= QG thermal partition function whose saddle matches classical JT black hole $M(T_H)$ (mass vs Hawking temperature)

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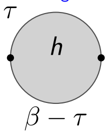

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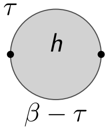

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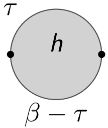

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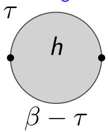

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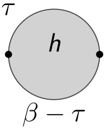

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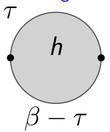

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$\frac{\Gamma(h \pm ik_1 \pm ik_2)}{\Gamma(2h)} = \int_{-\infty}^{+\infty} d\phi\, K_{2ik_1}(e^\phi)\, e^{2h\phi}\, K_{2ik_2}(e^\phi) =$(Clebsch-Gordan)^2

$K_{2ik_1}(e^\phi)$ are rep matrix elements in the principal series, $e^{2h\phi}$ is a discrete series rep matrix element ($j = -h$)

Gauge theory formulation of JT gravity: the BF model (2)

No coincidence!

1^{st} order formulation of JT gravity can be written in terms of $\mathfrak{sl}(2,\mathbb{R})$ BF theory Fukuyama-Kamimura '85, Isler-Trugenberger '89, Chamseddine-Wyler '89

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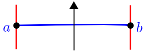
Structure of theory:

- Hilbert space $L^2(G)$ is determined by Peter-Weyl theorem: $\mathcal{H} = \{|R,a,b\rangle \,,\ R = \text{ unitary irrep of } G,\ a,b = 1..\text{dimR}\}$
- Coordinate basis $\{|g\rangle \,, g \in G\}$: $\langle g|R,ab\rangle = \sqrt{\text{dim R}}\, R_{ab}(g)$

Gravitational wavefunctions of JT gravity (1)

→ Representation matrices $R_{ab}(g)$ physically identified as two-boundary wavefunctions



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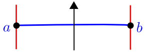

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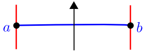
→ Representation matrices $R_{ab}(g)$ physically identified as two-boundary wavefunctions 

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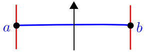

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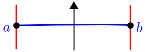
$\rightarrow$ Representation matrices $R_{ab}(g)$ physically identified as two-boundary wavefunctions 

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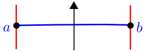
→ Representation matrices $R_{ab}(g)$ physically identified as two-boundary wavefunctions 

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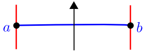

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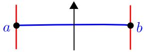
→ Representation matrices $R_{ab}(g)$ physically identified as two-boundary wavefunctions 

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→ Apply to JT gravity:

Algebra: $\mathfrak{sl}(2,\mathbb{R})$

$$[H, E] = E, \quad [H, F] = -F, \quad [E, F] = 2H$$

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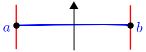

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$\hat{L}_i f(g_0) = \frac{d}{d\epsilon} f(e^{-\epsilon X_i} g_0)|_{\epsilon=0} \Rightarrow \hat{L}_i g = -X_i g$ and $\hat{R}_i g = g X_i$

$\rightarrow$ Apply to JT gravity:

Algebra: $\mathfrak{sl}(2,\mathbb{R})$

$$[H,E] = E, \quad [H,F] = -F, \quad [E,F] = 2H$$

Group: Gauss-Euler decomposition for $\mathrm{SL}(2,\mathbb{R})$ group element
$g = e^{\gamma F}\, e^{2\phi H}\, e^{\beta E}$ for coordinates (γ, ϕ, β)

Gravitational wavefunctions of JT gravity (2)

Regular representations: $\hat{L}_i g = -X_i g$ and $\hat{R}_i g = gX_i$:

$$\hat{L}_F = \partial_\gamma, \qquad \hat{R}_F = -\beta^2\partial_\beta - \beta\partial_\phi + e^{2\phi}\partial_\gamma,$$

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$\left(-\frac{1}{4}\partial_\phi^2 + e^{2\phi}\right)\psi(\phi) = k^2\psi(\phi)$ with solution $K_{2ik}\left(2e^\phi\right)$

The SYK-model: double-scaling limit

SYK model: N 0+1 dimensional Majorana fermions $\psi_i(t)$, satisfying $\{\psi_i, \psi_j\} = \delta_{ij}$ with all-to-all random interactions of p fermions Sachdev-Ye '92, Kitaev '15:

$H = \sum_{i_1 < \ldots < i_p} J_{i_1 \ldots i_p} \psi_{i_1} \ldots \psi_{i_p}$

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$Z(\beta) = \int_0^\pi d\theta \, (e^{\pm 2i\theta}; q^2)_\infty \, e^{-\beta 2\cos(\theta)}$ with $\rho(\theta) \sim (e^{\pm 2i\theta}; q^2)_\infty$

q-Pochhammer: $(a; q)_\infty = \prod_{k=0}^\infty (1 - aq^k)$

$q = e^{-\lambda}$

DSSYK: group structure (1)

Boundary two-point function:

$$\int_0^\pi d\theta_1\, \rho(\theta_1)\, e^{-\tau 2\cos(\theta_1)} \int_0^\pi d\theta_2\, \rho(\theta_2)\, e^{-(\beta-\tau)2\cos(\theta_2)} \frac{(q^{4h};q^2)_\infty}{(q^{2h}e^{\pm i\theta_1 \pm i\theta_2};q^2)_\infty}$$

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for non-commutative coordinates (γ, ϕ, β) satisfying

$$e^\phi \gamma = q\gamma e^\phi, \quad e^\phi \beta = q\beta e^\phi, \quad [\beta,\gamma] = 0$$

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where we used the notation $\left(\frac{d}{dx}\right)_q f(x) \equiv \frac{f(qx)-f(x)}{qx-x}$,
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→ Impose same (boundary) constraints as in JT
Left eigenfunction: $q^n H_n(\cos(\theta)|q^2), \quad \phi = -n \log q$
Right eigenfunction: $\frac{q^n}{(q^2;q^2)_n} H_n(\cos(\theta)|q^2)$

→ precisely matches with the structure of DSSYK!

