Understanding Ryu-Takayanagi as Entropy without invoking Holography

Upcoming work with Zhencheng Wang and Donald Marolf

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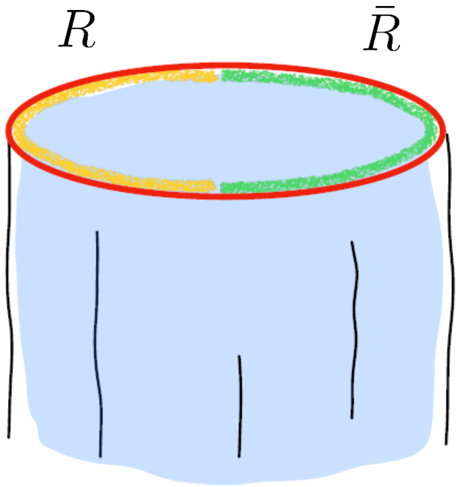
Outline

1. The problem of defining an **entropy for spacetime regions**



2. Holography and the **Ryu-Takayanagi formula**

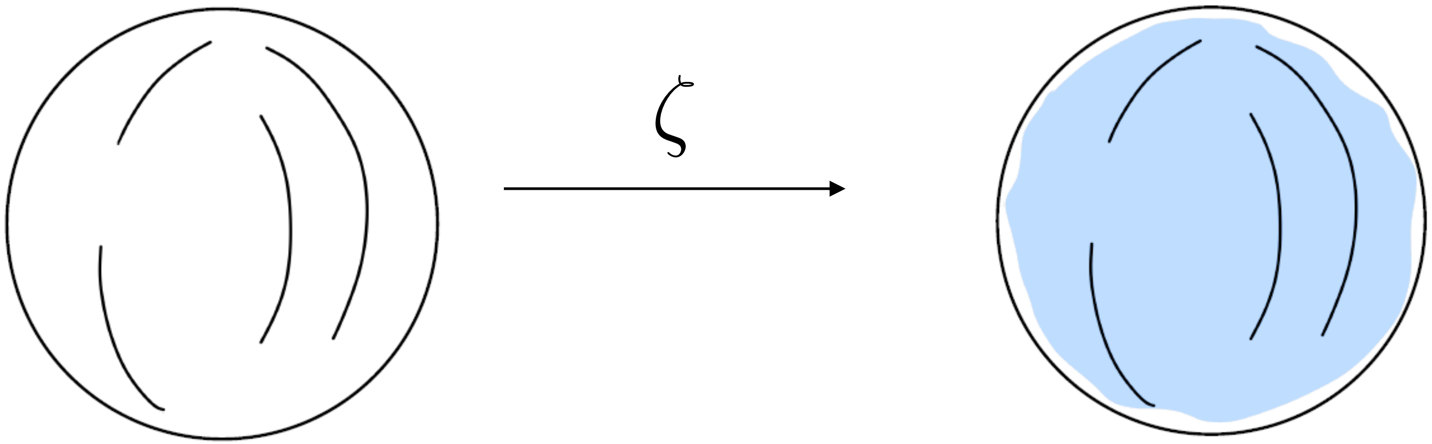
3. Our work: understanding gravitational entropy **without holography**

1

The problem of defining an **entropy for spacetime regions**

Gravitational path integral

Spacetime boundary



M

$g_M, \phi_M^{\mathrm{matter}}$

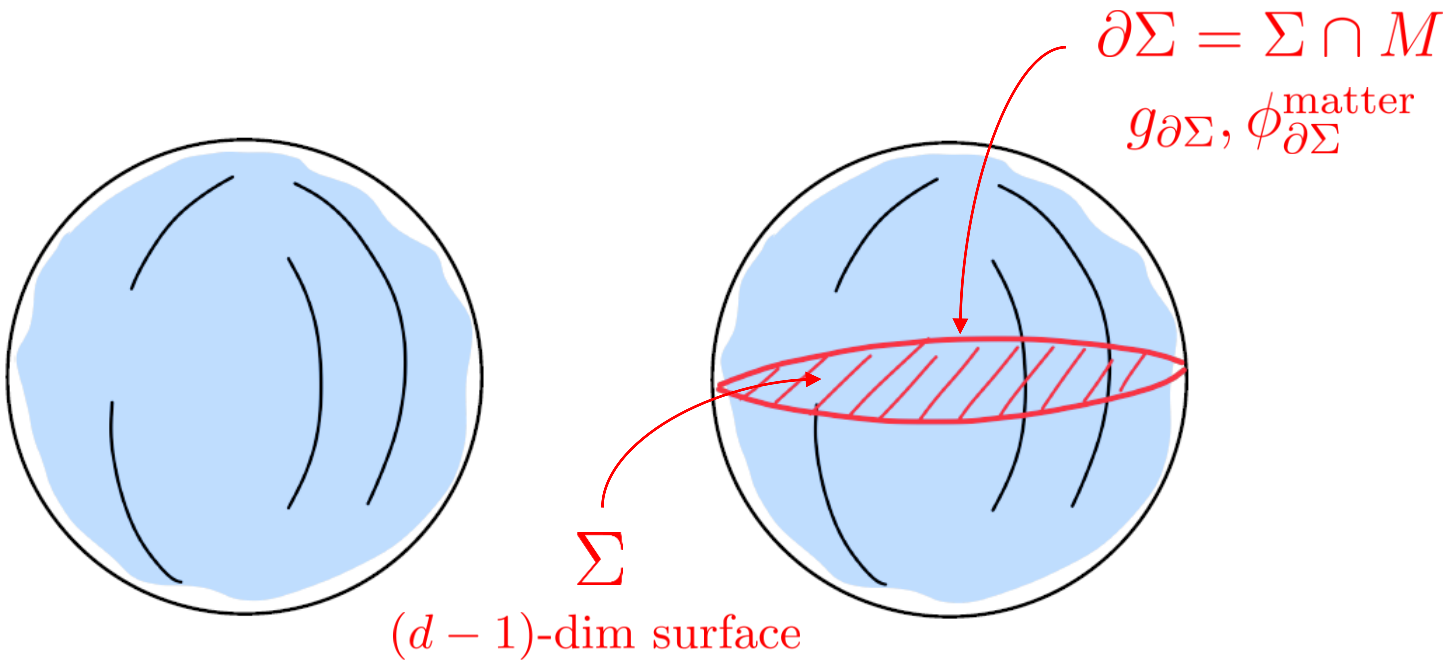
(boundary conditions)

