Multipartite Entanglement of Random Spin Networks in Quantum Gravity

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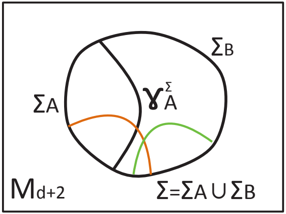
It-from-qubit: geometry/entanglement

Entanglement (reflected in separability, subsystems, gluing issues) has a big impact in quantum gravity since the last decade:

- holographic entanglement entropy in AdS/CFT [Ryu & Takayanagi 06]

$$S_A = \frac{Area(\gamma_A)}{4G_N}$$



- space-time bulk geometry reconstruction from the structure of correlations of the boundary state [Van Raamsdonk 09]
- local holography: *surface/state correspondence* for each codim-2 convex surface in gravitational spacetime; [Miyaji & Takayanagi 15]
- quantum black holes [Almheiri, Stanford & Maldacena, Engelhardt,...]

=> ? entanglement as spacetime fabric [Bianchi & Myers,12]

Quantum 3D Geometry in Loop Quantum gravity

- Loop Quantum Gravity: first order GR cast in Ashtekar-Barbero variables (E, A) canonically quantized: "quantum spacetime" realised via spin-networks kinematic Hilbert space
  [Ashtekar & Lewandowski, Rovelli, Thiemann, et al.]

$$H_\gamma = L^2(SU(2)^E/SU(2)^V)$$
$$\simeq \bigoplus_j \bigotimes_v^V H_v = \bigoplus_j \bigotimes_v^V Inv_{SU(2)}\left[\bigotimes_{e\in v} \mathcal{V}^{j_e}\right]$$

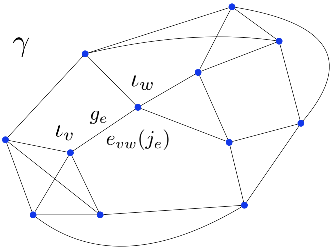
on graph $\gamma = (L, V)$ classically intended as collection of holonomies $g_e[A]$ of the $SU(2)$-Ashtekar-connection field embedded in 3D space

=> H_γ is spanned by $SU(2)$-invariant wave-functionals $\psi_\gamma(\{g_e\})$ interpreted as quantum geometry states

Tensor Network Description of SN

TN quantum $3D$ geometry state $\psi_\gamma(\{g_e\})$ can be abstractly constructed as a *contraction* of a generic state $|\psi\rangle$ in $H_V = \otimes_v^V H_v$ with the set of E edge states comprising the graph (SU(2)-spin network basis)

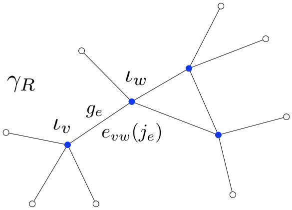
$$\psi_\gamma(\{g_e\}) = \sum_{\{j_e\}} \left(\bigotimes_{e=1}^{E} \langle e(g_e)| \right) |\psi\rangle$$



w/ bulk $|\psi\rangle = \sum_{\{\iota_v\}} \hat{\psi}(\{j\})_{\iota_1 \ldots \iota_V} |\iota_1\rangle \otimes \cdots \otimes |\iota_V\rangle$, $H_v = \text{span}\{|\iota_v\rangle\}$.

Boundaries => Bulk-to Boundary Maps States

Bounded region of 3D space (3-ball) R with ∂R spacetime corner is defined by an *open* graph $\gamma_R \subset \gamma$:



- wave-functions $\psi_{\tilde{\gamma}}(\{g_e\})$ take values into the boundary Hilbert space

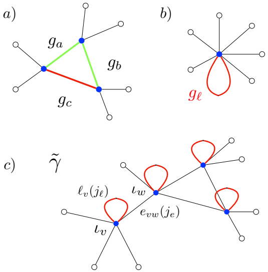
$$H_{\partial\gamma_R} = \bigoplus_{\{j_e\}} \bigotimes_{e\in\partial\gamma} \mathcal{V}^{j_e}$$

=> $\psi_{\gamma_R}(\{g_e\}) \in H_{\partial\gamma_R}$ defines bulk-to-boundary maps, encoding quantum correlations of the bulk into boundary states coefficients. [Chirco, Colafranceschi, Oriti 21; Chen & Livine 21, Langenscheidt et al. 22]

Curved Bounded Region: Gauge Fixing the Bulk

not just $SU(2)$-gauge symmetric tensor networks: carry non-trivial holonomies => curvature: how much of the bulk geometry can be recovered from boundary state correlations?

