A discrete discontinuity between the two phases of gravity

Sandipan Sengupta
Dept of Physics
Indian Institute of Technology Kharagpur
INDIA

[Based on:
Sengupta, arXiv: 2210.15980
Sengupta, PRD 107, 024010 (2023)]

Another phase (face!) of gravity

- Within Einstein's formulation of general relativity, the 4-metric $g_{\mu\nu}$ (e_μ^I) is assumed to be invertible: seems natural from the classical notion of a smooth (C^∞) spacetime

- However, invertible metrics need not be the true representation of spacetime near a singularity; Perhaps, smoothness is too 'nice'!

- Suggestive example: the celebrated BKL (1970) limit close to a spacelike singularity; Locally, the approximate classical soln reads:
$ds^2 \to -dt^2 + t^{2p_1}du^2 + t^{2p_2}dv^2 + t^{2p_3}dw^2$
$(\sum p_i = 1 = \sum p_i^2)$

- $\det g_{\mu\nu} \to 0$ as $t \to 0$ (singular limit)

Another phase (face!) of gravity

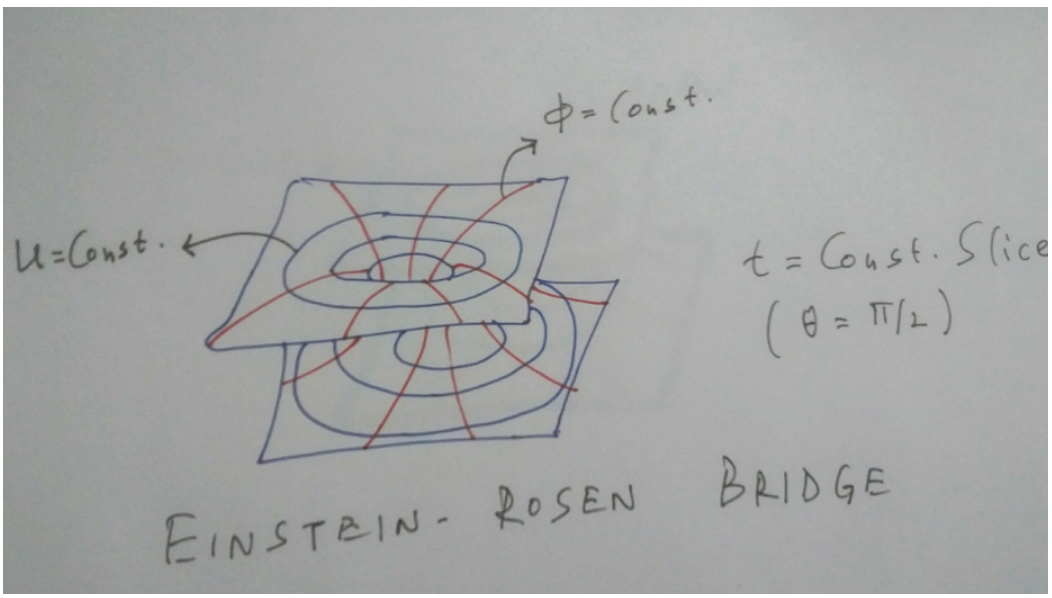
- In fact, $\det e = 0$ solns arise not only as a limit, but also as exact solutions within Hilbert-Palatini (first order) formulation [Tseytlin '82, Bengtsson '89, Horowitz '91, Kaul-Sengupta 2016, ..]

- Should be treated as saddle points of the quantum gravity path integral

- Such solutions should probably get reflected through any quantum formulation (e.g. LQG already anticipates a 'quantum' spacetime with deg spatial metric)

Old but intriguing ideas: Gravity with $g = 0$

• Einstein-Rosen bridge ('34) as a geometric realizn of elementary particles [classical]



• Torsion foam (Hanson-Regge ('78)) where a $e = 0$ phase shows up in Euclidean (quantum) gravity through torsion vortices, analogous to Abrikosov vortices in superconductors

The problem

- To understand the role of the $dete = 0$ phase in quantum gravity in concrete terms, one could explore a Hamiltonian analysis (followed by a quantzn)

- However, the standard ADM parametrization is built upon the inverse tetrad fields, and hence is not a suitable framework in this regard

Questions ..

- How to proceed then? (Variables? constraints? Algebra?...)

- What is the no. of local degrees of freedom in such a theory?

- Are the theories with $\det e = 0$ and $\det e \to 0$ (Carrollian or Levy Leblond-Sen Gupta limit of gravity) equivalent?

- Could the quantization of gravity in this phase be any simpler?

Hamiltonian gravity: Non invertible phase

- $\mathcal{L}(e, \omega) \ = \ \frac{1}{8}\epsilon^{\mu\nu\alpha\beta}\epsilon^{IJKL}e^I_\mu e^J_\nu R_{\alpha\beta}{}^{KL}(\omega)$.