$$\zeta(M) = \int_{\Phi \sim M} \mathcal{D}\Phi e^{-S[\Phi]}$$

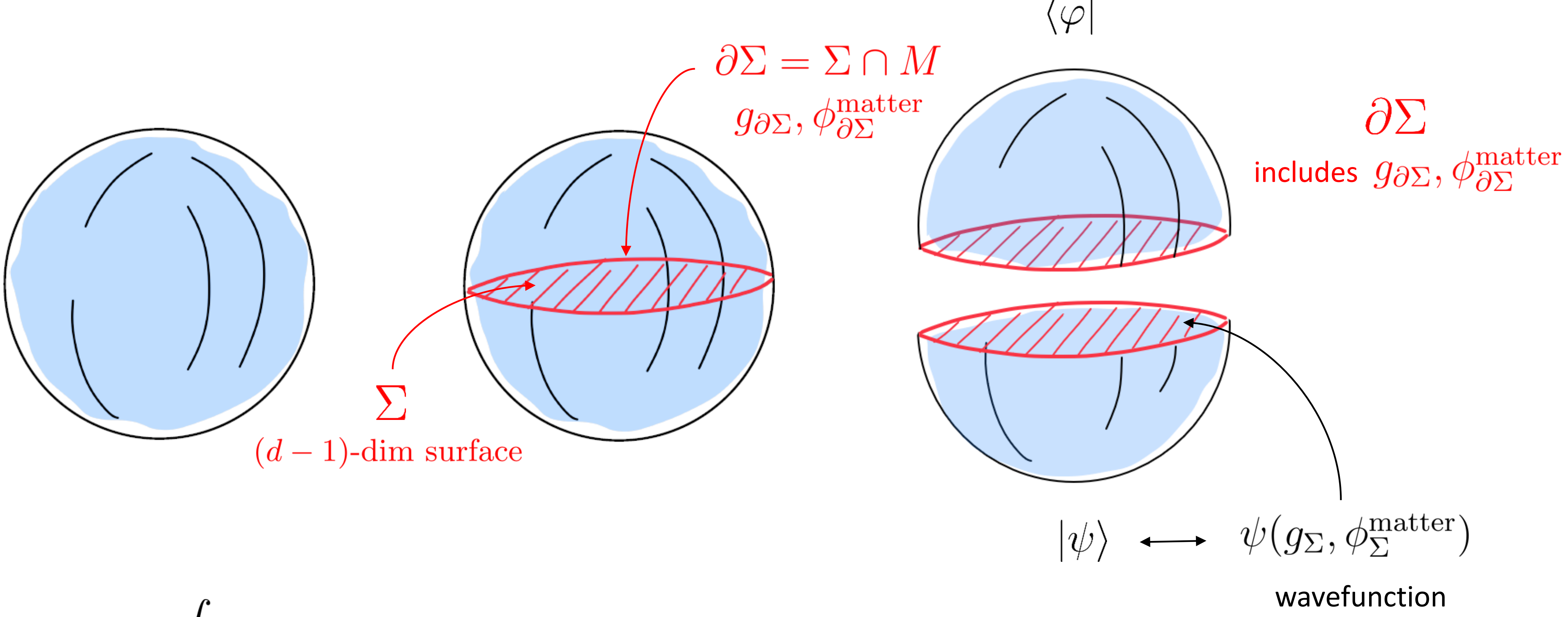
$\Phi = g, \phi^{\mathrm{matter}}$

M includes $g_M, \phi_M^{\mathrm{matter}}$

The path integral and the Hilbert space

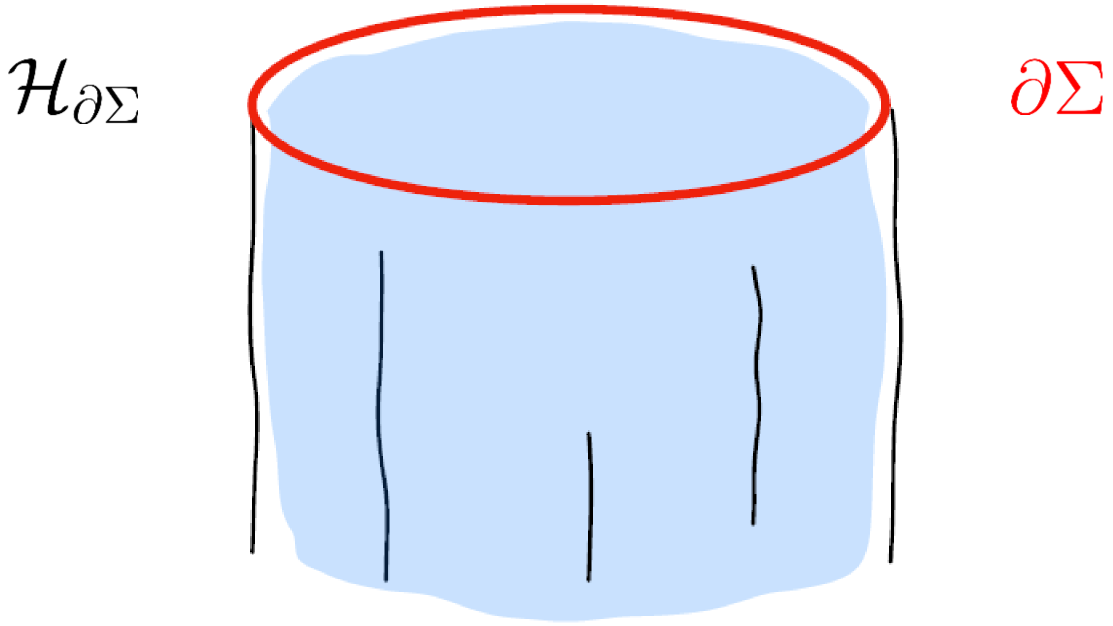


The path integral and the Hilbert space

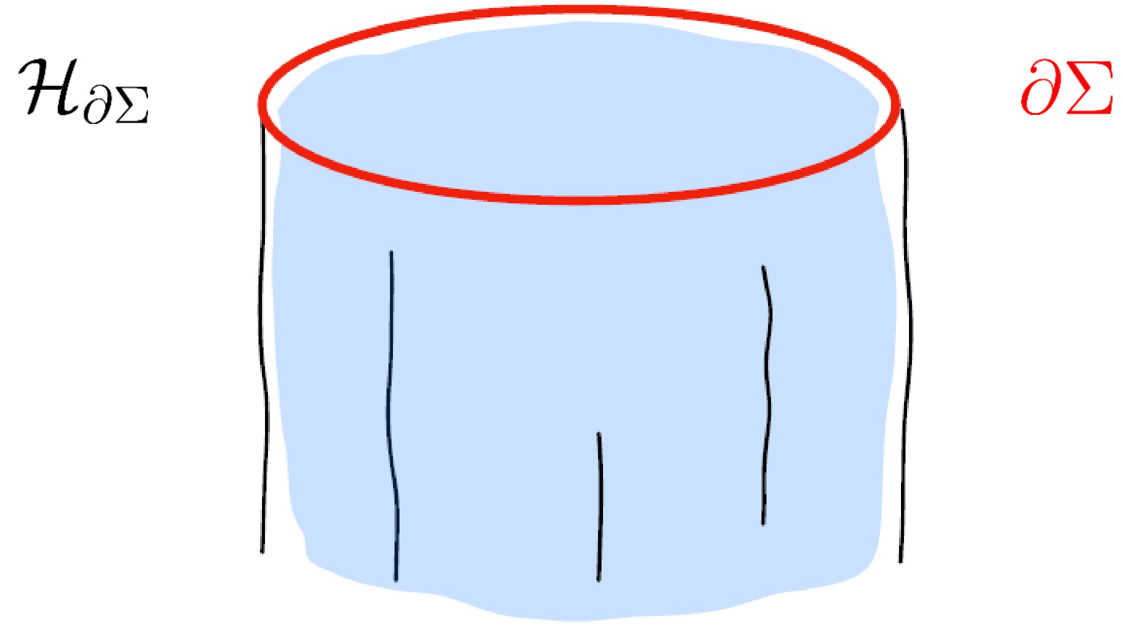


$$\zeta(M) = \int_{\Phi\sim M} \mathcal{D}\Phi e^{-S[\Phi]} = \langle\varphi|\psi\rangle \quad \text{on} \quad \mathcal{H}_{\partial\Sigma}$$

