Random Disks of Constant Curvature

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The problem

Motivated by recent developments in theoretical physics, we want to study models of 2D random geometries for which the curvature is fixed (positive, zero or negative).

Because the curvature is fixed, there is no degree of freedom in the bulk.
The randomness entirely comes from the fluctuating boundaries.

Let us note that this problem lies in between the two versions of 2D gravity / 2D random geometry that have been studied extensively in the past :

- The Liouville gravity, for which the bulk geometry fluctuate wildly (Brownian sphere, see the beautiful pics in Nicolas Curien talks)
- The topological gravity, for which the curvature is fixed and the boundaries are geodesics. There is then a finite number of moduli (Weil-Peterson measure, Kontsevitch-Witten, Mirzakhani, topological recursion)

We focus on the disk topology (generalizing is non-trivial, as we shall briefly discuss).

We consider mainly the flat (zero curvature) case (going to negative or positive curvature is straightforward for most of what we are going to explain today; we'll give some explicit information on negative curvature).

Let us now define very precisely the model of random curves we would like to study.

Let us use a lattice $\mathbb{Z}^2$, choose a base point 0, and consider the walks γ of length n, starting and ending at 0 (closed loops made of n steps). Let's call Γ_n the space of such loops.

On Γ_n we can consider various probability measures.

- the simplest is the uniform measure : simple random closed walks. The scaling limit yields Brownian bridges (the Wiener measure, the path integral for the partition function $\mathrm{tr}\, e^{-\beta H}$ in quantum mechanics, etc).
- we can impose that walks that have crossings have probability zero while all the others have the same probability; this yields the self-avoiding walk model (SAW) with a nice scaling limit to SLE for $\kappa = 8/3$. Other probability distributions on self-avoiding curves yield other values of κ.

We are going to introduce a very different probability measure.

We keep only the closed walks that bound a (distorted) disk.

We assign a multiplicity to each closed walk corresponding to the number of distinct disks that it bounds. This multiplicity can be computed algorithmically. Many loops have multiplicity zero.

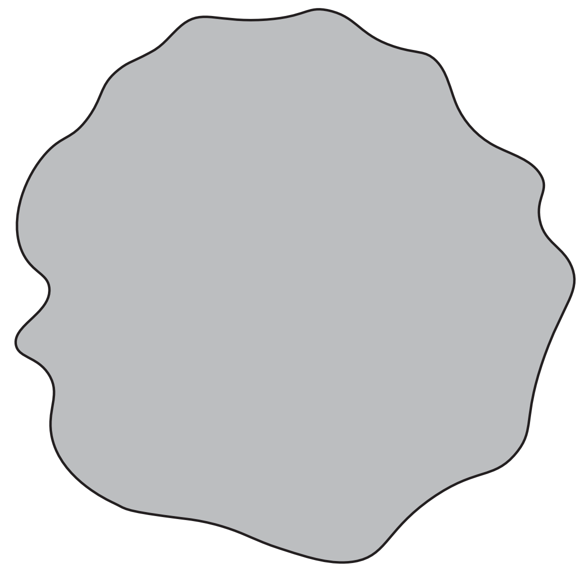
Goal: compute "partition functions" of the form

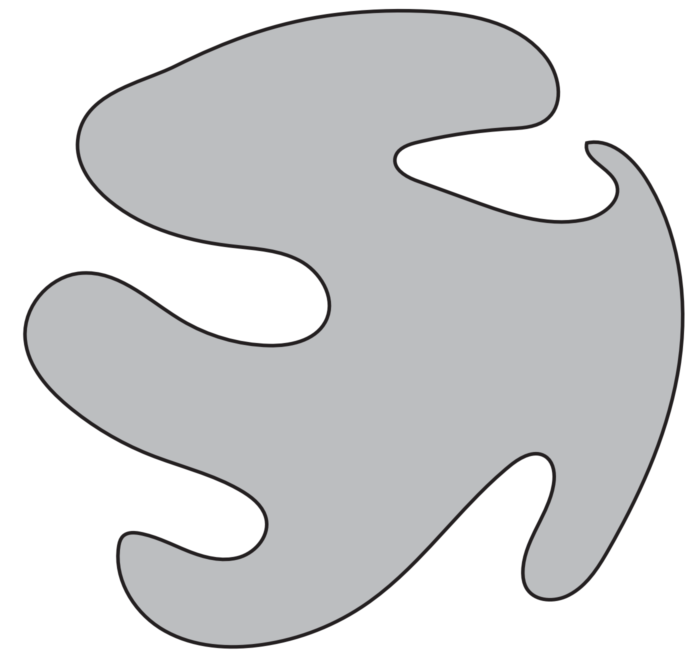
$$F_n(z) = \sum_{\gamma \in \Gamma_n} \nu(\gamma) z^{A(\gamma)} \qquad Z(z, w) = \sum_n F_n(z) w^n$$

A is the area of the disk bounded by the loop (number of lattice squares inside the curve).

