Gaussian states in group field theory and semiclassical properties

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Outline

1. GFT cosmology: emergence of bouncing FLRW
2. Semiclassical properties of quantum states
   - Coherent and squeezed states
3. Gaussian states for GFT
4. Algebraic approach and GS
5. Conclusions

GFT canonical quantisation [Oriti, Gielen, Sindoni, Wilson-Ewing, ...]

- *Simplicial* (discrete) gravity coupled to a massless scalar χ as matter
- Quantise **free** (single-mode) GFT using χ as clock: deparametrisation
- Construct Fock space (with $[\hat{a}(\chi), \hat{a}^\dagger(\chi)] = 1$)

$$\hat{a}(\chi)|0\rangle = 0\,, \qquad\qquad |\triangle\rangle = \hat{a}^\dagger(\chi)|0\rangle$$

- Relational Hamiltonian $\hat{H}$ evolves the volume operator $\hat{V}$

$$\hat{H} = -\frac{1}{2}\omega\left(\hat{a}^{\dagger 2} + \hat{a}^2\right)\,, \qquad\qquad \hat{V}(\chi) = v\,\hat{N}(\chi) = v\,\hat{a}^\dagger(\chi)\hat{a}(\chi)$$

ω GFT coupling and v volume of one GFT quantum

Bouncing FLRW cosmology (**what states** for $\langle\hat{V}\rangle$?)

$$\left(\frac{1}{\langle\hat{V}(\chi)\rangle}\frac{\mathrm{d}\langle\hat{V}(\chi)\rangle}{\mathrm{d}\chi}\right)^2 = 4\omega^2\left(1 + \frac{v}{\langle\hat{V}(\chi)\rangle} - \frac{v^2}{\langle\hat{V}(\chi)\rangle^2}\mathcal{I}(0)\right)$$

Agreement with GR $(V'/V)^2 = 12\pi G$ at late times requires identification $\omega^2 = 3\pi G$

with $\mathcal{I}(0)$ initial conditions

Semiclassical properties of quantum states

We demand semiclassical states to have **small quantum fluctuations**

$$\frac{(\Delta\hat{V}_\chi)^2}{\langle\hat{V}_\chi\rangle^2} \quad \text{and} \quad \frac{(\Delta\hat{H})^2}{\langle\hat{H}\rangle^2}$$

Can be made arbitrarily small

Saturation of the Robertson–Schrödinger (RS) uncertainty principle is sometimes also invoked as a semiclassical feature

$$(\Delta\hat{V}_\chi)^2(\Delta\hat{H})^2 \geq |\Delta(\hat{V}_\chi\hat{H})|^2 + \omega^2\langle\hat{C}_\chi\rangle^2$$

where $(\Delta\hat{A})^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$ and $\Delta(\hat{A}\hat{B}) = \frac{1}{2}\langle\{\hat{A},\hat{B}\}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$

Candidate states

- **(Fock) coherent states** $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ (CS)
- **Squeezed states** $|z\rangle = \hat{S}(z)|0\rangle$ (Perelomov-Gilmore CS) (SS)

where $\hat{D}(\alpha) = \exp\big(\alpha\hat{a}^\dagger - \overline{\alpha}\hat{a}\big)$ and $\hat{S}(z) = \exp\big(\frac{1}{2}(z\hat{a}^{\dagger 2} - \overline{z}\hat{a}^2)\big)$

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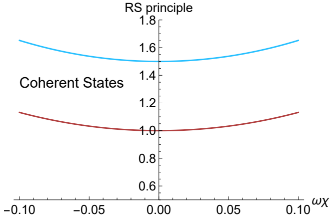
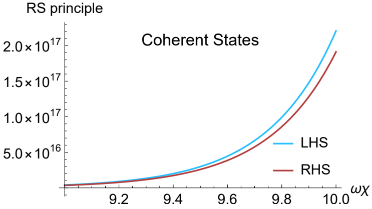
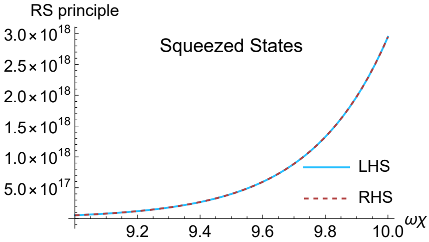
Robertson–Schrödinger inequality (CS and SS)

- Coherent states *do not* minimize the RS relation at *any time*





- Squeezed states *do* minimize the RS relation *at all times*



Does this mean anything?