The path integral and the Hilbert space

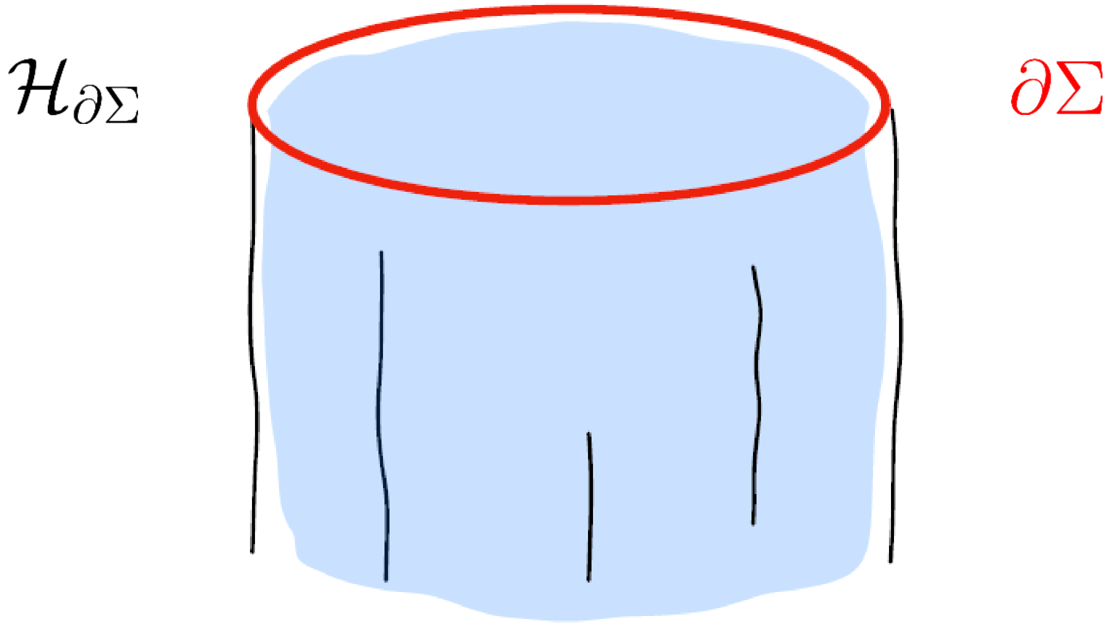


The path integral and the Hilbert space



$$\mathcal{H}_{QG} = \bigoplus_{\partial\Sigma} \mathcal{H}_{\partial\Sigma}$$

The path integral and the Hilbert space



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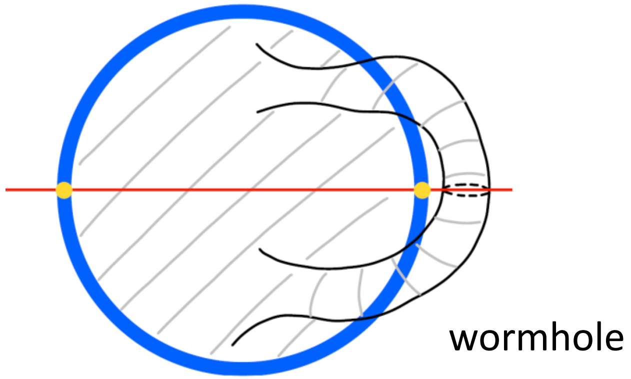