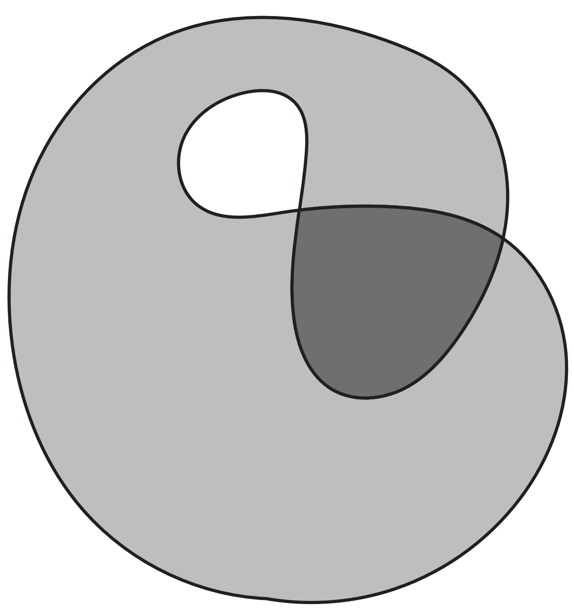
Show that there is a scaling, continuum limit. Identify the limit (a candidate will be proposed). Compute interesting "observables" (examples will be given).

