Composite Operators in Asymptotically Safe Quantum Gravity

Maximilian Becker[1]

[1]IMAPP, Radboud University Nijmegen

In collaboration with A. Kurov, and F. Saueressig

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Why study composite operators in the AS approach towards QG?

- Goal: predictive quantum field theory of gravity
- The Asymptotic Safety (AS) hypothesis: high-energy completion of gravity is provided by an interacting RG fixed point

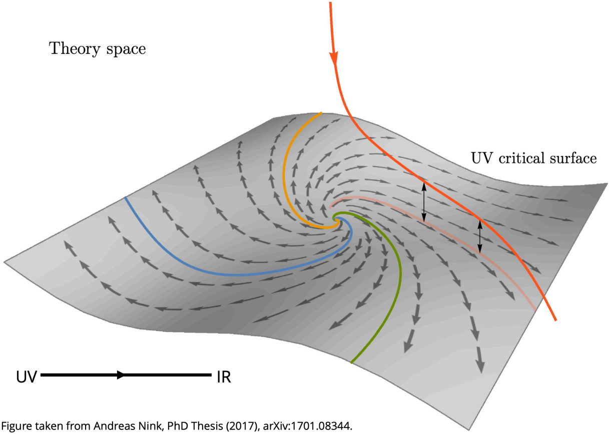

Figure taken from Andreas Nink, PhD Thesis (2017), arXiv:1701.08344.

Why study composite operators in the AS approach towards QG?

The AS scenario implies non-trivial quantum corrections to the scaling dimensions of operators (and correlation functions) that are characteristic for the corresponding universality class.

Given an interacting UV fixed point has been identified,

1. How many relevant parameters does the theory have?
2. How do we construct meaningful observables?

Both questions can be probed:

- via the Wetterich equation, a Functional Renormalization Group Equation (FRGE)
- via a composite-operator FRGE

Observables & geometric properties in the UV – how composite operators come into play

- Consider the following scaling argument (cf. Codello, d'Odorico '15):

$$G_{12}(r) \equiv \frac{1}{Z} \int \mathcal{D}g \, e^{-S} \int_{x,y} \frac{1}{\mathrm{Vol}} \sqrt{g_x} O_1(x) \sqrt{g_y} O_2(y) \, \delta(d_g(x,y) - r)$$

$$\Rightarrow G_{12}(\lambda r) = \lambda^{\frac{\Delta_1^g + \Delta_2^g - \Delta_{\mathrm{Vol}}^g}{\Delta_{d_g}^g} - 1} G_{12}(r)$$

- We need to compute the UV scaling properties of the geometric operators
- The Functional Renomalization Group (FRG) offers two avenues for their computation: the Wetterich equation (for (quasi-)local operators) and via composite operators
- Analogy: scaling dimensions from the KPZ equation

The Wetterich equation:

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathscr{R}_k\right)^{-1} k\partial_k \mathscr{R}_k\right]$$

- Typically solved with a truncation ansatz of the form $\Gamma_k = \sum_i \bar{u}_i(k) M_i$
- The RG equations take the form $k\partial_k u_i(k) = \beta_i(u(k))$

The UV fixed point

- A UV fixed point u^* is given by $\beta_i(u^*) = 0$
- Solution for the linearized theory: $u_i(k) = u_i^* + \sum_I c_I V_i^I (k_0/k)^{\theta_I}$
- The universal critical exponents θ_I are the eigenvalues of the stability matrix B, given by

$$\sum_j B_{ij} V_j^I = -\theta_I V_I \quad \text{and} \quad B_{ij} = \frac{\partial}{\partial u_j}\beta_i \Big|_{u=u^*}$$

- The relevant (attractive) directions are those with $\mathrm{Re}\,\theta_I > 0$

The composite-operator FRGE (Cf. Pawlowski '07; Igarashi, Itoh, Sonoda '10; Pagani '16):

$$\partial_t \left[\mathscr{O}_k\right]_i = -\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\left[\mathscr{O}_k\right]_i^{(2)}\left(\Gamma_k^{(2)} + R_k\right)^{-1}\partial_t R_k\right]$$

