# High-energy properties of the matter-graviton scattering in quadratic gravity

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@Radboud University（Netherlands）

Yugo Abe（National Institute of Technology, Miyakonojo College, Japan）

阿部裕悟　(都城高専, 日本)

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In collaboration with

Takeo Inami (RIKEN), Keisuke Izumi (Nagoya U, KMI)

# Plans

**Plans of my presentation**

1. **Introduction (Purpose＆Our 3-Questions)（7 page）**
2. **Perturbative S-matrix unitarity & Matter-graviton scattering in $R_{\mu\nu}^2$ gravity（8 page）**
3. **Summary & Outlook （1+1 page）**

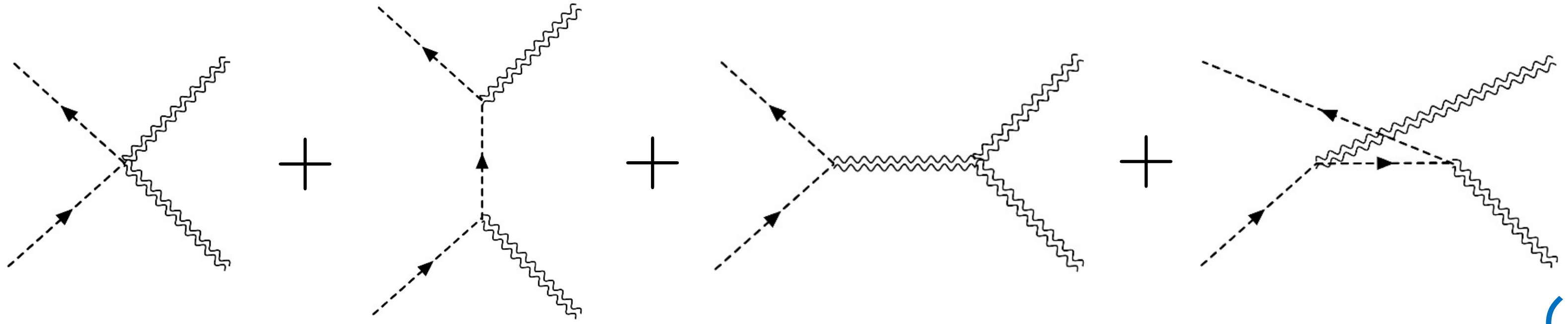
# Introduction：UV behavior of perturbative theory

・We know that Einstein gravity is non-renormalizable theory.

$\Leftrightarrow$ The perturbation theory of Einstein gravity breaks down at high-energy regions.

・Tree level high-energy unitarity of scattering amplitudes reflect UV properties of theory.

Consider elastic scalar matter-graviton scattering at tree-level, the sum of these amplitudes is



$\Rightarrow$ $\mathcal{M}(\phi g \to \phi g) \sim E^2$

（BERENDS, and GASTMANS　1974）

# Introduction : UV behavior of perturbative theory

- **We know that Einstein gravity is non-renormalizable theory.**

$\Longleftrightarrow$ **The perturbation theory of Einstein gravity breaks down at high-energy regions.**

- **Tree level high-energy unitarity of scattering amplitudes reflect UV properties of theory.**

**Including other amplitude calculations, it is found that all tree-level amplitudes grow like $E^2$.**

$$\left.\begin{aligned} \mathcal{M}(\phi g \to \phi g) &\sim E^2 \\ \mathcal{M}(\gamma g \to \gamma g) &\sim E^2 \\ \mathcal{M}(g g \to g g) &\sim E^2 \end{aligned}\right\} \quad \cancel{\text{const.} > \left|\mathcal{M}^{(\text{Tree})}(\alpha \to \beta)\right|} \quad @\ E \to \infty$$

**Perturbative unitarity (= unitarity bound) is not satisfied !**

$\Longleftrightarrow$ **The perturbative theory of Einstein gravity breaks down at the high energy limit.**

***Question 1. In quantum gravity,***
***can perturbative unitarity be used to determine a renormalizable theory ?***

# Introduction：UV behavior of perturbative theory

・Perturbative unitarity (＝Unitarity bound) can be verification of UV renormalizability.

**Quantum Field Theory (Gauge Theory)**

| Massive vector theory | → | Weinberg-Salam model |
|---|---|---|
| ・non-renormalizable | | ・renormalizable |
| ・NO Perturbative unitarity | | ・Perturbative unitarity is satisfied |

Perturbative Unitarity is cured by SSB.（Cornwall, Levin, and Tiktopoulos 1974）

・If we consider the history of particle physics, similar questions have arisen.

To UV complete the electroweak sector,
we needed to find a way to cure divergent scattering amplitudes like massive vector boson.
The Higgs boson was chosen in the Standard Model as a solution to these problems
and to provide an explicit origin for electroweak symmetry breaking.

# Introduction：UV behavior of perturbative theory

・Einstein gravity is non-renormalizable theory and lacks perturbative unitarity.
We need UV complete theory for evaluation of the quantum gravity corrects.
→ A few (many!?) alternatives has been studied.
Our focus on quantum quadratic gravity theory, because renormalizable is known.
We investigate how perturbative unitarity can be cured in the context of quantum gravity.