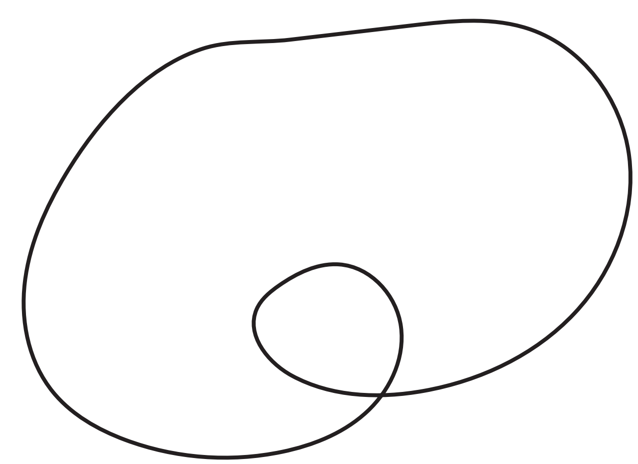
Examples





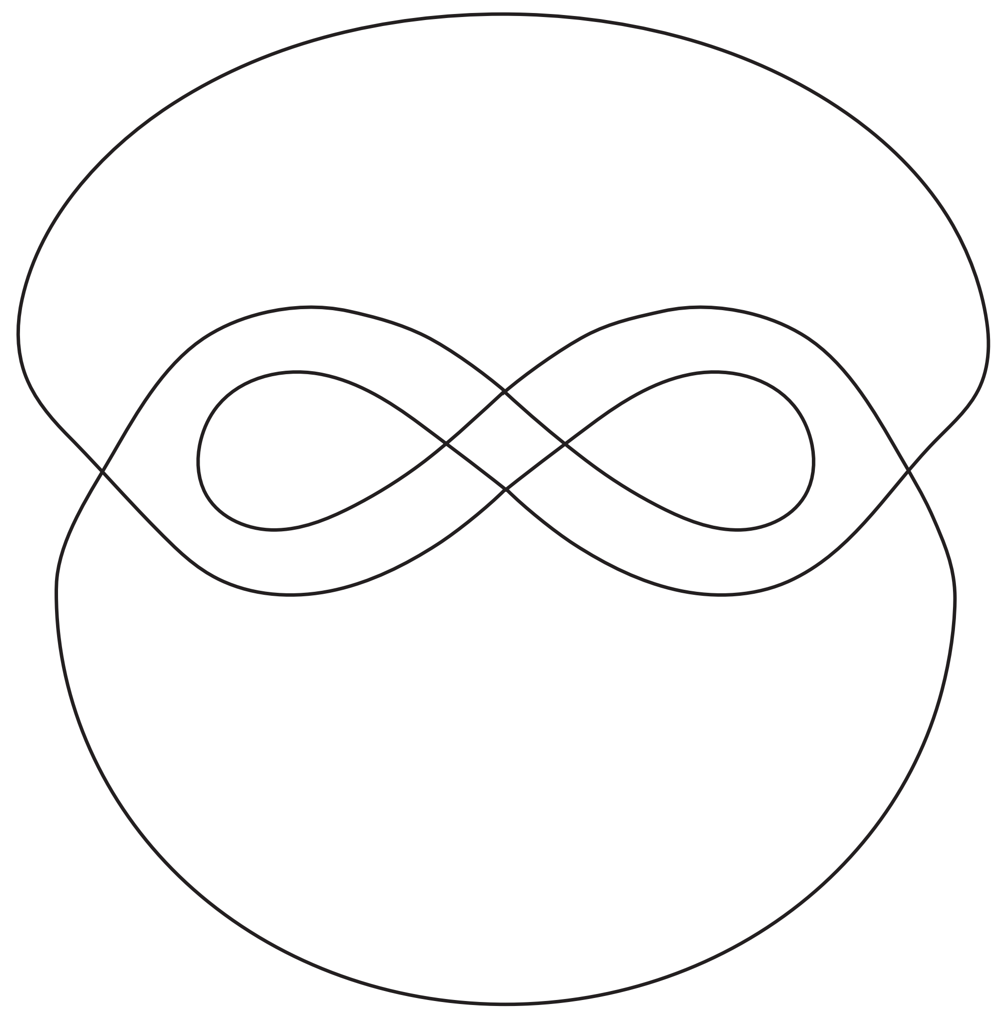
SAW are all included, with multiplicity one.