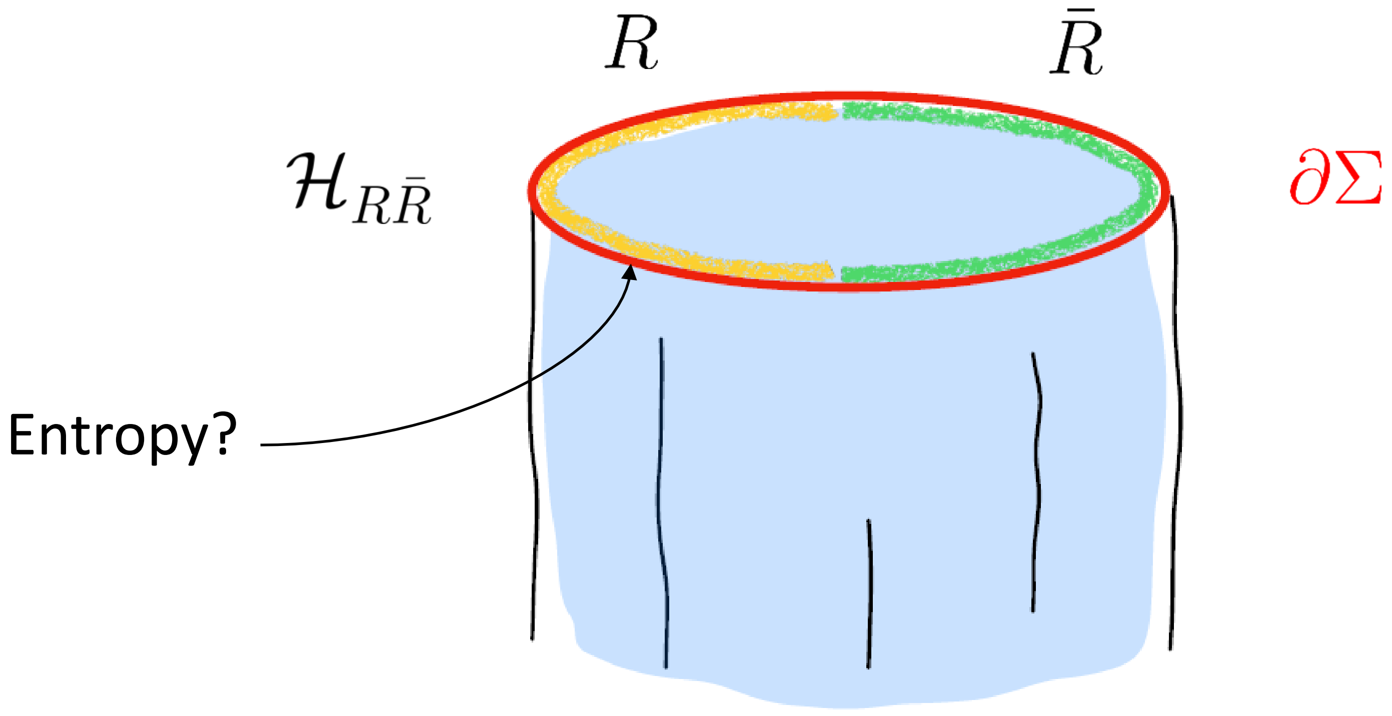


$\mathcal{H}_{\partial\Sigma=\emptyset} = \mathcal{H}_{\mathrm{BU}}$ Baby universe Hilbert space [*Giddings, Strominger, Coleman…*]

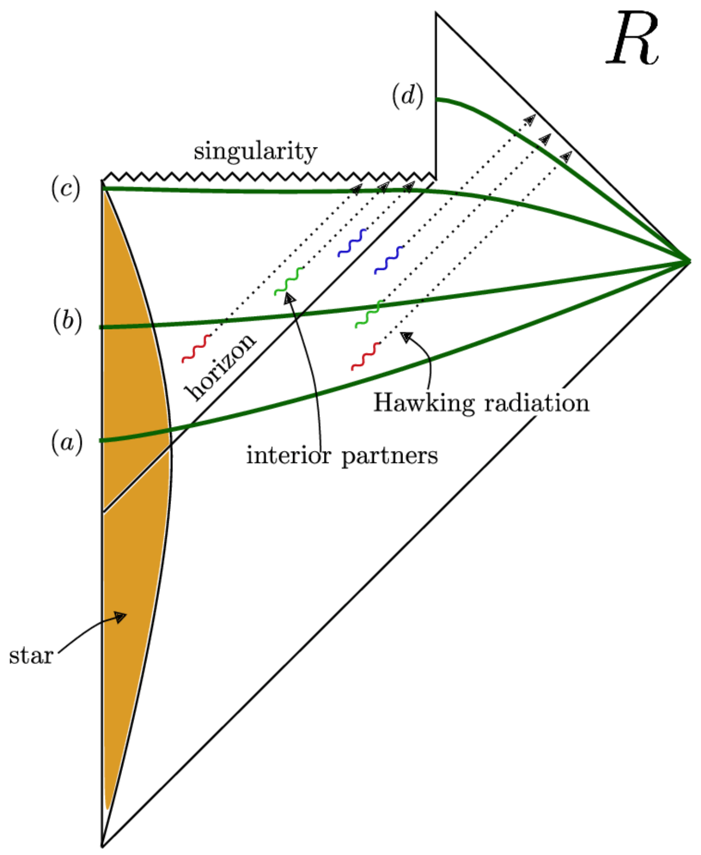
Notion of gravitational subsystem?



