Tensor eigenvalue distributions through field theoretical methods

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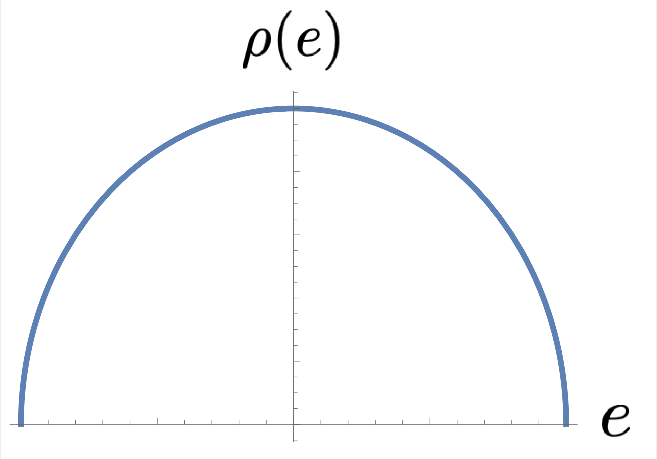
§ Introduction

Eigenvalue distributions of matrix models play important roles in understanding atoms, 2-dim quantum gravity, QCD, etc.

H ~ random matrix : Semicircle law E.Wigner 1958



Solving matrix models via $\rho(e)$

E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, 1978

Gross-Witten-Wadia transition, topological change of $\rho(e)$

D. J. Gross and E. Witten, S. R. Wadia, 1980

What roles eigenvalue/vector distributions can take in tensor models ?

Tensor eigenvalue/vector distributions were previously studied in

- Expectation numbers of real tensor eigenvalues

  P. Breiding, SIAM Journal on Applied Algebra and Geometry 1, 254-271 (2017).
  P. Breiding, Transactions of the American Mathematical Society 372, 7857-7887 (2019).

- Estimation of the largest eigenvalue

  O. Evnin, Lett. Math. Phys. 111, 66 (2021) doi:10.1007/s11005-021-01407-z [arXiv:2003.11220 [math-ph]].

- Extension of Wigner semicircle law

  R. Gurau, [arXiv:2004.02660 [math-ph]].

Real tensor eigenvalue distribution is the same as to count the critical points of the Hamiltonian (complexity) of the spherical *p*-spin model for spin glasses.

$$H = C_{abc} w_a w_b w_c, \quad w_a w_a = 1 \qquad C_{abc} : \text{random (Gaussian)}$$

$(p = 3)$

This has comprehensively been solved vie matrix model techniques in

Auffinger, A., Arous, G.B. and Černý, J. (2013), "Random Matrices and Complexity of Spin Glasses." Comm. Pure Appl. Math., 66: 165-201. https://doi.org/10.1002/cpa.21422

Accordingly, the end results of this talk are not new. However, the method we use is different, i.e., field theoretical, and give insights and extensions different from previous studies, as will be mentioned briefly at the summary.

§ Tensor eigenvalues/vectors

Consider real symmetric order-three tensor C_{abc} $(a, b, c = 1, \ldots, N)$

Tensor eigenvalues/vectors of C_{abc} :

$$C_{abc} v_b v_c = \zeta \, v_a \qquad \zeta : \text{Eigenvalue} \quad v_a : \text{Eigenvector}$$

L.Qi 2005, L.H.Lim 2005, D.Cartwright and B.Sturmfels 2013

There exist some differences from the matrix case:

- A system of N non-linear equations
- Not unique: can be rescaled by $\zeta \to c\,\zeta,\ v_a \to c\, v_a$
- Even if C_{abc} is real, ζ, v_a are not necessarily real.

Accordingly there are some different notions of eigenvalues/vectors.