- A set of composite (geometric) operators $[\mathscr{O}_k]_1, \ldots, [\mathscr{O}_k]_n$ can be incorporated into the FRG framework via the simple substitution $\Gamma_k \to \Gamma_k + \sum_i \varepsilon_i \cdot [\mathscr{O}_k]_i$
- Expand the renormalized composite operators in terms of the basis of bare composite operators, $[\mathscr{O}_k]_i[g,\bar{g}] = \sum_j Z_{ij}(k)\, \mathscr{O}_j[g,\bar{g}]$

Then,

$$\sum_{j=1}^{n} \bar{\gamma}_{ij}(k)\mathscr{O}_j = -\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1}\mathscr{O}_i^{(2)}\left(\Gamma_k^{(2)} + R_k\right)^{-1}\partial_t R_k\right]$$

- Anomalous dimension matrix $\bar{\gamma}_{ij}(k) = \sum_l (Z^{-1})_{il}(k)\partial_t Z_{lj}(k)$
- The renormalization behavior of the renormalized composite operators $[\mathscr{O}_k]_i$ becomes encoded into the anomalous dimension matrix $\gamma_{ij}(k)$

Two approximations are required to solve the composite-operator flow equation,

$$\sum_{j=1}^{n} \bar{\gamma}_{ij}(k)\mathscr{O}_j = -\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \mathscr{O}_i^{(2)} \left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_t R_k\right]$$

- Solution strategy:
  - The first truncation is the usual one for the EAA:

$$\Gamma_k = \sum_i \bar{u}_i(k) M_i$$

  - The second truncation is the one for the basis of composite operators:

$$[\mathscr{O}_k]_i = \sum_j Z_{ij}(k)\,\mathscr{O}_j$$

- In general, the size of the anomalous dimension matrix depends on the second, while its arguments depend on the first truncation:

$$\gamma_{ij}(k) \equiv \gamma_{ij}(\bar{u}(k))$$

Interpretation of the anomalous dimension matrix γ_{ij}

The anomalous dimension matrix γ_{ij} can be related to the stability matrix B_{ij} in the following way.

- Consider an $f(R)$-type first and second truncation (with $\rho = k^{-2}R$):
  - $\Gamma_k = \int \mathrm{d}^d x \sqrt{g} f_k(R) = \int \mathrm{d}^d x \sqrt{g} k^d \sum_{n=0}^{N_{\mathrm{prop}}} u_n(k)\rho^n$
  - $\mathscr{O}_n = \int \mathrm{d}^d x \sqrt{g} R^n$ with $n = 0, 1, \ldots, N_{\mathrm{scal}}$
- Then, the standard Wetterich equation takes the form

$$\partial_t u_i + (d-2i)u_i = \omega_i(u) + \sum_j (\partial_t u_j)\bar{\omega}_{ji}(u)\,,$$

yielding the RG equations (cf. Falls et al. '14)

$$\partial_t u_i = \big(-(d-2j)u_j + \omega_j(u)\big)\big(\delta_{ji} - \bar{\omega}_{ji}(u)\big)^{-1} = \beta_i(u)\,,$$

and stability matrix

$$B \equiv \partial\beta(u^*) = \big(-D + \partial\omega(u^*)\big)\big(\mathbb{1} - \bar{\omega}(u^*)\big)^{-1}\,,$$

where $D_{ij} = (d-2i)\delta_{ij}$.

- On the other hand, take a derivative of the RHS w.r.t. the couplings u_i and evaluate at the fixed point:

$$\partial \text{RHS}\Big|_{u=u^*} \equiv \partial\omega(u^*) \equiv \gamma(u^*) + \delta\gamma(u^*)\,,$$

thus,

$$B \equiv \partial\beta(u^*) = \big(-D + \gamma(u^*) + \delta\gamma(u^*)\big)\big(\mathbb{1} - \bar{\omega}(u^*)\big)^{-1}.$$

- Hence, there are two different ways of obtaining the theory's critical exponents, depending on whether the couplings' anomalous dimensions $\eta(u)$ on the RHS is differentiated or not:

1. Eigenvalues of $B \equiv \partial\beta(u^*)$ – here, $\eta(u)$ on the RHS is differentiated
2. Eigenvalues of $-D + \gamma(u^*)$ – here, $\eta(u) \equiv \eta^*$ on the RHS is fixed

Negative eigenvalues of B and $-D+\gamma$ at the UV fixed point, sorted by their real part, for selected values of N_{scal} and N_{prop}. Relevant directions are those with $\mathrm{Re}\,\theta > 0$. Results have been obtained in the physical gauge (for $d = 4$).