A priori $\mathcal{H}_{R\bar{R}}$ does not factorize over R and $\bar{R}$

Notion of subsystem $\sim \mathcal{H}_R$? Entropy associated to it?

An interesting scenario: the information paradox



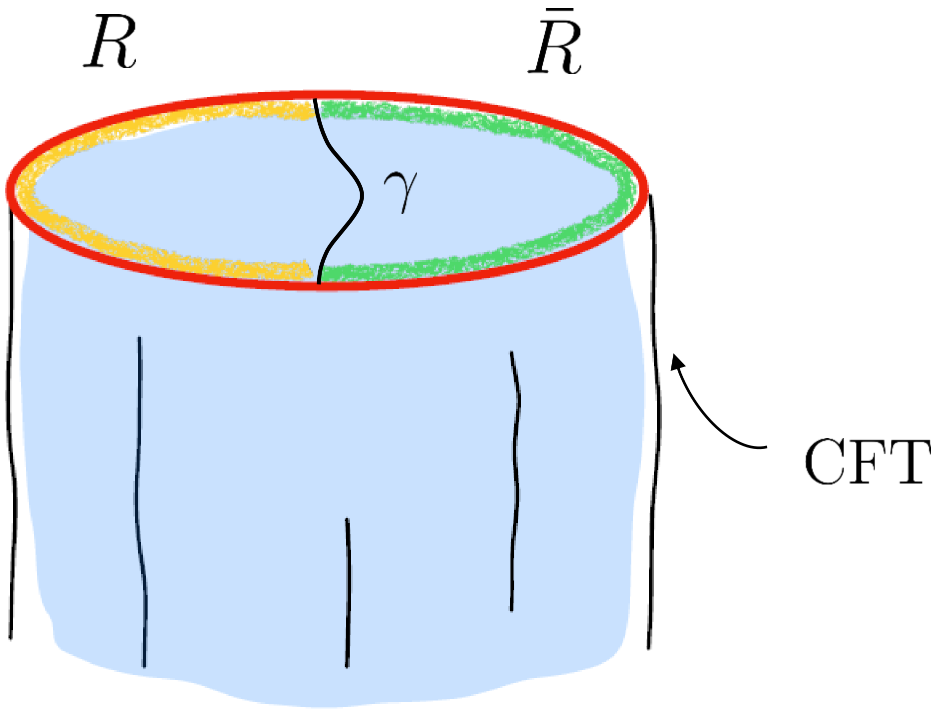
2

Holography and the **Ryu-Takayanagi formula**

The Ryu-Takayanagi formula

If the (bulk) gravitational theory has a holographic dual (boundary) theory, the Hilbert space takes the form

$$\mathcal{H}_{R\bar{R}} = \bigoplus_{\alpha} \mathcal{H}_R^{\alpha} \otimes \mathcal{H}_{\bar{R}}^{\alpha}$$



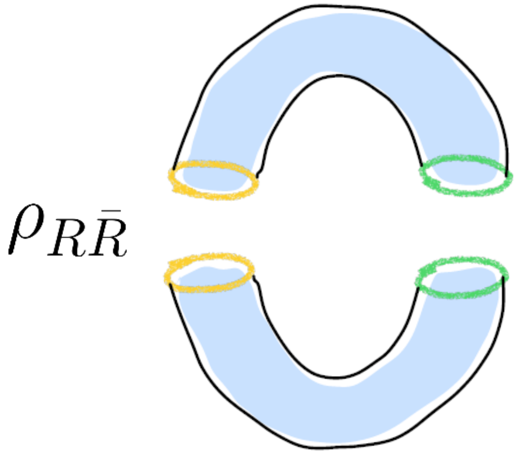
Ryu-Takayanagi formula

$$S(\rho_R) = \text{ext}_{\gamma} \left(\frac{A_{\gamma}}{4G} + S_{\text{matter}} \right)$$

Ryu-Takayanagi from the gravitational path integral

Lewkowycz-Maldacena calculation (2013)

- Replica trick $S(\rho_R) = \lim_{n\to 1} \frac{1}{n-1} \log \mathrm{Tr}\rho_R^n$
- Gravitational path integral
- Holography $\mathcal{H}_{R\bar{R}} = \bigoplus_\alpha \mathcal{H}_R^\alpha \otimes \mathcal{H}_{\bar{R}}^\alpha$



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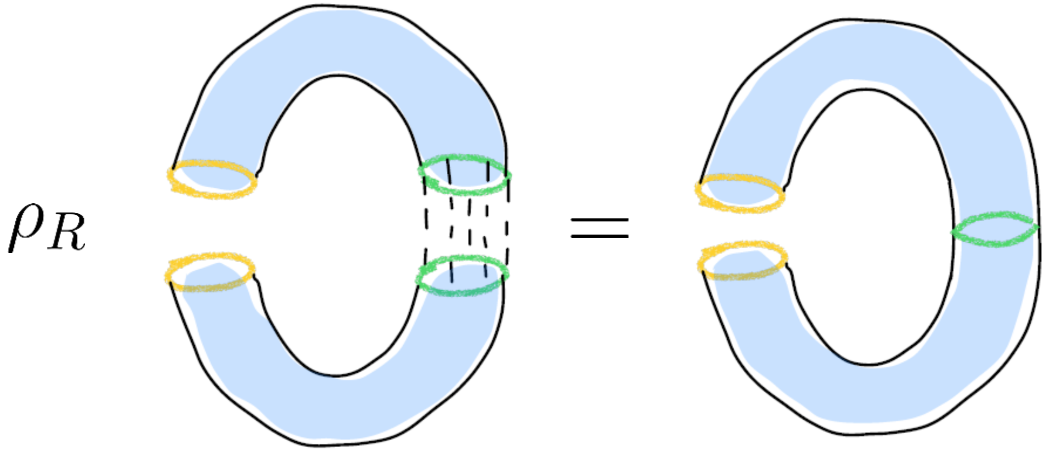