Ex. Z-eigenvalue (Qi) : $\zeta\ (\geq 0)$ with $v \in \mathbb{R}^N,\ |v| = 1$

In this talk we compute the distributions of real eigenvectors and eigenvalues:

C_{abc} : symmetric real tensor, Gaussian distribution

Real eigenvector distribution → Real eigenvalue distribution

$$C_{abc}v_b v_c = v_a$$

$$v \in \mathbb{R}^N$$

$$C_{abc}w_b w_c = \zeta w_a \qquad w_a = \frac{v_a}{|v|}$$

$$\zeta = \frac{1}{|v|}$$

(Z-eigenvalues)

What is new in this talk is that we use field theoretical methods instead of matrix models.

§ Eigenvector distributions

- For a given C_{abc}

$$\rho(v, C) = \sum_{i=1}^{\#\mathrm{sol}(C)} \delta^N(v - v^i) \qquad C_{abc} v_b^i v_c^i = v_a^i \quad v^i \in \mathbb{R}^N$$

$$= |\det M| \prod_{a=1}^{N} \delta(C_{abc} v_b v_c - v_a)$$

$$M_{ab} = \frac{\partial}{\partial v_a}(v_b - C_{bcd} v_c v_d) = \delta_{ab} - 2C_{abc} v_c \ \ : \text{Jacobian}$$

- For a Gaussian distributed C_{abc}

$$\rho(v) = \langle \rho(v, C) \rangle_C = A^{-1} \int_{\mathbb{R}^{\#C}} dC \, e^{-\alpha C^2} \rho(v, C) \qquad C^2 = C_{abc} C_{abc} \qquad \alpha > 0$$

$$= A^{-1} \int_{\mathbb{R}^{\#C}} dC \, e^{-\alpha C^2} |\det M| \prod_{a=1}^{N} \delta(C_{abc} v_b v_c - v_a)$$

§ Field theoretical expression

Rewrite $|\det M|$

(I) Signed distribution (Just forget $|\cdot|$)

$$\det M = \int d\bar{\psi} d\psi \, e^{\bar{\psi} M \psi} \qquad \bar{\psi}, \psi : \text{fermions}$$

(II) Distribution

- $$|\det M| = \lim_{R\to 1/2, \epsilon\to +0} \left\{\det(M^2 + \epsilon I)\right\}^R$$

  For integer R

  $$\left\{\det(M^2 + \epsilon I)\right\}^R = (-1)^{NR} \int d\bar{\psi} d\psi d\bar{\varphi} d\varphi \, e^{-\bar{\varphi}^i \varphi_i + \epsilon \bar{\psi}^i \psi_i - \bar{\psi}^i M \psi_i - \bar{\varphi}^i M \varphi_i}$$

- $$|\det M| = \lim_{\epsilon\to +0} \frac{\det(M^2 + \epsilon\, I)}{\sqrt{\det(M^2 + \epsilon\, I)}} \quad \begin{matrix} \longrightarrow \text{fermions} \\ \longrightarrow \text{bosons} \end{matrix}$$

Rewrite

$$\prod_{a=1}^{N} \delta(C_{abc}v_b v_c - v_a) = (2\pi)^{-N} \int_{\mathbb{R}^N} d\lambda\, e^{i\lambda_a(v_a - C_{abc}v_b v_c)}$$

Then generally we have

$$\rho_{\cdot}(v) = \int dC\, d\lambda\, d\bar{\psi} d\psi\, d\phi \cdots \; e^{S}$$

$$S = -\alpha C^2 + i\lambda_a(v_a - C_{abc}v_b v_c) - (\bar{\psi}, \psi, \phi, \cdots)^2 - (\bar{\psi}, \psi, \phi, \cdots) \underset{\substack{\uparrow \\ C}}{M} (\bar{\psi}, \psi, \phi, \cdots)$$

$$= (C, \lambda) \begin{pmatrix} -\alpha & * \\ * & 0 \end{pmatrix} \begin{pmatrix} C \\ \lambda \end{pmatrix} + (C, \lambda) \begin{pmatrix} * \\ * \end{pmatrix} + \cdots$$

C and λ can be integrated out, since these are at most quadratic.