	B	$-D+\gamma$	B	$-D+\gamma$	B	$-D+\gamma$	B	$-D+\gamma$	$-D+\gamma$
$(N_{\text{scal}}, N_{\text{prop}})$	(2,2)	(2,2)	(3,3)	(3,3)	(4,4)	(4,4)	(6,6)	(6,6)	(6,4)
$\mathrm{Re}\,\theta_1$	1.26	4.03	2.67	1.96	2.83	3.17	2.39	0.084	1.06
$\mathrm{Im}\,\theta_1$	-2.44	-1.40	-2.26	-1.61	-2.42	-3.14	-2.38	-3.96	-4.13
θ_2	27.02	0.89	2.07	-6.39	1.54	-5.09	1.51	6.82	16.83
θ_3			-4.42	-305.82	-4.28	-64.03	-4.16	-588.10	24.06
$\mathrm{Re}\,\theta_4$					-5.09	-534.47	-4.68	41.06	41.44
$\mathrm{Im}\,\theta_4$					0	0	6.08	-21.20	0
θ_5							–	–	-651.07
θ_6							-8.68	-1654.03	-1586.38

Results – Observations

- Both methods agree qualitatively for small $N_{\mathrm{scal}} = N_{\mathrm{prop}} \lesssim 2$
- The results are sensitive to the full information carried by the propagator

Solving the Wetterich equation:

- The critical exponents derived from B become Gaussian, $\theta_n \xrightarrow{n\to\infty} \theta_n^{\mathrm{Gaussian}} = 4 - 2n$ (cf. Falls et al. '14)

Solving the composite-operator flow equation:

- The critical exponents derived from $-D+\gamma$ become (unacceptably) large
  - The two points above have a technical origin: For $N \geq 3$, the 2-point function evaluated at the fixed point has a pole in R inside the unit circle
  - This pole creates very large coefficients in the Taylor expansion in R around $R=0$, that we need to perform to read off γ, ω and $\bar{\omega}$ from the corresponding powers of R
  - In B, the ratio of these larges coefficients drives the critical exponents into a Gaussian regime, whereas for $-D+\gamma$ these coefficients are taken at face value

- The eigenvalues of $-D+\gamma(u^*)$ can also be interpreted as the full geometric scaling dimensions of the operators $[\mathscr{O}_k]_1(r),\dots,[\mathscr{O}_k]_n(r)$ in the fixed point regime (given that these depend on some length scale r). (Cf. Pagani '16)
- In particular, for a single composite operator one has:

$$[\mathscr{O}_k]_{k\to\infty}(r) \sim r^{d-\gamma(u^*)}$$

- Applied to the volume operator, $\int \mathrm{d}^d x\sqrt{g}$, i.e., $N_{\mathrm{scal}}=0$, we obtain a stable result in the physical gauge for $N_{\mathrm{prop}}\geq 3$ of

$$\gamma = 1.9614$$

- Thus for the full spacetime volume, we observe a dimensional reduction from $d=4$ down to $4-\gamma\approx 2$

Thanks for your attention.

Literature about composite operators in the AS approach:

- C. Pagani, 1603.07250
- C. Pagani and M. Reuter, 1611.06522
- M. Becker and C. Pagani, 1810.11816
- M. Becker, C. Pagani, and O. Zanusso, 1911.02415
- W. Houthoff, A. Kurov, and F. Saueressig, 2002.00256
- A. Kurov and F. Saueressig, 2003.07454

Contact: Maximilian Becker, M.Becker@hef.ru.nl