- Space-time split ($\epsilon^{tabc} \equiv \epsilon^{abc}$):

$\mathcal{L} \ = \ \frac{1}{2}\Pi^a{}_{IJ}\partial_t\omega_a{}^{IJ} + P^a_I\partial_t e^I_a - e^I_t C_I \ - \ \omega_t{}^{IJ}G_{IJ}$

$(\mathcal{L} = p\dot{q} - \mathcal{H})$

- Primary constraints $[F(q,p) \approx 0]$:

$e^I_t, e^I_a, \omega_t{}^{IJ}$ have no velocities:

$P_I = \frac{\partial\mathcal{L}}{\partial \dot{e}^I_t} \approx 0, \ P^a_I = \frac{\partial\mathcal{L}}{\partial \dot{e}^I_a} \approx 0, \ \Pi_{IJ} = \frac{\partial\mathcal{L}}{\partial \dot{\omega}_t{}^{IJ}} \approx 0$

- Further,

$\Pi^a{}_{IJ} - \frac{1}{2}\epsilon^{abc}\epsilon_{IJKL}e^K_b e^L_c \approx 0 \equiv \chi^a{}_{IJ}$

$\epsilon^{bcd}\epsilon_{IJKL}e^I_a e^J_b e^K_c e^L_d \approx 0 \equiv \Phi_a$

- $\det e = 0$ $\Rightarrow$ one cannot invert the relation $\Pi(e)$ to obtain $e(\Pi)$ and eliminate e variables [Unlike invertible tetrad or HP case]

Secondary constraints

- $\mathcal{L} \;=\; \frac{1}{2}\Pi^a{}_{IJ}\partial_t\omega_a{}^{IJ} + P^a_I\partial_t e^I_a - e^I_t C_I \;-\; \omega_t{}^{IJ}G_{IJ}$

- $\mathcal{H} = e^I_t C_I \;+\; \omega_t{}^{IJ}G_{IJ}$

- Secondary constraints:
$C_I \equiv -\frac{1}{4}\epsilon^{abc}\epsilon_{IJKL}e^J_a R_{bc}{}^{KL} \approx 0$ (Ham+Diff)
$G_{IJ} \equiv -\frac{1}{2}D_a\left[\epsilon^{abc}\epsilon_{IJKL}e^K_b e^L_c\right] \approx 0$ (Rotn+Boost)

Hamiltonian gravity: Non invertible phase

- Primary Ham density:

$$\mathcal{H} = e_t^I C_I + \frac{1}{2}\omega_t{}^{IJ} G_{IJ} + \mu_a{}^{IJ}\chi^a{}_{IJ} + \mu_a^I P_I^a + \mu^a \Phi_a \approx 0$$

- Recall $\chi^a{}_{IJ} \equiv \Pi^a{}_{IJ} - \frac{1}{2}\epsilon^{abc}\epsilon_{IJKL}e_b^K e_c^L \approx 0$ [18]

Project out a set of 6 which involve Π only :

$C^{ab} \equiv \frac{1}{2}\epsilon^{IJKL}\Pi^a{}_{IJ}\Pi^b{}_{KL} \approx 0$

$(\chi^a{}_{IJ}[18] \equiv (\hat{\chi}^a{}_{IJ}[12] + C^{cd}[6]))$

- Non-trivial Poisson brackets:

$[C_I, C^{ab}], [P_I^a, \hat{\chi}^b{}_{JK}],\ [C_I, P_J^a],\ [G_{IJ}, \chi^a{}_{KL}],$
$[G_{IJ}, P_K^a],\ [P_I^a, \chi^b{}_{JK}],\ [P_I^a, \Phi_b]$

- Time evolution of $C^{ab} \approx 0$:

$$[\textstyle\int d^3x\ \mathcal{H}, C^{ab}]\ \approx 0\ \approx\ 4e_t^I\left[\epsilon^{cd(a}\Pi^{b)}{}_{IJ}D_c e_d^J\right]$$

det e=0 as a solution

$[\int d^3x\ \mathcal{H}, C^{ab}] \approx 4 e^I_t \left[\epsilon^{cd(a}\Pi^{b)}{}_{IJ} D_c e^J_d\right] \approx 0$

- For inv tetrad ($e^I_t \neq 0$), one obtains the secondary constraint $\left[\epsilon^{cd(a}\Pi^{b)}{}_{IJ} D_c e^J_d\right] \approx 0$ (implies vanishing of torsion in vacuum)

- However, there is another possible solution ($\det e^I_\mu = 0$): $e^I_t \approx 0$ and no secdry constraint; Torsion nonvanishing in general

- The pair (C^{ab}, C^I) is then second-class:
$\left[C_I, C^{ab}\right] = \epsilon^{cd(a}\Pi^{b)}{}_{IJ} D_c e^J_d \neq 0$
(In contrast with invertible case)