We obtain an effective theory of bosons and fermions with quartic interactions.

§ Computation of the effective field theories

(I) Signed distribution ($\det M$)

$$\rho(v) = 3^{(N-1)/2}\pi^{-N/2}\alpha^{N/2}v^{-2N}e^{-v^2/\alpha}\int d\bar{\psi}d\psi\, e^S$$

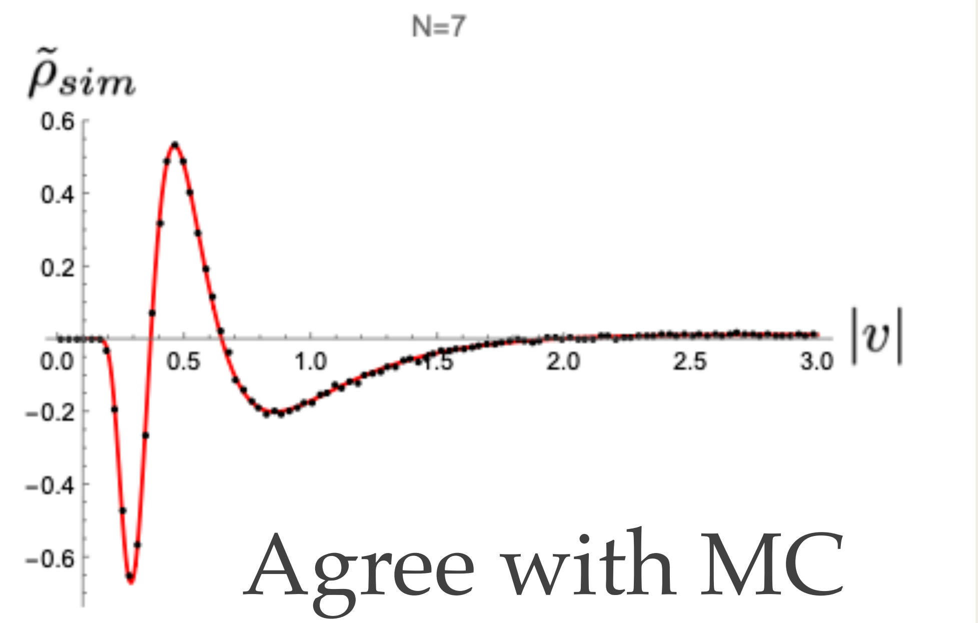
$$S = -\bar{\psi}_{\|}\psi_{\|} + \bar{\psi}_{\perp}\cdot\psi_{\perp} - \frac{v^2}{6\alpha}(\bar{\psi}_{\perp}\cdot\psi_{\perp})^2$$

$\|$: parallel to v, $\perp$: transverse to v

Exactly computed as

$$\rho(v) = -3^{1/2}2^{-1+N/2}\alpha\,\pi^{-N/2}e^{-\alpha/v^2}|v|^{-N-2}U\left(1-\frac{N}{2},\frac{3}{2},\frac{3\alpha}{2v^2}\right)$$

Confluent hypergeom. fn.
of the second kind



Agree with MC

(II) Distribution ($|\det M|$)

S is more complicated.

$$S = K_B + K_F + V_F + V_B + V_{BF}$$

σ_a, ϕ_a: bosons, $\bar{\psi}_a, \psi_a, \bar{\varphi}_a, \varphi_a$: fermions $(a = 1, 2, \cdots, N)$

$$K_B = -\sigma^2 - 2i\sigma\phi - \epsilon\phi^2$$

$$K_F = -\bar{\varphi}\varphi - \bar{\psi}\varphi - \bar{\varphi}\psi - \epsilon\bar{\psi}\psi$$

$(\bar{\psi}\psi) = \bar{\psi}_a\psi_a$, etc.