Relative uncertainties (CS and SS)

Can the state parameters $\alpha = |\alpha| e^{\mathrm{i}\vartheta}$, $z = re^{\mathrm{i}\psi}$ make fluctuations small?

- At $\chi = 0$

$$\frac{(\Delta \hat{V})^2_{\mathsf{C}}}{\langle \hat{V} \rangle^2_{\mathsf{C}}} = \frac{1}{|\alpha|^2}\,, \qquad \frac{(\Delta \hat{H})^2_{\mathsf{C}}}{\langle \hat{H} \rangle^2_{\mathsf{C}}} = \frac{4|\alpha|^2 + 2}{4|\alpha|^4 \cos(2\vartheta)^2} \qquad ✓$$

$$\frac{(\Delta \hat{V})^2_{\mathsf{S}}}{\langle \hat{V} \rangle^2_{\mathsf{S}}} = 2\coth^2 r\,, \qquad \frac{(\Delta \hat{H})^2_{\mathsf{S}}}{\langle \hat{H} \rangle^2_{\mathsf{S}}} = 2 + 2\sec^2 \psi \operatorname{csch}^2(2r) \qquad ✗$$

- Turning on time evolution

$$\frac{(\Delta \hat{V}_\chi)^2_{\mathsf{C}}}{\langle \hat{V}_\chi \rangle^2_{\mathsf{C}}} \sim \frac{1}{|\alpha|^2} \frac{\sin_{2\vartheta} \sinh_{4\omega\chi} + \cosh_{4\omega\chi}}{(\sin_{2\vartheta} \sinh_{2\omega\chi} + \cosh_{2\omega\chi})^2} \xrightarrow{\chi \to \pm\infty} \frac{2}{|\alpha|^2 |(1 \pm \sin_{2\vartheta})} \qquad ✓$$

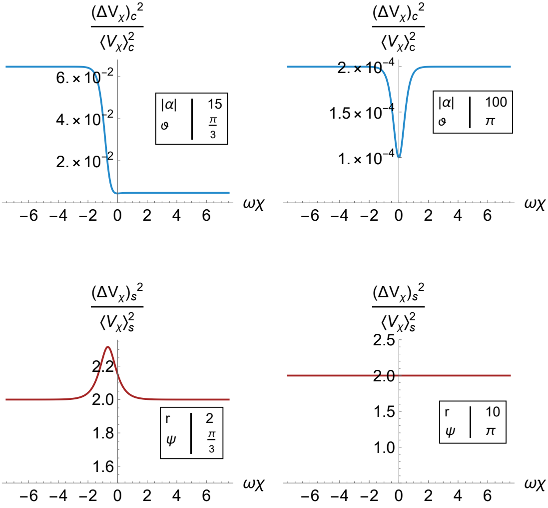
$$\frac{(\Delta \hat{V}_\chi)^2_{\mathsf{S}}}{\langle \hat{V}_\chi \rangle^2_{\mathsf{S}}} = 2\, \frac{\sin_\psi \sinh_{2r} \sinh_{2\omega\chi} + \cosh_{2r} \cosh_{2\omega\chi} + 1}{\sin_\psi \sinh_{2r} \sinh_{2\omega\chi} + \cosh_{2r} \cosh_{2\omega\chi} - 1} \xrightarrow{\chi \to \pm\infty} 2 \qquad ✗$$

Relative uncertainties (CS and SS)

Can the sta… …ons small?