**Quantum Gravity**

Einstein gravity
・non-renormalizable
・NO Perturbative unitarity
（BERENDS, and GASTMANS 1974）

Quadratic gravity （$R_{\mu\nu}^2$ gravity）
・renormalizable
・Perturbative unitarity is satisfied？
or What is required？（This Talk！）

Another approaches

**String**
Perturbative unitarity require Higher-mass/Higher-spin states
（Arkani-Hamed, and Yu-tin Huang 2016～）

# Introduction : $R_{\mu\nu}{}^2$ gravity theory has ghost mode

**Ghost problem：Graviton propagator has negative norm in $R_{\mu\nu}{}^2$ gravity.**

$$S = \int d^4x\sqrt{-g}\left(-\frac{1}{16\pi G}R + \alpha R^2 + \beta R_{\mu\nu}^2\right)$$

**These terms improves UV behavior in perturbative quantum gravity.**

**However,**

**Graviton Propagator：**  $\propto \dfrac{1}{-k^2 + \dfrac{k^4}{M^2}} = -\dfrac{1}{k^2} + \underline{\dfrac{1}{k^2 + M^2}}$

**ghost mode exist！**

**This unphysical mode violates "unitarity".**

***Question 2.　Does $R_{\mu\nu}{}^2$ gravity satisfy with perturbative unitarity？***

# Introduction : unitarity bound and renormalizability

- Tree-level unitarity is a good tool to investigate perturbative UV completion.

high energy behavior of tree level amplitudes $\simeq$ UV renormalizability

Unitarity bound（Perturbative unitarity） → Evaluation of renormalizability

**No counter example!**

| Theory | Unitarity bound | UV renormalizability |
|---|---|---|
| QED | satisfied | renormalizable |
| Yang-Mills theory | satisfied | renormalizable |
| Weinberg-Salam model | satisfied | renormalizable |
| 4-Fermi theory | not satisfied | non-renormalizable |
| Massive vector theory | not satisfied | non-renormalizable |
| Einstein gravity | not satisfied | non-renormalizable |

*Question 3. Is $R_{\mu\nu}{}^2$ gravity, which has ghost, a counter example in the correspondence between unitarity bound and UV renormalizability ?*

# Introduction：our investigation

**3-Questions**

*Question 1.　In quantum gravity,*
*can perturbative unitarity be used to determine a renormalizable theory？*

*Question 2.　Does $R_{\mu\nu}^2$ gravity satisfy with perturbative unitarity？*

*Question 3.　Is $R_{\mu\nu}^2$ gravity, which has ghost, a counter example*
*in the correspondence between unitarity bound and UV renormalizability？*

**Focus**

**Understanding tree-level approximation of S-matrix unitarity**

**Our Answer**

**We can extend perturbative unitarity to negative norm theory**

**Tree level amplitudes of matter-graviton scattering in $R_{\mu\nu}^2$ gravity**

**Our new idea：Perturbative S-matrix unitarity**

# Plans

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# Tree-level approximation of S-matrix unitarity

- **I will attempt to explain how the unitarity bound is derived from the optical theorem.**

**From** $\mathcal{S}\mathcal{S}^\dagger = 1$**, we get the optical theorem.**

$\mathcal{S} = 1 + i\mathcal{T}$

$$\mathcal{S}\mathcal{S}^\dagger = 1 \;\Leftrightarrow\; -i\left(\mathcal{T} - \mathcal{T}^\dagger\right) = \mathcal{T}\mathcal{T}^\dagger \;\Rightarrow\; 2\,\mathrm{Im}(T_{\alpha\alpha}) = \sum_\gamma |T_{\alpha\gamma}|^2$$ **: Optical Theorem**

**We consider the perturbative expansion of T-matrix.** $\therefore\ 2\mathrm{Im}T^{(1\text{-loop})}_{\alpha\alpha} = \sum_\gamma \left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2$

**If the theory behaves perturbatively,**

$$\left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right| \gg \left|T^{(1\text{-loop})}_{\alpha\alpha}\right| \;\Rightarrow\; \left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right| \gg \frac{1}{2}\sum_\gamma \left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2 \;\Rightarrow\; \mathrm{const.} > \left|T^{(\mathrm{Tree})}_{\beta\alpha}\right|$$

**Unitarity bound**
**(@high energy limit)**

**In scattering amplitudes point of view, this is evaluation of** $\mathrm{const.} > \left|\mathcal{M}^{(\mathrm{Tree})}(\alpha \to \beta)\right|$.

(**Ex. W-S model's unitarity bound** $\mathrm{const.} > \left|\mathcal{M}^{(\mathrm{Tree})}(W^+W^- \to W^+W^-)\right|$ **is satisfied@HE limit.**)

# What does unitarity mean?

Usual "unitarity" used in QFT means the following two elements

- S-matrix unitarity

$$S^\dagger S = 1$$

&

- positive norm

$$\langle \psi | \psi \rangle > 0$$ (for any $|\psi\rangle$)

Ghost mode violates "unitarity" $=$ a lack of positivity.