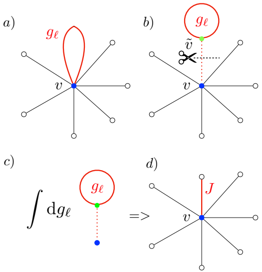
pin any bulk graph can be reduced via gauge fixing: holonomy loops attached to the "coarse-grained" vertices in the graph; [Freidel & Livine,03]



=> construct $\psi_{\gamma_R}(\{g_e, g_\ell\}) \in H_{\partial\gamma_R}$ as *proxies* of a bounded quantum $3D$ region with uniformly distributed curvature.

Quantum Space Kirigami: Coarse Graining the Bulk

Reduced description: retain only closure defects via kirigami



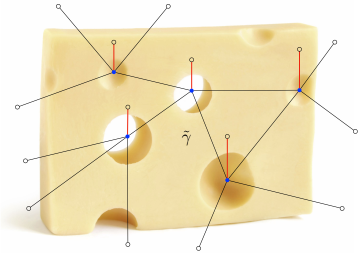
- (a,b) single loopy vertex unfolding:

$$\mathsf{H}_v^L = \bigoplus_{J,\{k\}} \left\{ \mathsf{Inv}_{SU(2)} \left[\bigotimes_{e \in v} V^{j_e} \otimes V^J \right] \otimes \mathsf{Inv}_{SU(2)} \left[V^J \otimes \left(V^k \otimes \overline{V}^k \right) \right] \right\}$$

- (c,d) coarse graining: partial trace via integration over the loop holonomy => loopy vertex tagged vertex encoding closure defect [Charles & Livine, 16; Livine,18]

Maasdammer: Extended Bulk-to-Boundary Map

output we get a ψ_τ functional now defined as a collection of tagged vertices glued together via edges, in an *extended* boundary space containing the tags:



$$H_{\partial\tilde{\gamma}} = \bigoplus_{\{j_e\}} \bigotimes_{e\in\partial\tilde{\gamma}} \mathcal{V}^{j_e} \bigotimes_{v\in\tilde{\gamma}} \mathcal{V}^{J_{\text{tag}}(\{j_e\})}$$

! the coarse grained state ψ_τ generalises the bulk-to-boundary map to $H_{\partial\tilde{\gamma}}$ effectively including reduced curvature information.

Geometry/Entanglement Correspondence

- the extended boundary state in $H_{\partial\tilde{\gamma}}$

$$|\psi_\tau\rangle = \sum_{\{j_e\}} \left(\bigotimes_{e=1}^{E} \langle e\,(g_e)| \right) |\psi_b\rangle$$

  is written as

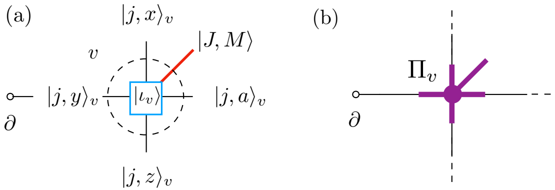
$$|\psi_\tau\rangle = \textstyle\sum_{\{j,M_v,m_e\}} C(\{\iota_v, J_v, j_e\})_{\{M_v,m_e\}} \bigotimes_v |J_v, M_v\rangle \bigotimes_{e\in\partial R} |j_e, m_e\rangle$$

- the coefficients C encode info about quantum correlations of bulk intertwiners, graph connectivity, and curvature (closure defects).