- At $\chi = 0$ …

- Turning c…



Relative uncertainties (CS and SS)

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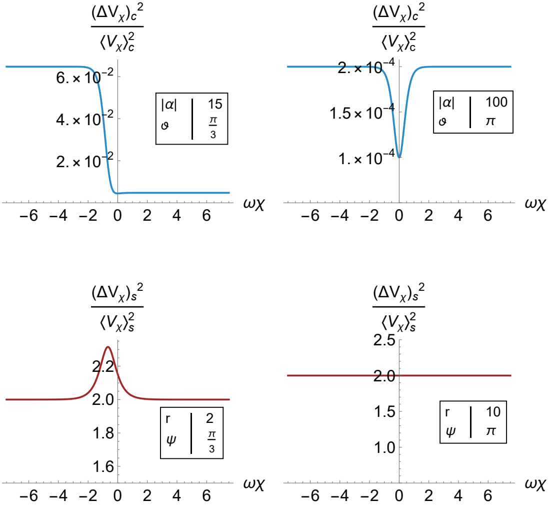


So we still only rely on Fock coherent states!

Gaussian states (GS) in GFT cosmology

- Associated with Gaussian characteristic functions
- Fully characterised by first and second canonical moments
- Gibbs states of second order Hamiltonians
- Can *always* be expressed as **displaced squeezed thermal states**

$$\hat{\rho}_G(\alpha, z, \beta) = \hat{D}(\alpha)\hat{S}(z)\hat{\rho}_\beta \hat{S}(z)^\dagger \hat{D}(\alpha)^\dagger$$

where $\hat{\rho}_\beta = \dfrac{e^{-\beta \hat{a}^\dagger \hat{a}}}{\mathrm{tr}\left(e^{-\beta \hat{a}^\dagger \hat{a}}\right)}$ with β "inverse temperature" (or ?)
[Assanioussi, Kotecha, Oriti]

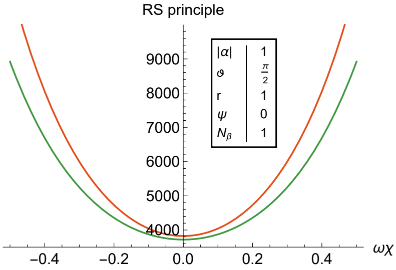
- Compute expectation values (can also use *thermofield dynamics*)

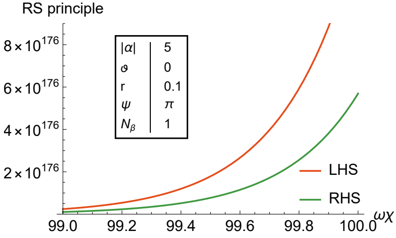
$$\langle \hat{\mathcal{O}} \rangle_\mathsf{G} = \mathrm{tr}\left(\hat{\mathcal{O}}\ \hat{D}(\alpha)\hat{S}(z)\hat{\rho}_\beta \hat{S}(z)^\dagger \hat{D}(\alpha)^\dagger\right) \qquad |\alpha|, \vartheta, r, \psi, \beta \in \mathbb{R}$$

- Find variances and covariances too, e.g. $(\Delta \hat{V})^2_\mathsf{G}$, $\Delta(\hat{V}\hat{H})_\mathsf{G}$, etc
- Finally turn on χ-evolution, and we are ready to check **semiclassicality**

Robertson–Schrödinger inequality (GS)

Much like coherent states, Gaussian states *do not* saturate the RS principle *at any time* (here $N_\beta = \mathrm{tr}\left(\hat{\rho}_\beta\, \hat{a}^\dagger \hat{a}\right) = (e^\beta - 1)^{-1}$)





Of course, the minimisation can happen by fine tuning $\alpha = 0$ and $\beta \to \infty$ which trivially returns the squeezed state case

Relative uncertainties (GS) – Analytics

- At $\chi = 0$

$$\frac{(\Delta \hat{V})^2_{\mathsf{G}}}{\langle \hat{V} \rangle^2_{\mathsf{G}}} \sim \frac{\coth_{\beta/2}}{|\alpha|^2}\left[\cosh_{2r} + \sinh_{2r}(\cos_{2\vartheta}\cos_\psi + \sin_{2\vartheta}\sin_\psi)\right] \quad \checkmark$$