Our expectation is the following relationships

In perturbation theory, regardless of the norm positivity,

S-matrix unitarity $\Longleftrightarrow$ UV renormalizability

# Lack of positivity

- **What does the lack of "positivity" give the discussion of unitarity？**

**Optical theorem of a positive norm theory：** $2\ \mathrm{Im}(T_{\alpha\alpha}) = \sum_{\gamma} |T_{\alpha\gamma}|^2$

**Optical theorem of a negative norm theory：** $2\ \mathrm{Im}(T_{\alpha\alpha}) = \sum_{\gamma} n_X\ |T_{\alpha\gamma}|^2$

**The coefficient is determined by the number of ghosts included in the total number of states in the initial state $\alpha$ and the arbitrary state $\gamma$.**

$$\begin{cases} \text{even :}\ n_X = 1 \\ \text{odd :}\ n_X = -1 \end{cases}$$

- **This coefficient arising a problem when deriving unitarity bound from the OT.**

# Positive norm vs Negative norm

- **a positive norm theory**

**Optical theorem** : $2\,\mathrm{Im}(T_{\alpha\alpha}) = \sum_\gamma |T_{\alpha\gamma}|^2$

$$\Rightarrow \quad \left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right| \gg \underline{\frac{1}{2}\sum_\gamma \left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2}$$

sum of all positive values

$\Rightarrow$ If we choose a specific state $\alpha$, this is part of the summation.

$$\therefore \quad \underline{\sum_\gamma \left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2 > \left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right|^2}$$

This inequality holds
because all the terms on LHS are positive!

$\Rightarrow$ $\mathrm{const.} > \left|T^{(\mathrm{Tree})}_{\beta\alpha}\right|$ : **Unitarity bound**

(evaluation of perturbation theory)

- **a negative norm theory**

**Optical theorem** : $2\,\mathrm{Im}(T_{\alpha\alpha}) = \sum_\gamma n_X\,|T_{\alpha\gamma}|^2$

$$\Rightarrow \quad \left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right| \gg \underline{\frac{1}{2}\sum_\gamma n_X \left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2}$$

There is a negative term in the summation.

$$\cancel{\sum_\gamma n_X \left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2 > \left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right|^2}$$

The inequality for a particular state does not hold！
Because negative terms appears in the summation on LHS.

$\Rightarrow$ **Unitarity bound cannot be derived！**

＝Unitarity bound does not make sense！

What should we consider？

# If negative norms exist?

- <u>a positive norm theory</u>

**Optical theorem** ：$2\ \mathrm{Im}(T_{\alpha\alpha}) = \sum_\gamma |T_{\alpha\gamma}|^2$

$$\Rightarrow \quad \left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right| \gg \frac{1}{2}\sum_\gamma \left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2$$

- <u>a negative norm theory</u>

**Optical theorem** ：$2\ \mathrm{Im}(T_{\alpha\alpha}) = \sum_\gamma n_X\, |T_{\alpha\gamma}|^2$

$$\Rightarrow \quad \left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right| \gg \frac{1}{2}\sum_\gamma n_X \left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2$$

- **These inequalities can be derived from the optical theorem in which theories.**

**If we consider a negative norm theory, for evaluation of $S^\dagger S = 1$,**

**we should evaluate this inequality** $\left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right| \gg \frac{1}{2}\sum_\gamma n_X \left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2$**.**

**Whether this inequality is satisfied @ high-energy limit**

**= a generalization of unitarity bound**

**We refer to it as perturbative S-matrix unitarity.**

# Perturbative S-matrix unitarity

**Perturbative theory**

Unitarity bound (perturbative unitarity) $\simeq$ Evaluation of renormalizability

We extended this relation to negative norm theory！

**Our new conjecture**

| | Tree level amplitudes | | | | loop |
|---|---|---|---|---|---|
| (External lines) | Unitarity bound | | Perturbative S-matrix unitarity | | UV renormalizability |
| not including negative norm | satisfied | $\longrightarrow$ can guarantee | satisfied | $\longleftrightarrow$ | renormalizable |
| | not satisfied | $\longrightarrow$ | not satisfied | $\longleftrightarrow$ | non renormalizable |
| including negative norm | not satisfied | $\cancel{\longleftrightarrow}$ unrelated | satisfied | $\longleftrightarrow$ | renormalizable |
| | | | not satisfied | $\longleftrightarrow$ | non renormalizable |

Verification of renormalizability is possible by evaluating tree-level scattering amplitudes.

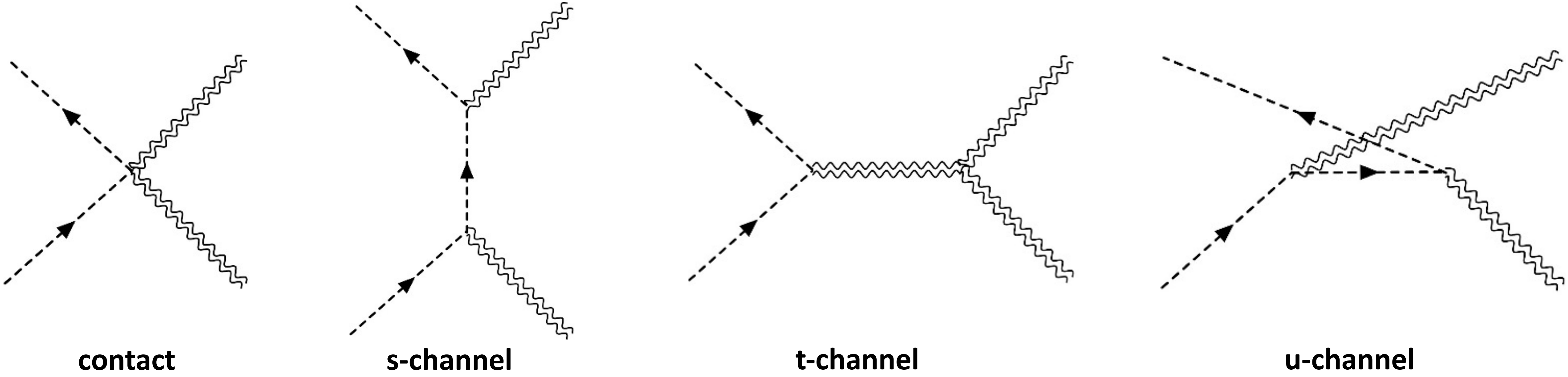
# Scalar matter–graviton scattering in $R_{\mu\nu}{}^2$ gravity