- TAKE: boundary observer measure - partially coarse grain the bulk via a *random measurement* on the bulk state $|\psi_b\rangle$ on $\bigotimes_v H_v^J$

Random Measurement on Bulk Vertices

- *random measurement* realized by projecting onto a set of *i.i.* (Haar) random vertex states $|f_v\rangle$ ($|f_v\rangle = U_v |0\rangle$, U_v Haar random unitary)



on each extended vertex space $K_v^J = V^J \otimes \bigotimes_{j_e \in v} V^{j_e} \otimes M_J^{\{j_e\}}$,

- the random extended-boundary density matrix is given by a trace over bulk indices,

$$\rho_\tau = \mathrm{Tr}[\rho_b \otimes \rho_E \Pi]$$

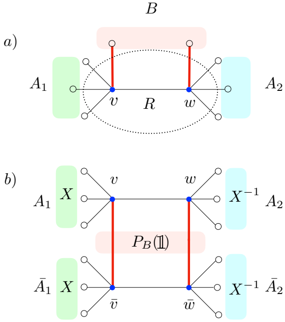
now with the extra insertion of $\Pi = \bigotimes_v^{V_R} \Pi_v = \bigotimes_v^{V_R} |f_v\rangle\langle f_v|$

Boundary Entanglement via Tripartition

Recipe for measuring boundary entanglement:

1 – Assumptions:

- curvature (closure defects) *uniformly* distributed throughout the graph $\tilde{\gamma}_R$.
- all edges and tags spins $\{j_e\}, \{J_v\}$ fixed to single value j and J respectively



2 – Tripartition of the extended boundary system into three subsystems A_1, A_2, B. Correspondingly, we look at the tripartite Hilbert space

$$H_{\tau}^{\{j,J\}} = \bigotimes_{e \in A_1} V^{j_e} \otimes \bigotimes_{e \in A_2} V^{j_e} \otimes \bigotimes_{i \in B} V^{J_i}$$

Entanglement Witness: Logarithmic Negativity

3 – Focus on *reduced* mixed state $\rho_{A_1A_2} = \mathrm{Tr}_B[\rho_\tau]$: negativity provides a computable measure of entanglement for mixed states

PPT the amount of entanglement of the state can be quantified by counting the number of negative eigenvalues of $\rho_{A_1A_2}^{T_{A_2}}$ by [Peres,96; Horodecki,96]

$$N(\rho_{A_1A_2}) \equiv \frac{\|\rho_{A_1A_2}^{T_{A_2}}\|_1 - 1}{2} = \sum_{i:\lambda_i<0} |\lambda_i| \,,$$

where $\|\cdot\|_1$ is the trace norm, and the logarithmic negativity is defined as

$$E_N(\rho_{A_1A_2}) \equiv \log\|\rho_{A_1A_2}^{T_{A_2}}\|_1 \,.$$

rmk both N and E_N are entanglement monotone under general PPT preserving operations (faithful measure) but not generally a necessary and sufficient separability criterion.

Rényi, Replicas & Typicality

4 – Look first at the k-th Rényi negativity of $\rho_{A_1 A_2}$,

$$N_k(\rho_{A_1 A_2}) := \mathrm{Tr}\left[\left(\rho_{A_1 A_2}^{T_{A_2}}\right)^k \Big/ (\mathrm{Tr}[\rho_{A_1 A_2}])^k\right]$$

then recover the logarithmic negativity in the $k \to 1$ limit.

5 – random measurement on the bulk state => we shall compute Rényi negativity in expected value,

$$\mathbb{E}_\mu\left[N_k(\rho_{A_1 A_2})\right] \equiv \overline{N_k(\rho_{A_1 A_2})}$$

6 – TYPICALITY (1): in the large spin regime, our measure is well approximated by its *average*

$$\mathrm{N}_k(\rho_{A_1 A_2}) \simeq \overline{N_k(\rho_{A_1 A_2})}$$

Rényi, Replicas & Typicality

7 – TYPICALITY (2): we can approximate averaged Rényi negativity by the ratio of expected values of the k-th moment and the k-th power of the partition function of $\rho_{A_1A_2}^{T_{A_2}}$,

$$\overline{N_k(\rho_{A_1A_2})} \simeq \frac{\overline{\text{Tr}\left[\left(\rho_{A_1A_2}^{T_{A_2}}\right)^k\right]}}{\overline{\left(\text{Tr}\left[\left(\rho_{A_1A_2}\right)\right]\right)^k}} \equiv \frac{\overline{Z_1^{(k)}}}{\overline{Z_0^{(k)}}}$$