Degrees of freedom in $\det e = 0$ phase

- $\mathcal{H} = e_t^I C_I + \frac{1}{2}{\omega_t}^{IJ} G_{IJ} + {\mu_a}^{IJ} {\chi^a}_{IJ} + \mu_a^I P_I^a + \mu^a \Phi_a$

- Constraints: $C^I, G^{IJ}, \chi^a_{IJ} (\equiv C^{ab}, \hat{\chi}^a_{IJ}), P_I^a, \Phi_a$

- $({\hat{\chi}^a}_{IJ}, P_K^b)$: 12 2nd-cl pairs
(C^{ab}, C^I): 4 2nd-cl pairs + 2 1st-cl constraints
(G^{IJ}, Φ_a): 6+3=9 1st-cl (having zero brackets with C^{ab})

- Altogether, (e_a^I, P_J^b) [12] and $({\omega_a}^{IJ}, \Pi^b_{KL})$ [18] are 30 canonical pairs, subject to 11 first-class and 16 second-class pairs of constraints

- $30 - (16 + 11) = 3$ d.o.f per spacetime point!

- An important contrast to the invertible phase of first-order gravity in vacuum (2 d.o.f)

Time-gauge constraints

- Fix the boost freedom through time gauge: $\chi_i = 0$ ($\Rightarrow \zeta_i = \partial_a E^a_i$)

- Constraints acquire a simpler form:

$$
\begin{aligned}
G^i_{rot} &\equiv -\epsilon^{ijk} Q^j_a E^a_k \approx 0, \\
C_0 &\equiv -\frac{\sqrt{E}}{2}[E^{[a}_i E^{b]}_j (\bar{R}_{ab}{}^{ij}(\bar{\omega}) + Q^i_a Q^j_b) \\
&\quad + N^{ij}N_{ji} - N^i{}_i N^j{}_j] \approx 0, \\
C_i &\equiv \sqrt{E}\left[E^{[a}_i E^{b]}_k \bar{D}_a Q^k_b - N_{ki} G^k_{rot}\right] \approx 0, \\
\pi_{kl} &\approx 0,
\end{aligned}
$$

- Contrast with time-gauge constraints of the invertible phase:
vector (C_i) and rotation (G^i) constraints same, but
$C_0 \equiv -\frac{\sqrt{E}}{2} E^{[a}_i E^{b]}_j (\bar{R}_{ab}{}^{ij}(\bar{\omega}) + Q^i_a Q^j_b)$,
no π constraints

Final phase space

$$C_0 \equiv -\frac{\sqrt{E}}{2}[E_i^{[a}E_j^{b]}(\bar{R}_{ab}^{\ ij}(\bar{\omega}) + Q_a^i Q_b^j) + N^{ij}N_{ji} - N^i{}_i N^j{}_j] \approx 0$$

- The 2nd-cl pair may be solved by fixing the torsional scalar as: $N^{ij}N_{ji} - N^i{}_i N^j{}_j = E_i^{[a}E_j^{b]}(\bar{R}_{ab}^{\ ij}(\bar{\omega}) + Q_a^i Q_b^j)$ and setting the associated momenta to zero

- The rest of the momenta π_{kl} may now be set to zero strongly; This does not affect any of the constraints

The fate of Hamiltonian constraint

- The scalar (Hamiltonian) constraint C_0 is already solved now; Simply disappeared through this gauge fixing!

- Final phase space: (Q_a^i, E_j^b) subject to first-class constraints
$G^i_{rot} \equiv -\epsilon^{ijk} Q^j_a E^a_k \approx 0,\ C_i \equiv \sqrt{E} E_i^{[a} E_k^{b]} \bar{D}_a Q_b^k \approx 0$

Main results: Discontinuity in d.o.f.

- We find that the Hamiltonian theory of gravity at the $det e = 0$ phase exhibits 3 local d.o.f. irrespective of whether the null eigenvalue lies along time or space direction

- However, for any finite e, the d.o.f is 2. Same for the limit $e \rightarrow 0$ (Sengupta, PRD 107, 024010 (2023)), known as 'Carrollian' gravity

- Thus, the limit $\det e \rightarrow 0$ has a discrete discontinuity!

- Analogous to vDVZ discontinuity for massive gravity (recall that $m_g = 0$ and $m_g \rightarrow 0$ correspond to d.o.f. 2 and 5, respectively)

Main results: Hamiltonian constraint

- The disappearance of the Hamiltonian constraint in the noninvertible phase of gravity is an inviting feature: Formally, any functional invariant under internal rotations and spatial diffeomorphisms are solutions (upto regularization and ordering ambiguities)

- Interpretational aspects of the candidate solutions for quantum states should be explored: what is the connection of these $\det e = 0$ states to the Einstein phase?

- Possible contexts of relevance: short distance or strong gravity, early universe, BKL behaviour in cosmology, ...

- May help us understand if the noninvertible phase indeed has a physical importance

Thank you!