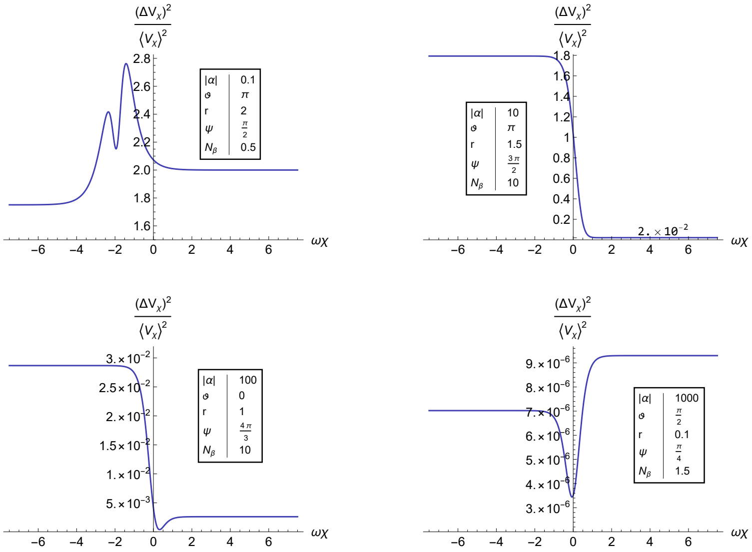
$$\frac{(\Delta \hat{H})^2_{\mathsf{G}}}{\langle \hat{H} \rangle^2_{\mathsf{G}}} \sim \frac{\coth_{\beta/2}}{|\alpha|^2 \cos^2_{2\vartheta}}\left[\cosh_{2r} + \sinh_{2r}(\cos_{2\vartheta}\cos_\psi - \sin_{2\vartheta}\sin_\psi)\right] \quad \checkmark$$

- At late times $\chi \to \pm\infty$

$$\frac{(\Delta \hat{V})^2_{\mathsf{G}}}{\langle \hat{V} \rangle^2_{\mathsf{G}}} \sim \frac{2}{|\alpha|^2}\frac{\coth_{\beta/2}(\cosh_{2r} \pm \sinh_{2r}\sin_\psi)}{(1 \pm \sin_{2\vartheta})} \quad \checkmark$$

All these expressions are expanded for large $|\alpha|$, which indeed is crucial to make quantum fluctuations small and classify **GS as semiclassical**

- Generic χ


(ΔV_χ)^2 / ⟨V_χ⟩^2
|α| 0.1
ϑ π
r 2
ψ π/2
N_β 0.5
ωχ
|α| 10
ϑ π
r 1.5
ψ 3π/2
N_β 10
2. × 10^-2
|α| 100
ϑ 0
r 1
ψ 4π/3
N_β 10
|α| 1000
ϑ π/2
r 0.1
ψ π/4
N_β 1.5

Algebraic approach and GS

- Change **quantisation scheme**: $\varphi \in \mathbb{C}$ and $[\hat{\varphi}(\chi), \hat{\varphi}^\dagger(\chi')] = \delta(\chi - \chi')$
- Build *kinematical* Hilbert space via abstract ladder operators $\hat{\varphi}$ and $\hat{\varphi}^\dagger$
- Dynamics is defined *through* quantum states (i.e., $|\Psi\rangle$ physical if)

$$\left\langle \Psi \left| \frac{\delta S[\hat{\varphi}, \hat{\varphi}^\dagger]}{\delta \hat{\varphi}^\dagger} \right| \Psi \right\rangle = 0 \qquad \text{or stronger} \qquad \left. \frac{\delta S[\hat{\varphi}, \hat{\varphi}^\dagger]}{\delta \hat{\varphi}^\dagger} \right| \Psi \Big\rangle = 0$$