- **We study $R_{\mu\nu}{}^2$ gravity action with scalar matter**

$$S=\int d^4x\sqrt{-g}\left(-\frac{1}{16\pi G}R+\alpha R^2+\beta R_{\mu\nu}^2+\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi-\frac{1}{2}m^2\phi^2-\frac{1}{4!}\lambda\phi^4+\xi\phi^2 R\right)$$

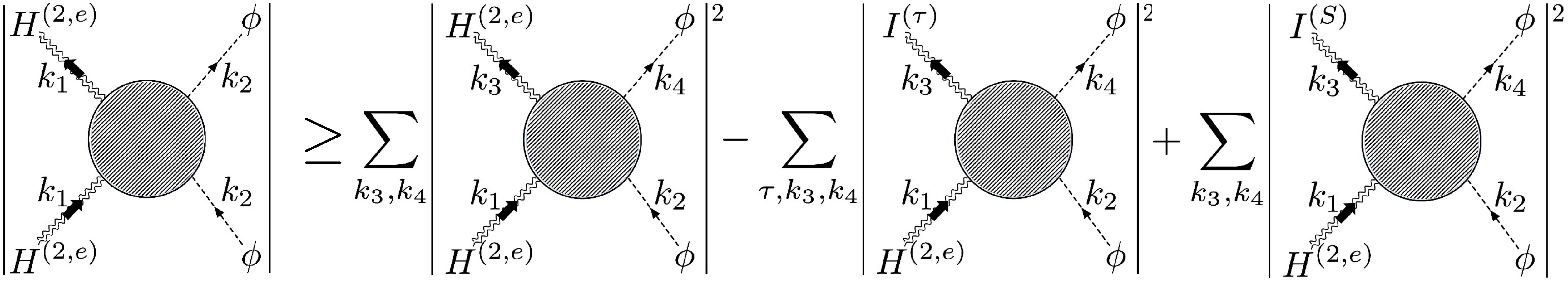
**Graviton field：** $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu},\ \ \eta_{\mu\nu}$ **is flat metric.**

**Consider elastic scalar matter-graviton scattering：** $\phi+h_{\mu\nu}\to\phi+h_{\mu\nu}$



# Our results of scalar matter-graviton scattering

**inequalities of two-particle scattering** **(at Tree level)**




**massless graviton (2DOF)** **massive ghost (5DOF)** **scalar graviton (1DOF)**

**(LHS)** $\sim \mathcal{O}(k^2)$ **(RHS)** $\sim \mathcal{O}(k^2)$ ： $\mathcal{O}(k^4)$ **term is non-trivial cancellation！**

$(k \sim E)$

$\boxed{\mathcal{O}(E^2) \quad \geq \quad \mathcal{O}(E^2)}$ **is satisfied！ (@$E \to \infty$)** ⇒ **Perturbative S-matrix unitarity is satisfied.**

**including the coefficient**

**(Scattering processes in other initial states have similar results. )**

# Summary

Tree level amplitudes ($\phi + h_{\mu\nu} \to \phi + h_{\mu\nu}$ scattering) | loop level

| | Unitarity bound | Perturbative S-matrix unitarity | UV renormalizability |
|---|---|---|---|
| Einstein gravity + scalar | not satisfied ➡ | not satisfied ⬌ | non renormalizable |
| $R_{\mu\nu}{}^2$ gravity + scalar | | satisfied ⬌ | renormalizable |

## Summary of our work

- Not only in particle physics but also in a quantization of gravity, tree level scattering is very useful to evaluate UV behavior of perturbation.
- In particular, the behavior of perturbative S-matrix unitarity at high energy limit is deeply related to UV renormalizability of a perturbative theory including a negative norm state.

# Outlook

- **Gravitational quantum corrections to Higgs/inflaton physics.**

  **Scalar matter-graviton scattering：**$\phi + h_{\mu\nu} \to \phi + h_{\mu\nu}$

  **is necessary to Higgs-graviton/inflaton-graviton scattering.**

- **Ghost mode is necessary to perturbative S-matrix unitarity. However, can it be experimentally observed?**

  **Discusses the experimental treatment of ghost mode.**
  **Experimentally, the ghost mode is not in asymptotic state,**
  **thus, it cannot be observed independently.**
  **It means that the graviton scattering needs to consider**
  **the gravitational parton shower as the exclusive scattering.**
  **「Ultra-Planckian scattering from a QFT for gravity」（Bob Holdom 2021）**

# Back up : $R_{\mu\nu}{}^2$ gravity theory

- **Higher derivative gravity theory (Stelle gravity/Quadratic gravity/$R_{\mu\nu}{}^2$ gravity)**

$$S=\int d^4x\sqrt{-g}\left(-\frac{1}{16\pi G}R+\alpha R^2+\beta R_{\mu\nu}^2\right)$$ : **renormalizable**