8 – REPLICA technique: linearise the partial transpose matrix,

$$\text{Tr}\left[\left(\rho_{A_1A_2}^{T_{A_2}}\right)^k\right] = \text{Tr}\left[\rho_{A_1A_2}^{\otimes k}\, P_{A_1}(X) \otimes P_{A_2}(X^{-1})\right]$$
$$= \text{Tr}\left[\rho_R^{\otimes k}\, P_{A_1}(X) \otimes P_{A_2}(X^{-1}) \otimes P_B(\mathbb{1})\right] .$$

$P_I(\sigma)$ unitary representation of the permutation σ, $I = A_1, A_2, B$, with X, X^{-1} and $\mathbb{1}$ the cyclic, anti-cyclic and identity permutations.

Rényi, Replicas & Typicality

9 – trace linearity: averaging only concerns random tensors

$$\overline{Z_1^{(k)}} = \mathrm{Tr}\left[\rho_b^{\otimes k} \otimes \rho_E^{\otimes k} \left(\bigotimes_v \overline{(|f_v\rangle\langle f_v|)^{\otimes k}}\right) \cdot P_{A_1}(X) \otimes P_{A_2}(X^{-1}) \otimes P_B(\mathbb{1})\right]$$

- Schur's lemma: the average results in a sum over unitary representations of the permutation group g_v acting on the k copies of the extended vertex space K_v^J

$$\overline{(|f_v\rangle\langle f_v|)^{\otimes k}} = \frac{(D_v - 1)!}{(D_v + k - 1)!} \sum_{g_v \in S_k} P_v(g_v)$$

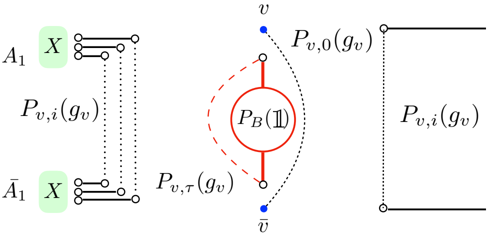
with dimension $D_v = \dim(K_v^J)$.

=> $\overline{Z_1^{(k)}} = \mathcal{C}\,\mathrm{Tr}\left[\rho_b^{\otimes k} \otimes \rho_E^{\otimes k} \left(\bigotimes_v \sum_{g_v \in S_k} P_v(g_v)\right) P_{A_1}(X) \otimes P_{A_2}(X^{-1}) \otimes P_B(\mathbb{1})\right]$

Mapping to generalised Ising Model

10 – compute $\overline{Z_1^{(k)}}$ via diagrams:

bulk for k copies of the density matrix, the action of the propagators factorise in the bulk

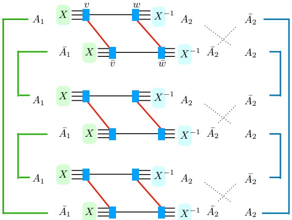


bdry dangling indices glued via $P_{A_1}(X) \otimes P_{A_2}(X^{-1})$

Mapping to generalised Ising Model

we get

$$N_k(\rho_{A_1A_2}) = \frac{\overline{Z_1^{(k)}}}{\overline{Z_0^{(k)}}} = \sum_{\{g_v\}} e^{-A_{1/0}^{(k)}}$$



$k = 3$

- with $A_{1/0}^{(k)}$ (normalized) action of a *generalized* Ising-like model:

$$A_1^{(k)}\left[\{g_v\}\right] = A_\iota + \sum_{e^i_{vw}\in\mathcal{E}_R} \Delta(g_v, g_w)\log d_{j^i_{vw}} + \sum_{e^i_{v\bar{v}}\in A_1} \Delta(g_v, X)\log d_{j^i_{v\bar{v}}}$$

$$+ \sum_{e^i_{v\bar{v}}\in A_2} \Delta(g_v, X^{-1})\log d_{j^i_{v\bar{v}}} + \sum_{v} \Delta(g_v, \mathbb{1})\log D_{J_v}$$

- Cayley distances $\Delta(g,h) = k - \chi(g^{-1}h)$ on the permutation group S_k, where $\chi(g)$ indicates the number of cycles in a permutation g.