$$V_F = -\frac{v^2}{6\alpha}\left((\bar{\psi}\varphi)^2 + (\bar{\varphi}\psi)^2 + 2(\bar{\psi}\bar{\varphi})(\varphi\psi) + 2(\bar{\psi}\psi)(\bar{\varphi}\varphi)\right)$$

$$V_B = -\frac{2v^2}{3\alpha}(\sigma^2\phi^2 + (\sigma\phi)^2)$$

$$V_{BF} = \frac{2iv^2}{3\alpha}((\bar{\psi}\sigma)(\varphi\phi) + (\bar{\varphi}\sigma)(\psi\phi) + (\bar{\psi}\phi)(\varphi\sigma) + (\bar{\varphi}\phi)(\psi\sigma))$$

But we can obtain

- Exact analytic expressions for small N (,R) in terms of error fn.

$$G_{N=8} = \pi^{\frac{13}{2}}\left(\frac{\sqrt{2}e^{-\frac{1}{8z}}(1+210z^2-2100z^3+12600z^4+25200z^5)}{15z^{\frac{3}{2}}} + (1-42z+420z^2-840z^3)\,\gamma\left[\frac{1}{2},\frac{1}{8z}\right]\right). \qquad z = v^2/6\alpha$$

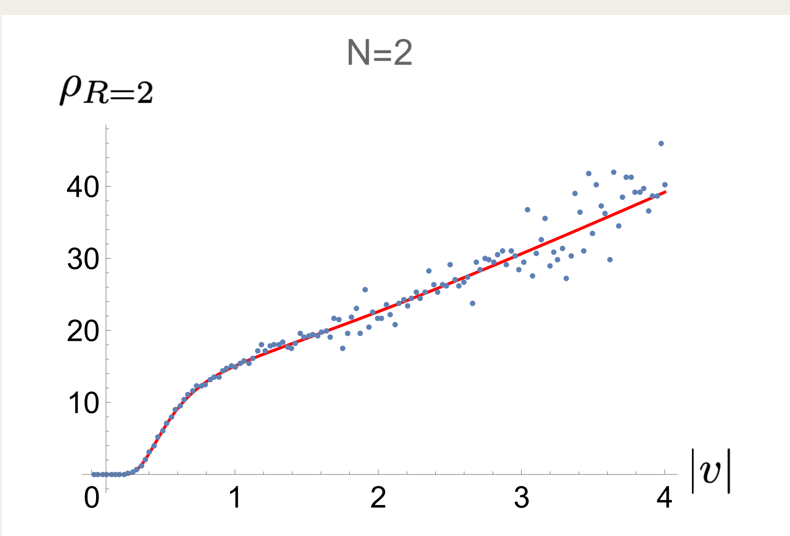
- The expression for large-N computed through Schwinger-Dyson equation. It turns out that the eigenvalue distribution for large-N is given by Gaussian.