**a finite number of counter terms**

**The mass dimension of the coupling constant :** $[\alpha]=0,[\beta]=0\geq 0$

- **General Relativity (Eintstein gravity)**

$$S_{\mathrm{EH}}=-\frac{1}{16\pi G}\int d^4x\sqrt{-g}R$$ : **non-renormalizable**

**an infinite number of counter terms**

**The mass dimension of the coupling constant :** $[G]=-2<0$

**behavior of propagator**
- **Einstein gravity :** $\mathcal{O}(1/p^2)$
- **$R_{\mu\nu}{}^2$ gravity :** $\mathcal{O}(1/p^4)$ → **Improved UV behavior**

# Back up : Propagator of $R_{\mu\nu}{}^2$ gravity

**Graviton propagator in $R_{\mu\nu}{}^2$ gravity**

$$G_{\mu\nu,\alpha\beta} = \frac{2}{\beta p^4 + \kappa^{-2} p^2} P^{(2)}_{\mu\nu,\alpha\beta} + \frac{1}{2(3\alpha+\beta)p^4 - \kappa^{-2}p^2} P^{(0)}_{\mu\nu,\alpha\beta},$$

$$P^{(2)}_{\mu\nu,\alpha\beta} := \frac{1}{2}\left(\theta_{\mu\alpha}\theta_{\beta\nu} + \theta_{\mu\beta}\theta_{\alpha\nu}\right) - \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta},$$

$$P^{(0)}_{\mu\nu,\alpha\beta} := \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta},$$

**Propagator in Einstein gravity is regained by setting** $\alpha = \beta = 0$ **.**

$$\begin{aligned} G^{\mathrm{E}}_{\mu\nu,\alpha\beta} &= \frac{2\kappa^2}{p^2} P^{(2)}_{\mu\nu,\alpha\beta} - \frac{\kappa^2}{p^2} P^{(0)}_{\mu\nu,\alpha\beta} \\ &= \frac{\kappa^2}{p^2}\left(\theta_{\mu\alpha}\theta_{\beta\nu} + \theta_{\mu\beta}\theta_{\alpha\nu} - \theta_{\mu\nu}\theta_{\alpha\beta}\right) \end{aligned}$$

# Back up : Renormalizable theory

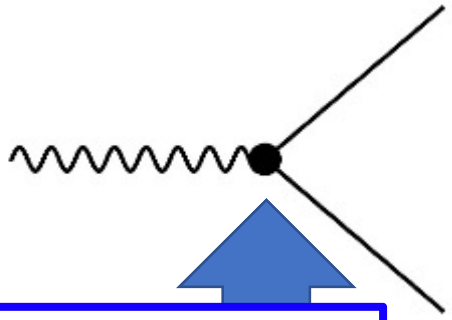
**Example. Quantum electrodynamics (QED) ➡ Unitarity bound is satisfied.**

**terms in the Lagrangian**

**kinetic term of $A_\mu$**

**kinetic term of $e$**

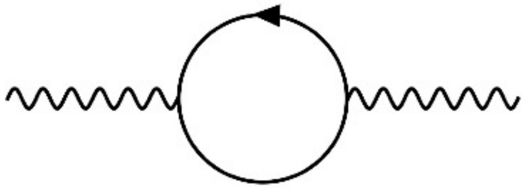
**interaction term**



**The divergences of 2- and 3-point from quantum correction can be absorbed by theoretical parameters.**

**divergent operators**

**2-point amplitude**

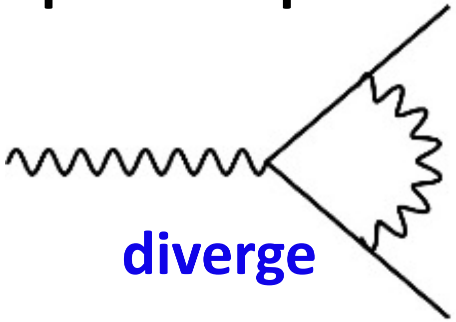


**diverge**

**2-point amplitude**

**diverge**

**3-point amplitude**



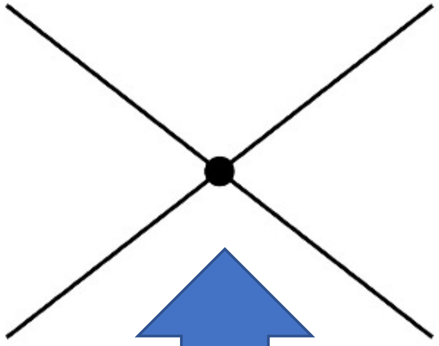
**diverge**

**(4-point amplitude or more does not diverge : finite)**

# Back up : Non-renormalizable theory

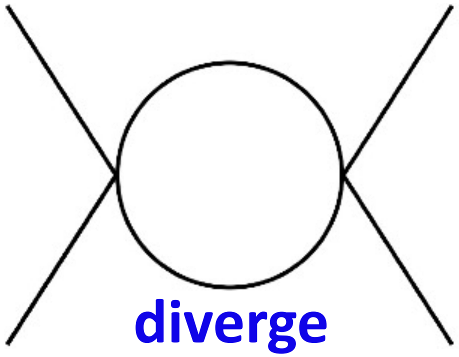
**Example. 4-Fermi theory** ➡ **Unitarity bound is not satisfied.**