- Imposing either of these gives conditions for state parameters
- Fock coherent states $|\sigma\rangle = \exp\left(\int \mathrm{d}\chi\, \sigma(\chi) \hat{\varphi}^\dagger(\chi)\right) |0\rangle$ *exact* solution

$$\langle \hat{V}(\chi) \rangle_\sigma = v \langle \hat{\varphi}^\dagger(\chi) \hat{\varphi}(\chi) \rangle_\sigma = v |\sigma(\chi)|^2$$

state function (here $\sigma(\chi)$) gives notion of dynamics

- $\langle \hat{V}(\chi) \rangle_\sigma$ satisfies Friedmann equation as before (or very very similar)
- Other options (e.g. *dipoles*)? Or more generally **Gaussian states?**
- **No other exact solution and no condition on the parameters!**

Conclusions

- We introduced (mixed) **Gaussian states** in GFT cosmology (working in the deparametrised approach)
- GS include every other previously studied state as a subcase
- GS are the **most general** family of states preserved under time evolution (for second-order Hamiltonians)
- GS have **semiclassical** features according to the small quantum fluctuations criterion
- Specifically, both Volume and Hamiltonian can have **small relative uncertainties** at all times (including the late time limit)
- **Algebraic approach**: GS don't seem to work, are we stuck with coherent states? $\Rightarrow$ Further investigation necessary!

Extra: Thermofield formalism

- Doubling the Fock space one can define a *pure* state $|0_\beta\rangle$ such that

$$\mathrm{tr}\left(\hat{\rho}_\beta \hat{\mathcal{O}}\right) = \langle 0_\beta|\hat{\mathcal{O}}|0_\beta\rangle$$

- Fictitious (tilde) system with $[\hat{\tilde{a}}, \hat{\tilde{a}}^\dagger] = 1$ and $\hat{\tilde{a}}|\tilde{0}\rangle = 0$, then

$$|0,\tilde{0}\rangle = |0\rangle \otimes |\tilde{0}\rangle\,, \qquad \hat{a}|0,\tilde{0}\rangle = \hat{\tilde{a}}|0,\tilde{0}\rangle = 0$$

- Introduce thermality via Bogoliubov transformation

$$|0_\beta\rangle = \hat{T}(\theta_\beta)|0,\tilde{0}\rangle\,, \qquad \hat{T}(\theta_\beta) = e^{\theta_\beta(\hat{a}^\dagger\hat{\tilde{a}}^\dagger - \hat{a}\hat{\tilde{a}})}$$

- Link with density matrix formalism

$$\frac{1}{e^\beta - 1} = \mathrm{tr}\left(\hat{\rho}_\beta \hat{a}^\dagger \hat{a}\right) = \langle 0_\beta|\hat{a}^\dagger\hat{a}|0_\beta\rangle = \sinh^2_{\theta_\beta}$$

- Thermofield analogue of Gaussian state

$$|\Psi_G;\beta\rangle = \hat{D}(\alpha)\hat{S}(z)\hat{T}(\theta_\beta)|0,\tilde{0}\rangle = \hat{D}(\alpha)\hat{S}(z)|0_\beta\rangle$$

Extra: "Static" contributions in algebraic approach

- GS parameters: displacement σ, squeezing ξ and "temperature" β
- Algebraic approach: determine evolution of such parameters using

$$\left\langle \frac{\delta S[\hat{\varphi}, \hat{\varphi}^\dagger]}{\delta \hat{\varphi}^\dagger} \right\rangle_{\sigma,\xi,\beta} = 0$$

- However only obtain a condition on the displacement parameter since

$$\langle \hat{\varphi} \rangle_{\sigma,\xi,\beta} = \sigma(\chi)$$

- The other parameters drop out trivially!
- Higher order Schwinger–Dyson equations are hard to solve

$$\left\langle \frac{\delta \hat{\mathcal{O}}}{\delta \hat{\varphi}^\dagger} - \hat{\mathcal{O}} \frac{\delta S[\hat{\varphi}, \hat{\varphi}^\dagger]}{\delta \hat{\varphi}^\dagger} \right\rangle = 0\,, \qquad \text{e.g. with} \quad \hat{\mathcal{O}} = \hat{\varphi},\, \hat{\varphi}^\dagger$$

- One can then only assume other parameters are time-independent

$\Rightarrow$ obtain constant contributions to the effective Friedmann equation