Quantum Geometrodynamics Revived – Continuum Limit –

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Quantum Gravity 2023, Nijmegen
July 10, 2023

in collaboration with Susanne Schander (previous talk)

arXiv:2305:09650, arXiv:2305.10097 and forthcoming publications.

Quantum Geometrodynamics Revived

Quantum Geometrodynamics faced several mathematical and conceptual problems over the years.
See e.g., DeWitt '67, Isham '91, and Kiefer '07

We propose a new approach:

1. Restrict classical theory to a lattice with finitely many d.o.f.
2. Canonical quantization with representation that enforces positive definiteness
3. Quantization of discretized constraints: Weyl quantization, Approximate Gauge Transformations
4. Continuum Limit

Here, focus on 3 and 4.

Generalized Weyl Quantization

Weyl quantization can be generalized to our new representation of the CCR:

$$Q[f] = \iint_{\mathbb{R}\times\mathbb{R}_+} \tilde{f}(\xi,\kappa) e^{\frac{1}{2}i\xi\kappa} e^{i\xi\hat{q}} U(\kappa) \mathrm{d}\xi \mathrm{d}\kappa + \text{h.c.}$$

This ensures

$$Q\left[(aq+bp)^n\right] = (a\hat{q}+b\hat{p})^n + (a\hat{q}+b\hat{p}^\dagger)^n.$$

Can be used to quantize lattice constraints involving difficult expressions, such as inverse square roots:

$$\hat{H}_n[N_n] = Q[H_n[N_n]]$$

Classical Approximate Gauge Transformations

If constraints form a Lie algebra (e.g. the diffeomorphism constraints), a better way is available. General form of continuum constraint:

$$D[f] = \int_{\mathbb{T}} \mathcal{D}(\phi(x), \partial\phi(x), \pi(x), \partial\pi(x)) f(x) \mathrm{d}x$$

Satisfies first class Poisson bracket algebra:

$$\{D[f], D[g]\} = D[F(f, \partial f, g, \partial g)]$$

Use lattice discretization $\phi_n(x) = \sum_{k=1}^{N_n} \phi_{nk} \chi_{X_k}(x)$. Lattice constraints are given by:

$$D_n[f_n] = \sum_{k=1}^{N_n} \mathcal{D}(\phi_{nk}, \Delta^n \phi_{nk}, \pi_{nk}, \Delta^n \pi_{nk}) f_{nk} \eta_n$$

Algebra on the lattice:

$$\{D_n[f_n], D_n[g_n]\} = D_n[F_n(f_n, \Delta^n f_n, g_n, \Delta^n g_n)] + \eta_n G_n(f_n, \Delta^n f_n, g_n, \Delta^n g_n)$$

Classical Approximate Gauge Transformations

Solve Hamilton's equations of motion on the lattice:

$$\frac{\mathrm{d}\phi_n[g_n]}{\mathrm{d}s} = \{\phi_n[g_n], D_n[f_n]\}$$

Solution only depends on inital data for ϕ_{nk} if $D_n[f_n]$ is of first order in π_{nk}. The Hamiltonian flow $\varphi_s^{D_n[f_n]}$ can be interpreted as an approximate gauge transformation.

Quantum Approximate Gauge Transformations

Define approximate gauge transformation in the quantum theory on the lattice:

$$\left(U\left(\varphi_s^{D_n[f_n]}\right)\psi_n\right)((\phi_{nk})_k) = \sqrt{\det\left(J_{\varphi_s^{D_n[f_n]}}((\phi_{nk})_k)\right)}\,\psi_n(\varphi_s^{D_n[f_n]}((\phi_{nk})_k))$$

Forms a unitary one–parameter group $\Rightarrow$ generator exists
See Thiemann '22 for related approach

Continuum Limit

The Weyl algebra on the lattice is spanned by the exponentiated canonical variables:

$$W_n = \overline{\mathrm{span}\{e^{\hat{\phi}_n[f_n]+\hat{\pi}_n[g_n]}\}}$$

Let $W = \varprojlim W_n$ be the inverse limit with identifications

$$\hat{\phi}_{n+1,2k} f_{n+1,2k} + \hat{\phi}_{n+1,2k+1} f_{n+1,2k+1} \equiv \hat{\phi}_{nk}(f_{n+1,2k} + f_{n+1,2k+1})$$

Choose a sequence ψ_n of states on every lattice. Define

$$\omega_n \left(e^{\hat{\phi}_n[f_n]+\hat{\pi}_n[g_n]} \right) := \left\langle \psi_n, e^{\hat{\phi}_n[f_n]+\hat{\pi}_n[g_n]} \psi_n \right\rangle .$$

If ω_n forms Cauchy sequence, define

$$\omega \left(\lim_{n\to\infty} e^{\hat{\phi}_n[f_n]+\hat{\pi}_n[g_n]} \right) := \lim_{n\to\infty} \omega_n \left(e^{\hat{\phi}_n[f_n]+\hat{\pi}_n[g_n]} \right) .$$

Use GNS–construction to obtain continuum Hilbert space.

Gauge Transformations in the Continuum

Continuum Limit of approximate lattice gauge transformations not easy, since $U_{D_n[f_n]}(s) W_n U_{D_n[f_n]}(-s)$ is not automatically contained in W_n.

If ψ is a state in the continuum Hilbert space, choose a sequence ψ_n such that ω_n converges to the algebraic state defined by ψ.

Expand gauge transformed lattice states using lattice Weyl algebra:

$$U_{D_n[f_n]}(s)\psi_n = \sum_k c_{nk} e^{\hat{\phi}_n[f_{nk}] + \hat{\pi}_n[g_{nk}]} \psi_n$$

Define gauge transformed continuum state as:

$$U_{D[f]}(s)\psi = \lim_{n\to\infty} \sum_k c_{nk} e^{\hat{\phi}_n[f_{nk}] + \hat{\pi}_n[g_{nk}]} \psi$$

Summary and Outlook

Summary

- Generalized Weyl transformation for quantizing H
- Discretized first class constraints generate approximate gauge transformations
- Quantize approximate gauge transformations
- Continuum limit of lattice theories using Cauchy criterion
- Method of taking continuum limit of approximate gauge transformations

Outlook

- Continuum limit of lattice Hamiltonian constraints
- Supply missing estimates (Use convergence results in numerical relativity)

See arXiv:2305:09650 and arXiv:2305.10097, and Susanne's talk.

Thank you for your attention!