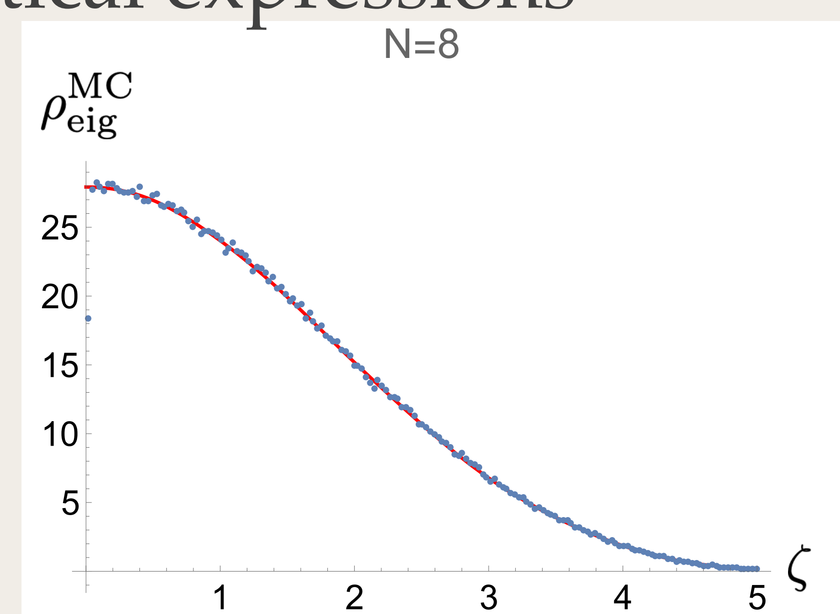
$$\rho(\zeta) \sim 2^{-\frac{N}{2}+2}\alpha^{\frac{1}{2}}\pi^{-\frac{1}{2}}\frac{\Gamma[N+1]}{\Gamma\left[\frac{N}{2}+1\right]\Gamma\left[\frac{N}{2}\right]}e^{-\frac{\alpha}{4}\zeta^2} \qquad \text{for } 1 \ll N$$

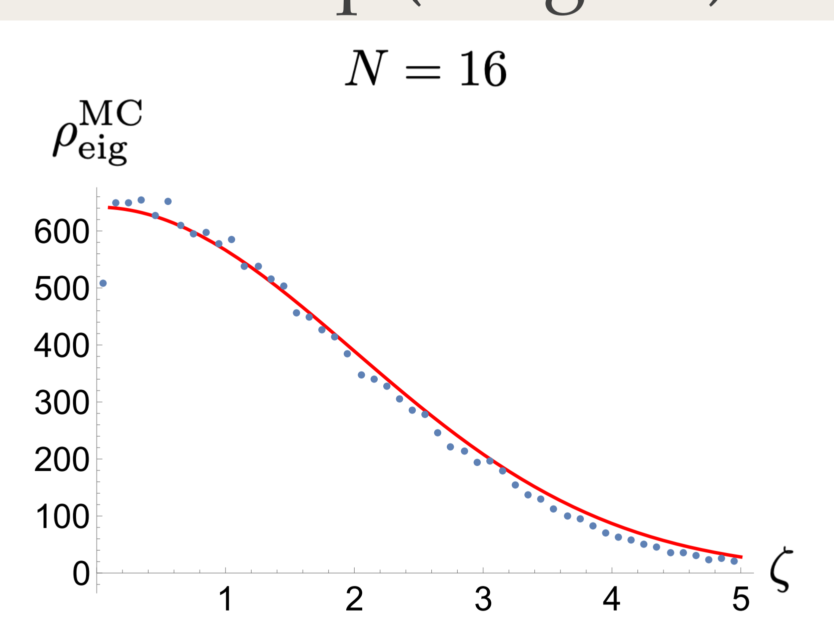
Compared with MC

Exact analytical expressions

S.-D. eq. (Large N)







§ Summary and future prospects

We have computed real eigenvalue/vector distributions for order-three real symmetric tensors with Gaussian distributions.

- Some exact analytical expressions
- The large-N limit of the eigenvalue distribution is given by Gaussian, which contrasts with Wigner's semicircle law in the matrix model.

Extensions with similar procedures

- Correlations among eigenvectors. In matrix models, eigenvalues are repulsive, how about tensor models ?

- Why integrable ? Obtain exact formulas of eigenvalue/vector distributions for any N, R

- Introduce allowances to eigenvector equation *(with N.Delporte, R.Toriumi)*

  $C_{abc}v_b v_c = v_a + \eta_a \quad \eta_a$: Gaussian noise

- Introduce backgrounds *(with Z. Mirzaiyan)*

  $C_{abc} \to Q_{abc} + C_{abc} \qquad Q_{abc}$: fixed

- Complex eigenvalues *(With S.Majumder)*

- Analysis of tensor rank decompositions