Quantum Geometrodynamics Revived – Conceptual Setup –

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in collaboration with Thorsten Lang (next talk – stay tuned!)

arXiv:2305:09650, arXiv:2305.10097 and forthcoming publications.

What is Quantum Geometrodynamics ...

Wheeler, DeWitt, Bergmann, Dirac,... see also Arnowitt et al. '08, and Kiefer '07

- Einstein GR in its Hamiltonian (ADM) form.
- Spatial metric q_{ab} and momentum p^{cd}.
- Constrained system $D_i(q,p) = 0, H(q,p) = 0$.
- Quantization à la Dirac $D_i(\hat{q},\hat{p})\psi = 0$, $H(\hat{q},\hat{p})\psi = 0$.

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... and why does it need to be reanimated?

- Constraints yield ill–defined QFT expressions.
- Operator–ordering and Dirac consistency.
  Tsamis & Woodard '87
- No Hilbert space available.
- Positive definiteness? Klauder 99', Isham & Kakas '84
- Problem of time, ... Kiefer '07, Isham '91

Forward Solution: Discretization

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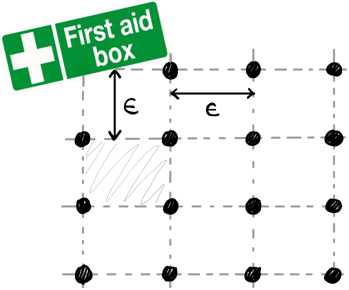

Restrict phase space to piecewise constant fields

$$q_{ab}(x) = \sum_X q_{ab}^X \chi_X(x)$$

with periodic boundary conditions.

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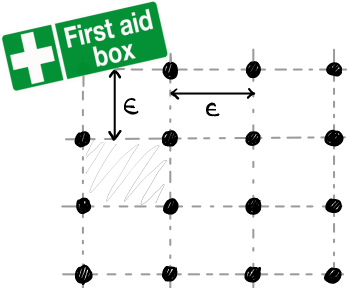

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Calculate constraints on the lattice (Ambiguous!)

$$H(N) := \epsilon^3 \sum_X N^X \left(\frac{1}{\sqrt{q}} \left(q_{ac} q_{bd} - \frac{1}{n-1} q_{ab} q_{cd} \right) p^{ab} p^{cd} - \sqrt{q} R \right)^X$$

$$D(\mathcal{N}) := \epsilon^3 \sum_X \mathcal{N}_X^a \left(-2\Delta_b (q_{ac} p^{cb}) + (\Delta_a q_{bc}) p^{bc} \right)^X$$

Other discretization schemes include Regge calc., CDT and spinfoams (among others), see e.g., Williams '09, Loll '98, '17, Ambjørn '22, Perez '13, Rovelli & Vidotto '20

Constraint Algebra on the Lattice

$$\begin{aligned}\{D(\mathcal{V}), D(\mathcal{W})\} &= D(\mathcal{L}_{\mathcal{V}}\mathcal{W}) + \epsilon\, A_{DD}(\mathcal{V}, \mathcal{W}),\\ \{H(f), H(g)\} &= D(V) + \epsilon\, A_{HH}(f, g),\\ \{D(\mathcal{V}), H(f)\} &= H(\mathcal{L}_{\mathcal{V}} f) + \epsilon\, A_{DH}(f, \mathcal{V}),\end{aligned}$$

where $\mathcal{L}_{\mathcal{V}}\cdot$ is the Lie derivative represented on the lattice with respect to $\mathcal{V}$, and the lattice vector V is given by $V^a_X(q, f, g) := q^{ab}_X \left(f\Delta_b g - g\Delta_b f\right)^X$.*

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Anomalies appear: A_{DD}, A_{HH}, A_{DH}

- Unsurprising and expected – we break general covariance.
- Proportional to ϵ.
- New degrees of freedom.
- Need to examine continuum limit.

*Discrete constrained algebra in other approaches to quantum gravity: Bander '87, Bonzom & Dittrich '13, Friedman & Jack '86, Loll '98, Piran & Williams '86, and references therein.

Quantum Theory at a Point

Schrödinger representation

- Standard CCR $\left[\hat{q}^X_{ab}, \hat{p}^{cd}_Y\right] = \epsilon^{-3}\delta^{(c}_a\delta^{d)}_b\delta^X_Y$
- States with support on non–positive definite metrics!

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A new representation using the Cholesky decomposition

$$q = u^\mathsf{T} u, \quad u \in \mathrm{UT}_+(3, \mathbb{R}),$$

where $\mathrm{UT}_+(3, \mathbb{R})$ is the Lie group of upper triangular matrices with pos. diagonal elements.*

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- Hilbert space: $\mathcal{H}_X = L^2(\mathrm{UT}_+(3, \mathbb{R}), \rho(u)\mathrm{d}u)$.
- Representation of $\hat{q}_{ab}$: $(\hat{q}_{ab}\psi)(u) = q_{ab}(u)\psi(u)$

*Thiemann '23 independently used this triangular gauge.

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Representation of Momenta

First, define generators of shifts in positive q–directions

$$U(s)\hat{q}_{ab}U(s)^{\dagger} = \hat{q}_{ab} + s_{ab}.$$

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The following $U(s)$ does the job:

$$(U(s)\psi)(u) = \sqrt{\frac{\det J_q(u)}{\det J_q(g_s(u))}\frac{\rho(g_s(u))}{\rho(u)}}\psi(g_s(u)),$$

where g_s is a diffeo on $\mathrm{UT}_+(3, \mathbb{R})$ with $g_s(u) = q^{-1}(q(u) + s)$.

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$\{U(s) \in B(\mathcal{H}), s \in \mathbb{R}^6\}$ forms a strongly continuous contraction semigroup.

Quantum Theory on the Lattice

This contraction semigroup admits the infinitesimal generators (Hille–Yosida)

$$i\hat{p}^{cd}\,\psi = \left(\frac{\mathrm{d}}{\mathrm{d}s_{cd}}U(s)\psi\right)_{s_{cd}=0}.$$

With these definitions: $\left[\hat{q}^X_{ab}, \hat{p}^{cd}_Y\right] = \epsilon^{-3}\delta^{(c}_a\delta^{d)}_b\delta^X_Y$.

Total lattice Hilbert space

$$\mathcal{H}_{\text{tot}} := \bigotimes_X L^2(\mathrm{UT}_+(3,\mathbb{R}), \rho(u_X)\,\mathrm{d}u_X)$$

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- All results also apply to $n \neq 3$ dimensions!
- Construction scheme applies to other physical systems.
- Coupling of fermions easily possible.
- Generalized Weyl quantization scheme available.*

*Thorsten's talk.

Summary and Outlook

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- Progress in quantum geometrodynamics stalled...
- ...we suggest some solutions:
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    - Including the constraints and their algebra.
    - Construction of a separable lattice Hilbert space
    - that ensures positive–definiteness of q.

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Outlook

- Examine continuum limit (work in progress).

See arXiv:2305:09650 and arXiv:2305.10097, and Thorsten's talk.

Thank you for your attention!