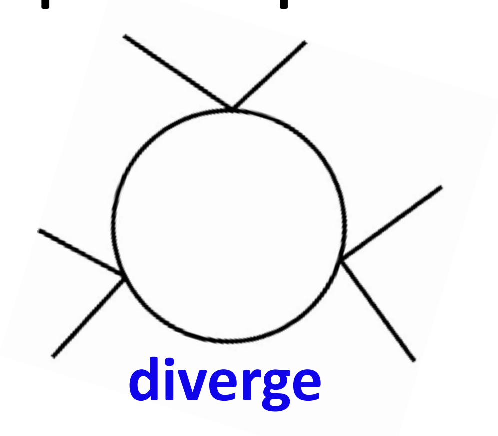
**terms in the Lagrangian** **interaction term**

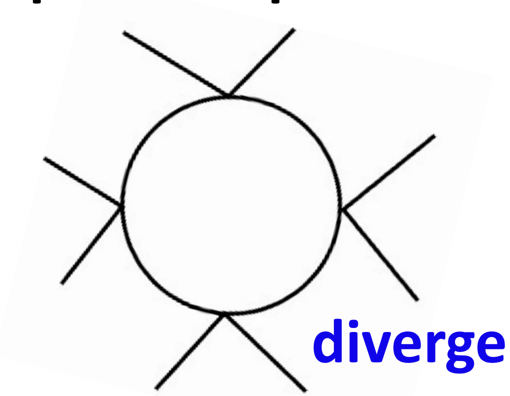


**can be absorbed・・・** **can not be absorbed！**

**divergent operators** **4-point amplitude** **6-point amplitude** **8-point amplitude**



# Back up : Renormalizability

If we calculating quantum correction as a perturbative theory,
divergences generally occur from Feynman diagrams with a loop.

At this time,
the number of divergent terms $\leq$ the number of terms in the original Lagrangian
all divergences can be absorbed.

$\Rightarrow$ Only a finite number of counter terms are sufficient： renormalizable
If we need to add infinite counter terms： non renormalizable

*Question,* Why do we need a renormalizable theory？

*Answer,* Higher-order quantum corrections can be controlled.
＝ The theory can give predictions at the quantum level！

# Back up : S-matrix unitarity

**The S-matrix also can be expressed as using the S-matrix operator $\mathcal{S}$ which is an interaction picture.**

$$S_{\beta\alpha} = \langle\ \beta\ ;\mathrm{out}|\ \alpha\ ;\mathrm{in}\rangle \equiv \langle\ \beta\ ;\mathrm{in}|\mathcal{S}|\ \alpha\ ;\mathrm{in}\rangle$$

**The S-matrix is a unitary matrix.**

$$\sum_{\gamma} S_{\beta\gamma}S^{*}_{\alpha\gamma} = \sum_{\gamma} S^{*}_{\gamma\beta}S_{\gamma\alpha} = \delta_{\beta\alpha}$$

$$\Rightarrow \quad \underline{\mathcal{S}\mathcal{S}^{\dagger} = \mathcal{S}^{\dagger}\mathcal{S} = \mathbf{1}}$$

**This means conservation of probability.**

**(One of the important properties of quantum theory)**

# Back up : T-matrix

**The S-matrix contains all scattering processes.**

$$\mathcal{S} = \mathbf{1} + i\underline{\mathcal{T}} \qquad (S_{\beta\alpha} = \delta_{\beta\alpha} + iT_{\beta\alpha})$$

**No Interaction process** **Interaction process**

**We are interested in this non-trivial process！**

**The unitarity of $\mathcal{S} = 1 + i\mathcal{T}$ implies the non-linear relation for $\mathcal{T}$.**

$$\mathcal{S}\mathcal{S}^{\dagger} = \mathbf{1} \quad \Leftrightarrow \quad -i\left(\mathcal{T} - \mathcal{T}^{\dagger}\right) = \mathcal{T}\mathcal{T}^{\dagger}$$

# Back up : Optical theorem

**The optical theorem is a straightforward consequence of the unitarity of the S-matrix.**

$$-i\left(\mathcal{T}-\mathcal{T}^{\dagger}\right)=\mathcal{T}\mathcal{T}^{\dagger}$$

$$\Rightarrow \quad -i\left(T_{\beta\alpha}-T^{*}_{\alpha\beta}\right)=\sum_{\gamma}T_{\beta\gamma}T^{*}_{\alpha\gamma}$$ **($\gamma$ : all intermediate states)**

**We consider that the initial state $\alpha$ and the final state $\beta$ are the same.**

**(= elastic forward scattering)**

$$-\,i\left(T_{\alpha\alpha}-T^{*}_{\alpha\alpha}\right)=\sum_{\gamma}T_{\alpha\gamma}T^{*}_{\alpha\gamma}$$

$$\therefore \quad 2\,\mathrm{Im}(T_{\alpha\alpha})=\sum_{\gamma}\left|T_{\alpha\gamma}\right|^{2}$$ **: The optical theorem**

# Back up : Tree unitarity

「Derivation of gauge invariance from high-energy unitarity bounds on the S matrix」
（Cornwall, Levin, and Tiktopoulos 1974）

**Using the perturbative expansion of $\mathcal{T}$**
**in coupling $g$ parametrizing the interaction strength,**

$$\mathcal{T} = g^n \cdot \mathcal{T}^{(\text{Tree})} + g^{2n} \cdot \mathcal{T}^{(\text{1-loop})} + g^{3n} \cdot \mathcal{T}^{(\text{2-loop})} + \cdots$$