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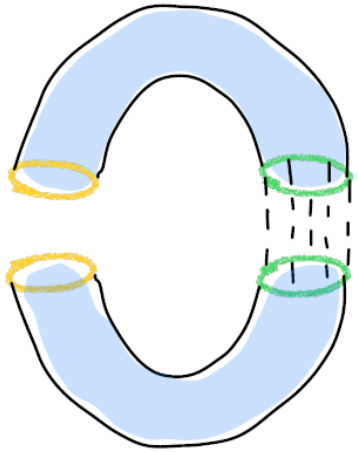
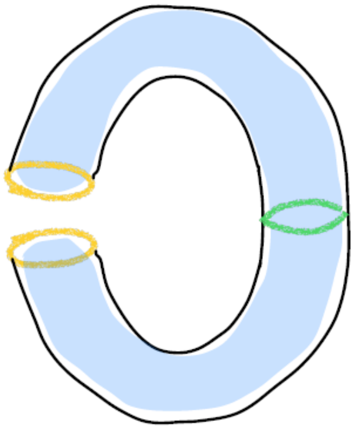
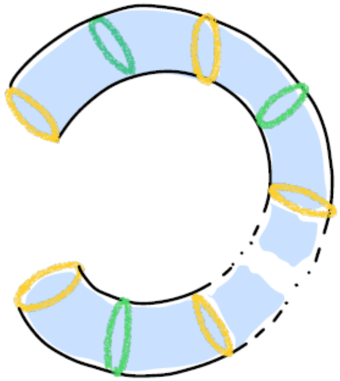
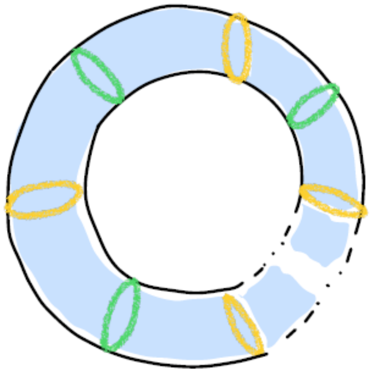
ρ_R  $=$ 

$\rho_R^n =$ 

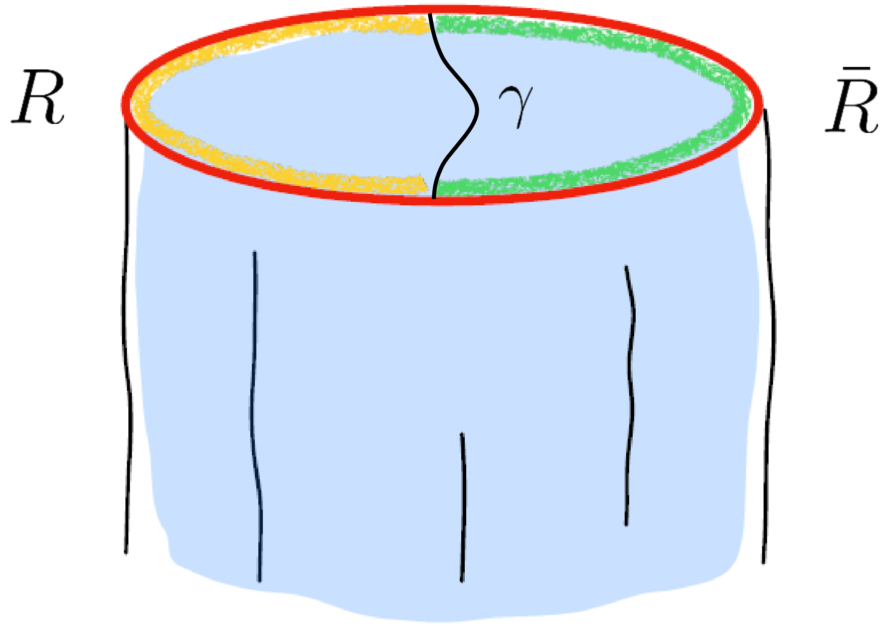
$\mathrm{Tr}\rho_R^n =$ 

→ **Ryu-Takayanagi formula**

semiclassical limit
(assuming bulk replica symmetry)

Without holography?

$$\mathcal{H}_{R\bar{R}} \stackrel{?}{=} \bigoplus_{\alpha} \mathcal{H}_R^{\alpha} \otimes \mathcal{H}_{\bar{R}}^{\alpha}$$



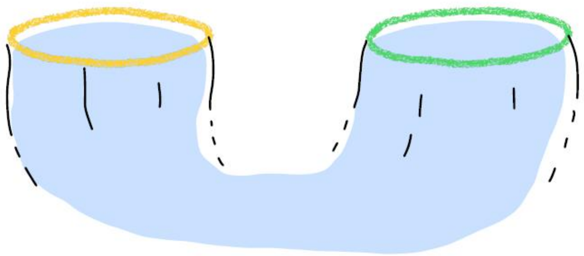
Is the Ryu-Takayanagi formula computing an entropy for a ***gravitational subsystem*** associated to R ?