Liouville gravity: Definition

Non-critical string from 2d conformal matter coupled to 2d gravity, or critical string with a 2d Liouville + matter + ghost CFT Polyakov '81, David '88, Distler-Kawai '89 . . .

Liouville gravity: $S_L + S_M + S_{\rm gh}$
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- Liouville action: $S_L = \frac{1}{4\pi}\int_\Sigma \left[(\hat{\nabla}\phi)^2 + Q\hat{R}\phi + 4\pi\mu e^{2b\phi}\right]$
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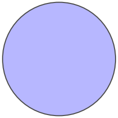
- S_M = arbitrary CFT with $c_M < 1$
- $S_{\rm gh}$ is usual bc-ghost theory with $c_{\rm gh} = -26$

Disk partition function and boundary two-point function

Amplitudes determined in TM-Turiaci '20 using older results of Zamolodchikov[2] et al.

Disk partition function:

$Z(\beta) =$ β  $= \int_0^\infty ds \, \sinh(2\pi b s) \sinh\left(\frac{2\pi s}{b}\right) e^{-\beta \cosh(2\pi b s)}$

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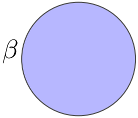
$Z(\beta) =$ 

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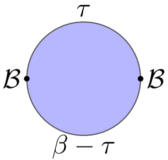

$$\mathcal{B} = c \, \Phi_M e^{\beta_L \phi}$$

$$\sim \int_0^{+\infty} ds_1 ds_2 \, \rho(s_1) \, \rho(s_2) \, e^{-\cosh 2\pi bs_1 \tau} e^{-\cosh 2\pi bs_2 (\beta - \tau)} \frac{S_b(h \pm is_1 \pm is_2)}{S_b(2h)}$$

where $h = b - \beta_L$, S_b is double sine function

$\rho(s) \sim \sinh(2\pi bs) \sinh\left(\frac{2\pi s}{b}\right)$

Quantum group interpretation of Liouville gravity

where $\frac{S_b(h \pm is_1 \pm is_2)}{S_b(2h)} \sim \int_{-\infty}^{+\infty} dx \; R_{s_2}^*(x) e^{2h\pi x} R_{s_1}(x)$ TM-Turiaci '20

where $R_s(x) =$
$e^{\pi i 2sx} \int_{-\infty}^{+\infty} \frac{d\zeta}{(2\pi b)^{-2i\zeta/b - 2is/b}} S_b(-i\zeta) S_b(-i2s - i\zeta) e^{-\pi i \epsilon(\zeta^2 + 2s\zeta)} e^{2\pi i \zeta x}$

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Based on so-called modular double of $\mathrm{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ quantum algebra Teschner ...

Quantum algebra: Two (commuting) copies of the $\mathrm{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ quantum algebra $[H, E] = E\,, \quad [H, F] = -F\,, \quad [E, F] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}$
with $q = e^{\pi i b^2}$ and $\tilde{q} = e^{\pi i b^{-2}}$

Quantum group interpretation of Liouville gravity

where $\frac{S_b(h \pm is_1 \pm is_2)}{S_b(2h)} \sim \int_{-\infty}^{+\infty} dx\, R^*_{s_2}(x) e^{2h\pi x} R_{s_1}(x)$ TM-Turiaci '20

where $R_s(x) =$
$e^{\pi i 2sx} \int_{-\infty}^{+\infty} \frac{d\zeta}{(2\pi b)^{-2i\zeta/b - 2is/b}} S_b(-i\zeta) S_b(-i2s - i\zeta) e^{-\pi i \epsilon(\zeta^2 + 2s\zeta)} e^{2\pi i \zeta x}$
→ Explain explicitly (i.e. are the blue functions rep. matrices?)

Based on so-called modular double of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ quantum algebra Teschner ...
Quantum algebra: Two (commuting) copies of the $U_q(\mathfrak{sl}(2,\mathbb{R}))$ quantum algebra $[H, E] = E\,, \quad [H, F] = -F\,, \quad [E, F] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}$
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Quantum group: $g = g_b(\gamma f)\, e^{2\phi H}\, g_b^*(\beta e)$ TM '22 where g_b is Faddeev's quantum dilogarithm, $e = (2 \sin \pi b^2) E$ and $f = (2 \sin \pi b^2) F$

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→ Above blue function $R_s(x)$ is indeed eigenfunction of associated Casimir eigenvalue problem

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Leads to unconventional models of holography that are not aAdS:

- The sinh dilaton gravity model has a curvature singularity at the boundary
- The sine dilaton gravity model can lead to positively curved regions in spacetime, allowing us to implement dS physics within the UV-complete model of DSSYK

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Thank you!