**At the two lowest orders in the coupling $g$ expansion $-i\left(\mathcal{T} - \mathcal{T}^\dagger\right) = \mathcal{T}\mathcal{T}^\dagger$ states**

$$\Rightarrow \begin{cases} \mathcal{T}^{(\text{Tree})} = \mathcal{T}^{\dagger(\text{Tree})} \\ -i\left(\mathcal{T}^{(\text{1-loop})} - \mathcal{T}^{\dagger(\text{1-loop})}\right) = \mathcal{T}^{(\text{Tree})}\mathcal{T}^{(\text{Tree})} \end{cases}$$

**The optical theorem then becomes** $2\mathrm{Im}T^{(\text{1-loop})}_{\alpha\alpha} = \sum_\gamma \left|T^{(\text{Tree})}_{\gamma\alpha}\right|^2$

# Back up : Tree unitarity

**If we consider that the perturbation theory holds correctly,** $\left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right| \gg \left|T^{(\text{1-loop})}_{\alpha\alpha}\right|$

$T^{(\text{1-loop})}_{\alpha\alpha}$ **is expressed by the real and the imaginary part.** $\left|T^{(\text{1-loop})}_{\alpha\alpha}\right|^2 > \left|\mathrm{Im}T^{(\text{1-loop})}_{\alpha\alpha}\right|^2$

**The optical theorem** $2\mathrm{Im}T^{(\text{1-loop})}_{\alpha\alpha} = \sum_{\gamma}\left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2$ **gives** $\mathrm{Im}T^{(\text{1-loop})}_{\alpha\alpha} > 0$

$$\left|T^{(\text{1-loop})}_{\alpha\alpha}\right| > \mathrm{Im}T^{(\text{1-loop})}_{\alpha\alpha}$$

$$\Rightarrow \quad \left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right| \gg \left|T^{(\text{1-loop})}_{\alpha\alpha}\right| > \mathrm{Im}T^{(\text{1-loop})}_{\alpha\alpha}$$

$$\Rightarrow \quad \left|T^{(\mathrm{Tree})}_{\alpha\alpha}\right| \gg \frac{1}{2}\sum_{\gamma}\left|T^{(\mathrm{Tree})}_{\gamma\alpha}\right|^2$$

**: Whether this inequality is satisfied.**
**@high-energy limit (＝Tree unitarity)**

# Back up : Tree unitarity

**Question.** **What does it mean that tree unitarity is not satisfied？**

① $\left|T_{\alpha\alpha}^{(\text{Tree})}\right| \gg \left|T_{\alpha\alpha}^{(\text{1-loop})}\right|$ **: This is false. (= violation of perturbation theory)**

② $2\text{Im}T_{\alpha\alpha}^{(\text{1-loop})} = \sum_{\gamma} \left|T_{\gamma\alpha}^{(\text{Tree})}\right|^2$ **: This is false. ~~(= violation of optical theorem)~~**

③ $\left|T_{\alpha\alpha}^{(\text{1-loop})}\right|^2 > \left|\text{Im}T_{\alpha\alpha}^{(\text{1-loop})}\right|^2$ **: This is false. ~~(= violation of Mathematics)~~**

**Answer.** **It means that the theory does not behave perturbatively.**

# Back up : Unitarity bound

$$\left| T^{(\mathrm{Tree})}_{\alpha\alpha} \right| \gg \frac{1}{2} \underbrace{\sum_{\gamma} \left| T^{(\mathrm{Tree})}_{\gamma\alpha} \right|^2}$$

**summation of all arbitrary state**

**If we choose a specific state $\alpha$, this is part of the summation.**

$$\sum_{\gamma} \left| T^{(\mathrm{Tree})}_{\gamma\alpha} \right|^2 > \left| T^{(\mathrm{Tree})}_{\alpha\alpha} \right|^2$$

$$\therefore \quad \left| T^{(\mathrm{Tree})}_{\alpha\alpha} \right| \gg \frac{1}{2} \sum_{\gamma} \left| T^{(\mathrm{Tree})}_{\gamma\alpha} \right|^2 > \frac{1}{2} \left| T^{(\mathrm{Tree})}_{\alpha\alpha} \right|^2$$

$$\therefore \quad \mathrm{const.} > \left| T^{(\mathrm{Tree})}_{\alpha\alpha} \right|$$

# Back up : Unitarity bound

$$\text{const.} > \left|T^{(\text{Tree})}_{\alpha\alpha}\right| \quad \Rightarrow \quad \text{const.} > \left|T^{(\text{Tree})}_{\alpha\alpha}\right| \gg \underline{\frac{1}{2}\sum_{\gamma}\left|T^{(\text{Tree})}_{\gamma\alpha}\right|^2}$$

summation of all arbitrary state

⇒ **If we choose a specific state $\beta$, this is part of the summation.**

$$\therefore \quad \text{const.} > \left|T^{(\text{Tree})}_{\alpha\alpha}\right| \gg \frac{1}{2}\sum_{\gamma}\left|T^{(\text{Tree})}_{\gamma\alpha}\right|^2 > \frac{1}{2}\left|T^{(\text{Tree})}_{\beta\alpha}\right|^2$$