3

Our work: understanding gravitational entropy **without holography**

Our results

Goal: understanding Ryu-Takayanagi as computing an entropy for gravitational subsystem without assuming holography

$\mathcal{A}_R$  $\mathcal{A}_{\bar{R}}$

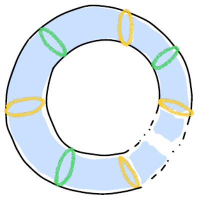
$\partial\Sigma =$ union of spatially-compact boundaries

We define von Neumann algebras of observables $\mathcal{A}_R, \mathcal{A}_{\bar{R}}$ acting on the Hilbert space $\mathcal{H}_{R\bar{R}}$ and show that, for a path integral satisfying a set of axioms, $\mathcal{A}_R, \mathcal{A}_{\bar{R}}$ decompose into type I factors.

$$\Longrightarrow \mathcal{H}_{R\bar{R}} = \bigoplus_i \mathcal{H}^i_R \otimes \mathcal{H}^i_{\bar{R}}$$

Trace operation tr on $\mathcal{A}_R, \mathcal{A}_{\bar{R}}$ defined as the evaluation of the gravitational path integral.

$\Longrightarrow \mathrm{tr}\rho_R^n$ where $\rho_R = \mathrm{tr}_{\bar{R}}(\rho_{R\bar{R}}) \in \mathcal{A}_R \Longrightarrow S(\rho_R) =$ **Ryu-Takayanagi formula**

Lewkowycz-Maldacena computation $\mathrm{Tr}\rho_R^n =$ 

Axioms for the gravitational path integral

Finiteness: $\zeta(M)$ is well-defined and finite for every smooth M

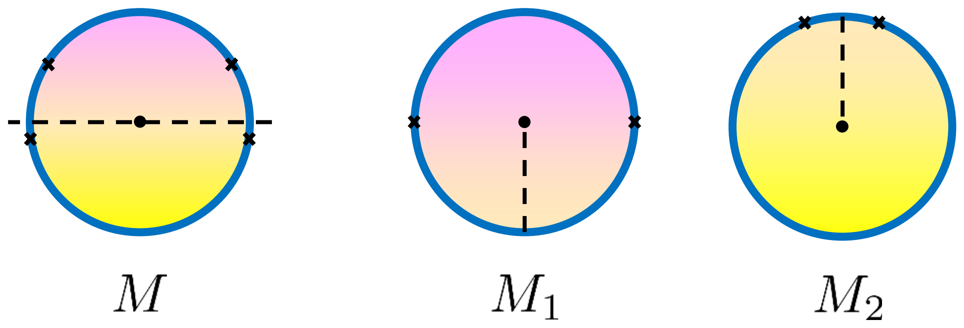
Reality: $[\zeta(M)]^* = \zeta(M^*)$

Reflection Positivity: $\zeta(MM^*) \geq 0$

Continuity: $\zeta(M_\epsilon)$ is a continuous function of ϵ

Factorization: $\zeta(M_1 \sqcup M_2) = \zeta(M_1)\zeta(M_2)$

Trace inequality: $\zeta(M) \leq \zeta(M_1)\zeta(M_2)$

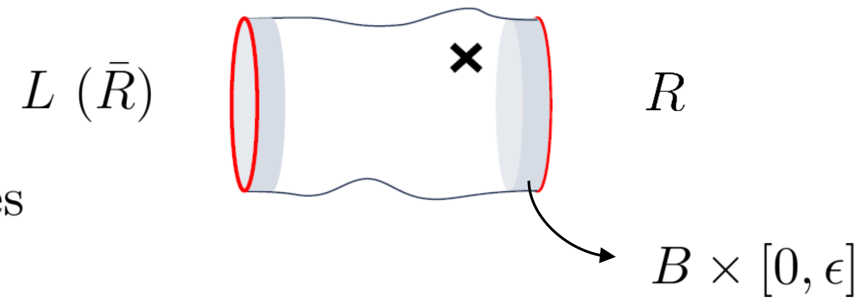


The trace inequality is ***not*** an independent axiom, it follows from the other axioms! [*Dong, Marolf, to appear*]

Surface algebras

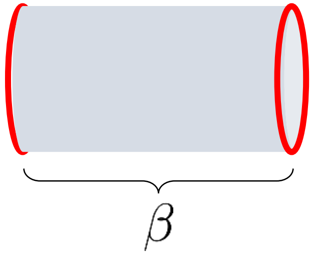
Rimmed surfaces

$B = (d-2)$-dim surface



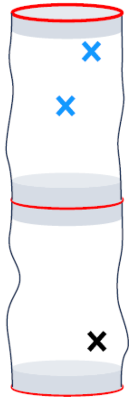
$X^B =$ set of $(d-1)$-dim *rimmed* surfaces with two B boundaries

Cylinder element: $C(\beta) = B \times [0, \beta]$

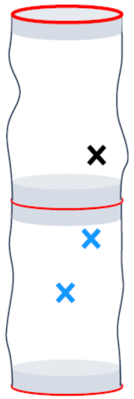


Multiplication = gluing

$a \cdot_R b$



$a \cdot_L b$

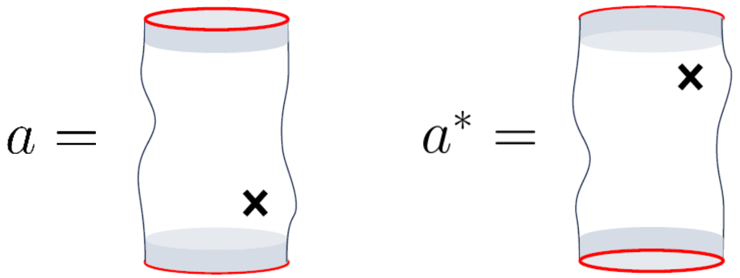


$\Longrightarrow$ surface algebras A_R^B and A_L^B

Surface algebras

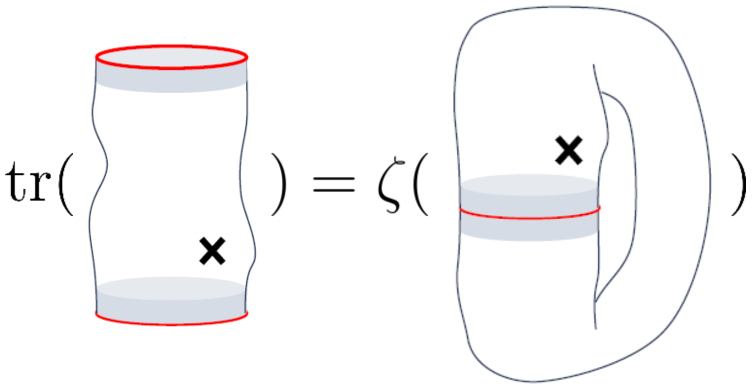
Conjugation map

acts by reversing the orientation, conjugating sources and exchanging the R and L labels