Mapping to generalised Ising Model

1st Result

- we can then formally divide the action in three terms as follows

$$A_1^{(k)} = A_{\text{topology}}^{(k)} + A_{\text{phys}}^{(k)}$$

* $A_{\text{topology}}^{(k)} = A_{\text{edges}}^{(k)} + A_{\text{tags}}^{(k)}$ – graph connectivity
* $A_{\text{phys}} = A_\iota$ – quantum correlations among bulk intertwiners.

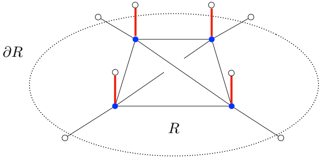
Focus on $A_{\text{topology}}^{(k)}$: define $\log d_j \equiv \beta$ and $\log D_J \equiv \beta_t$

$$N_k(\rho_{A_1 A_2}) \simeq \sum_{\{g_v\}} e^{-A_{\text{topology}/0}^{(k)}} = \sum_{\{g_v\}} e^{-(\beta\, H_e + \beta_t\, H_{\text{tags}})}$$

in the large spin (the strong coupling or "low temperature") regime, the leading contribution to N_k corresponds to dominant $\{g_v\}$ configurations that minimize the action!

Example: Cluster of four tags

Example: tagged open spin network graph corresponding to a cluster of four-valent tagged vertices, with $T = 4$ and $E_{\partial R} = 4$

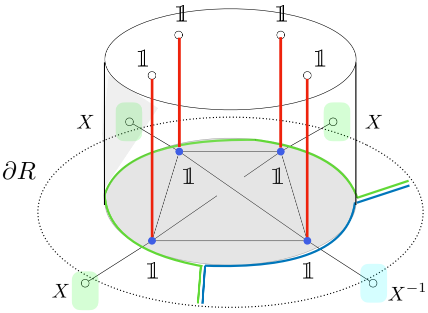


2nd Result: we find three configurations with minimal energy (equilibrium phases). The corresponding actions (for even $k > 0$)

- hole: cluster colored with $\mathbb{1}$: $\mathcal{A}_k^{(\mathbb{1})} = \beta E_{\partial R}\,(k-1)$;
- bipartite: cluster colored with $X(X^{-1})$:
  $\mathcal{A}_k^{(X)} = \beta\, S\,(k-2) + \beta_t\, T\,(k-1)$ w/ S *min*-cut btw X – X^{-1};
- island cluster colored with τ: $\mathcal{A}_k^{(\tau)} = \beta E_{\partial R}\,\left(\frac{k}{2}-1\right) + \beta_t\, T\,\frac{k}{2}$.

Hole Regime: Boundary Thermalization

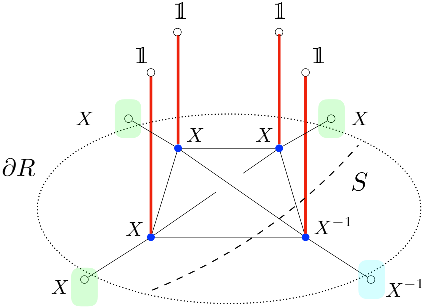
for $T > E_{\partial R}$



$$E_N^{(\text{hole})}(\rho_{A_1A_2}) \simeq -\lim_{k\to 1} \mathcal{A}_k^{(\mathbb{1})} = -\lim_{k\to 1} \beta E_{\partial R}(k-1) = 0$$

Bipartite Regime: Entanglement Entropy Area Law

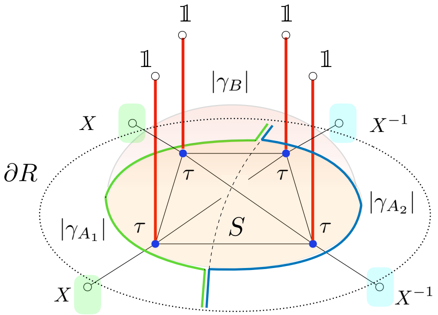
for $\beta E_{\partial R} > \beta_t T + 2\beta S$



$$E_N^{(\text{bipartite})}(\rho_{A_1A_2}) \simeq -\lim_{k\to 1} \mathcal{A}_k^{(X)} = -\lim_{k\to 1} [\beta S(k-2) + \beta_t T(k-1)] = \beta S$$