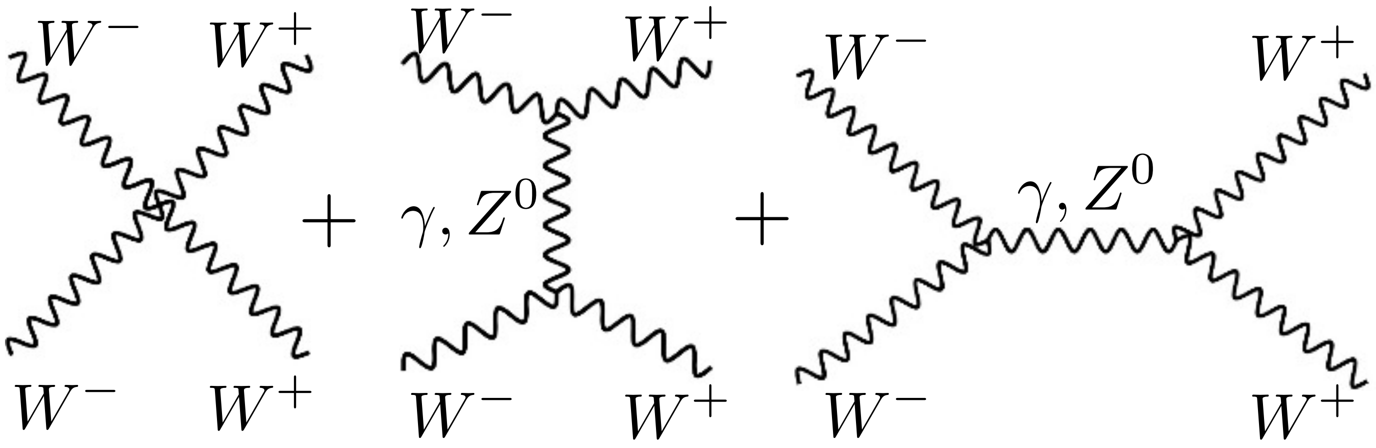
$$\therefore \quad \text{const.} > \left|T^{(\text{Tree})}_{\beta\alpha}\right|$$ **: Unitarity bound (@high-energy limit)**

**Unitarity bound is derived from originally $\mathcal{S}\mathcal{S}^\dagger = 1$.**

# Back up : Phenomenology

**Example. Massive vector theory → Weinberg-Salam model**

$W^+W^- \to W^+W^-$ **: Longitudinal mode**

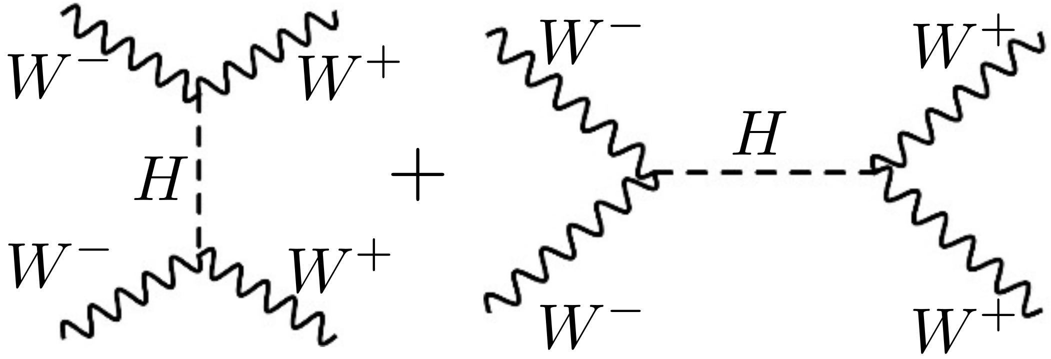



$$\mathcal{M}_{\mathrm{Longitudinal}}(W^+W^- \to W^+W^-)$$

$$\approx g_4\left(\frac{E^4}{M_W^4}\right) - g_3^2\left(\frac{E^4}{M_W^4}\right) + g_3^2\left(\frac{E^2}{M_W^2}\right)$$

$E^4$ **behavior canceled !**

**: Higgs exchanges**




$$\mathcal{M}_{\mathrm{Higgs\text{-}exchange}}(W^+W^- \to W^+W^-)$$

$$\approx -g_{HWW}^2\left(\frac{E^2}{M_W^2}\right)$$

$E^2$ **behavior canceled !**

**Unitarity bound is OK! (cured by SSB)**

$$\therefore\ \mathrm{const.} > \left|\mathcal{M}^{(\mathrm{Tree})}(W^+W^- \to W^+W^-)\right|$$

# Back up : Phenomenology

## Example. Massive vector theory → Weinberg-Salam model

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**Weak interactions at very high energies: The role of the Higgs-boson mass**

Benjamin W. Lee,* C. Quigg,† and H. B. Thacker
*Fermi National Accelerator Laboratory,*‡ *Batavia, Illinois 60510*

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We give an $S$-matrix-theoretic demonstration that if the Higgs-boson mass exceeds $M_c = (8\pi\sqrt{2}/3G_F)^{1/2}$, parital-wave unitarity is not respected by the tree diagrams for two-body scattering of gauge bosons, and the weak interactions must become strong at high energies. We exhibit the relation of this bound to the structure of the Higgs-Goldstone Lagrangian, and speculate on the consequences of strongly coupled Higgs-Goldstone systems. Prospects for the observation of massive Higgs scalars are noted.

**Unitarity bound limits the Higgs mass less than 1000[GeV]. (1977)**

**1983 Discovery of $W$ boson and $Z$ boson (SPS@CERN)**

**2012 Discovery of Higgs Boson (LHC@CERN)**