Trace operation

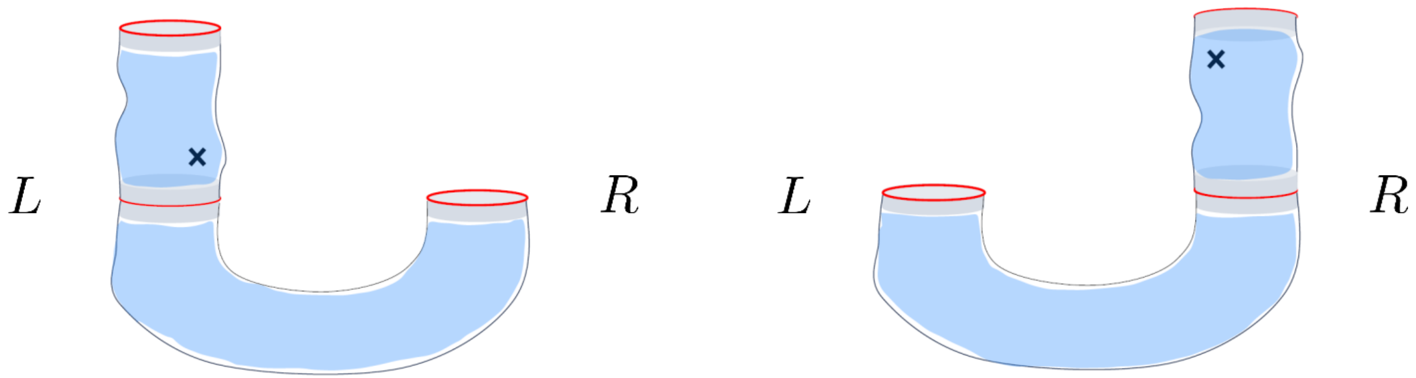
$\mathrm{tr} : A^B_{L/R} \to \mathbb{R}$ gluing + evaluation of the path integral



Reflection positivity: $\mathrm{tr}(aa^*) \geq 0$

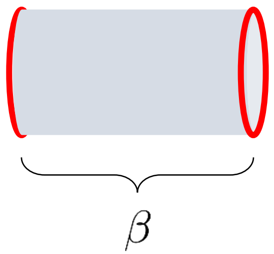
The von Neumann Algebras

Representation on the Hilbert space $\mathcal{H}_{RL}$



Quotient by null states and **closure** $\Longrightarrow$ von Neumann Algebras $\mathcal{A}_{L/R}$

$$\mathrm{tr}(a) = \lim_{\beta \to 0} \langle C(\beta)|a|C(\beta)\rangle$$

$C(\beta) =$ 

- Faithful $\mathrm{tr}(a) = 0$ iff $a = 0$
- Normal for any bounded increasing sequence a_n, $\mathrm{tr}\ sup\ a_n = sup\ \mathrm{tr}\ a_n$
- Semifinite $\forall a \in \mathcal{A}^+, \exists b < a$ such that $\mathrm{tr}(b) < \infty$

For a type I or type II factor, a *faithful, normal, semifinite trace* is **unique** up to an overall coefficient.

For type III such a trace does not exist

Type I factors and entropy

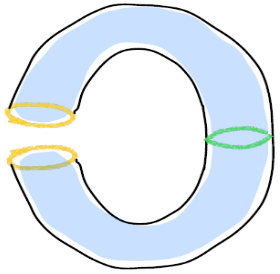
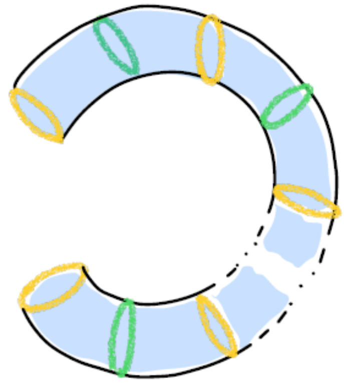
$\mathcal{A}_{L/R}$ non-trivial center $\longrightarrow$ $\mathcal{A}_{L/R} = \bigoplus_i \underbrace{\mathcal{A}^i_{L/R}}_{\text{trivial center}}$ direct sum over the spectrum of the center operators as represented on $\mathcal{H}_{LR}$

Trace inequality $\mathrm{tr}(ab) \leq \mathrm{tr}(a)\mathrm{tr}(b)$ for $a = b = P_i$ projector $\implies$ $\mathrm{tr}(P_i) \geq 1$ $\implies$ $\mathcal{A}^i_R$ type I factor!

Therefore,

$$\mathcal{H}_{LR} = \bigoplus_i \mathcal{H}^i_L \otimes \mathcal{H}^i_R$$

and

$\rho_R =$  $\rho^n_R =$ 

operators in the von Neuman algebra!

The Lewkowycz-Maldacena procedure then computes the entropy $-\mathrm{tr}(\rho_R \log \rho_R)$ which, in the semiclassical limit, is given by the **Ryu-Takayanagi formula**.

Thanks for the attention!