Island Regime: New from tripartite regime

for $\beta E_{\partial R} \in [\beta_t T,\ \beta_t T + 2\beta S]$



$$E_N^{(\text{island})}(\rho_{A_1A_2}) = \frac{1}{2}(\beta E_{\partial R} - \beta_t T) + \log\frac{8}{3\pi}$$

Island Regime and Curvature

3rd Remarkable (though expected in the typical regime): [Shapourian,21]

- ! entanglement phases defined solely in terms of the relative dimensions of the three subsystems A_1, A_2 and B, via the ratio of bulk curvature over boundary surface

$$q = \beta_t T / \beta E_{\partial R} = \frac{\log[\dim(B)]}{\log[\dim(A_1)\dim(A_2)]} \,,$$

  with $E_{\partial R} = E_{A_1} + E_{A_2}$.

- ! new from tripartition: for $1 - 2S/E_{\partial R} < q < 1$, the curvature environment mediates the entanglement of the boundary.

$$E_N^{(\text{island})}(\rho_{A_1 A_2}) \propto \frac{1}{2}\beta E_{\partial R}(1-q) = \frac{1}{2}\Big[\beta(|\gamma_{A_1}| + |\gamma_{A_2}|) - \beta_t |\gamma_B|\Big]$$

- ! E_N scales with the area of the cluster boundary with a negative correction which depends linearly on the number of tags.

Island Regime and Curvature

4rth Recall the role of T:

- for a Riemannian n-manifold $\mathcal{M}$ at a point p

$$\frac{\text{Area}(\partial B_\varepsilon(p) \subset \mathcal{M})}{\text{Area}(\partial B_\varepsilon(0) \subset \mathbb{R}^n)} = 1 - \frac{\mathcal{R}}{6n}\varepsilon^2 + O(\varepsilon^3)\,.$$

- typicality = large-spins regime $\simeq$ semiclassical limit: for $n = 3$, flat 3-ball, we have $\text{Area}(\partial B_\varepsilon(0)) \simeq \beta E_{\partial R}(\varepsilon)$
- in presence of tags, for curved 3-ball in $\mathcal{M}$, we assume

$$\text{Area}(\partial B_\varepsilon(p)) \simeq \beta E_{\partial R}(\varepsilon) + \kappa\, \beta_t\, T(\varepsilon)$$

!! we get a suggestive characterisation of the curvature in purely information theoretic terms

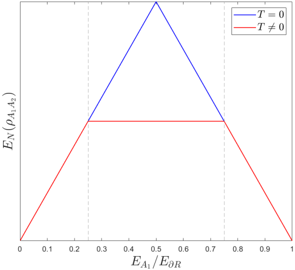
$$\mathrm{q}(\varepsilon) \simeq -\frac{\mathcal{R}}{18\,\kappa}\varepsilon^2$$

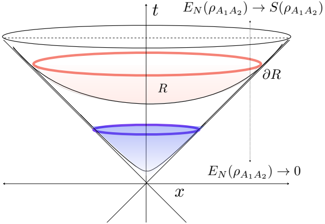
Summary

- Logarithmic negativity can be used to provide a measure of entanglement for tripartite spin-network states: in the Haar random case, this can extended to include sums over j.
- At the level of spin network basis states, the quantum correlations reflect the topology (connectivity) of the graph:
  - area scaling behaviour follows, if we disregard intertwiner entanglement
  - intertwiner entanglement is captured by a bulk term (the analogous of the bulk quantum correction to RT formula in AdS/CFT)
- more that area scaling information get encoded in the negativity measure: corrections due to the environment can be use to translate the curvature of the region into information-theoretic terms.

Speculative Summary

Interesting interplay of: Unitarity, Typicality, Monogamy of quantum entanglement





Left generalized Page curve of the logarithmic negativity

Right Milne spacetime sliced by $3D$ surfaces with hyperbolic embedding.

Thank You!