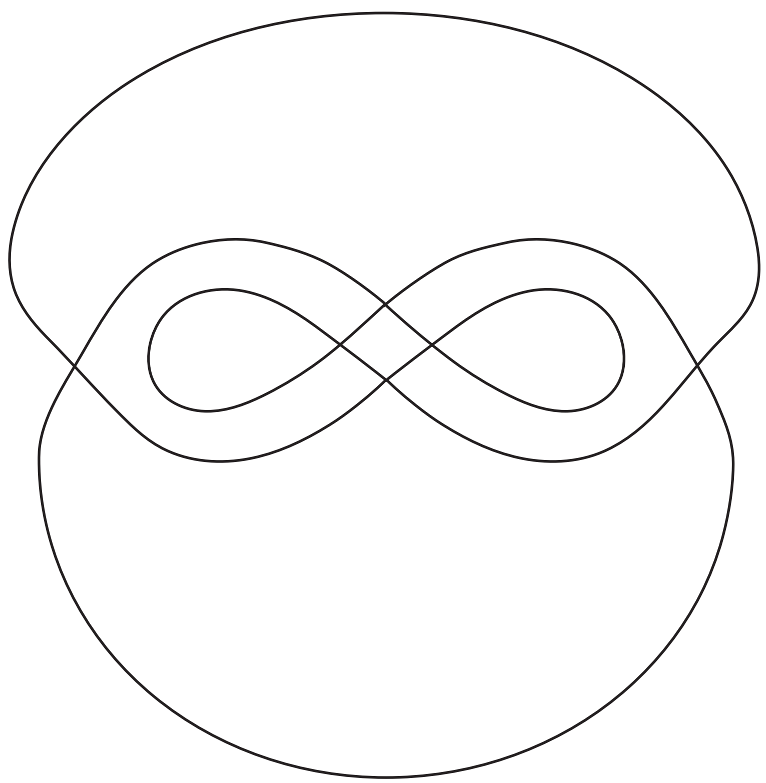


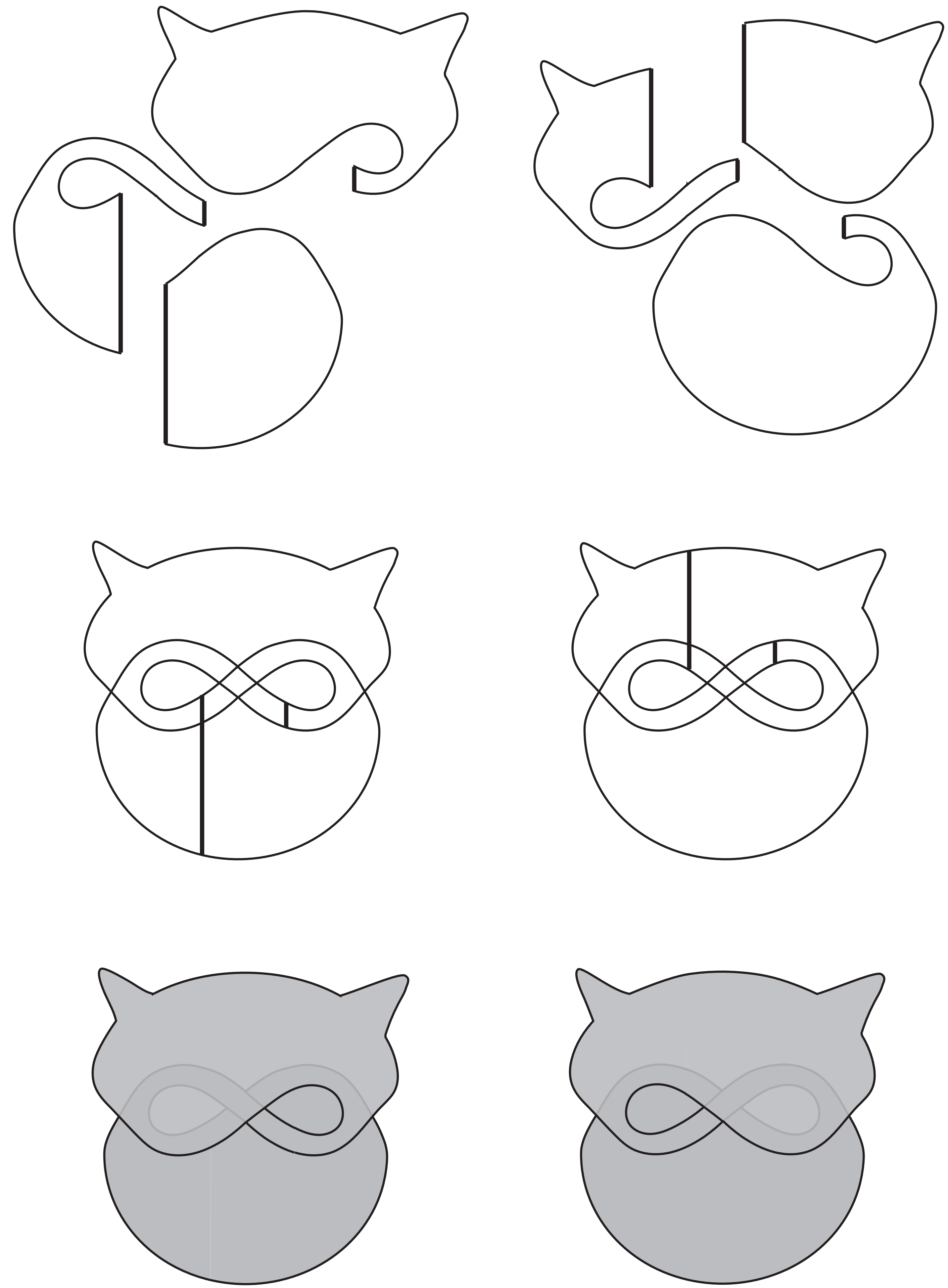


Some curves with self-intersections are included, but not all (arbitrary Brownian bridges do not work !)

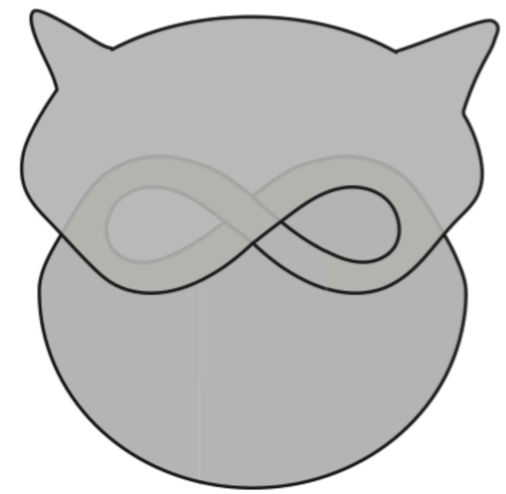
What about this example (the Milnor curve)?







Simple constraints on curves that bound a disk

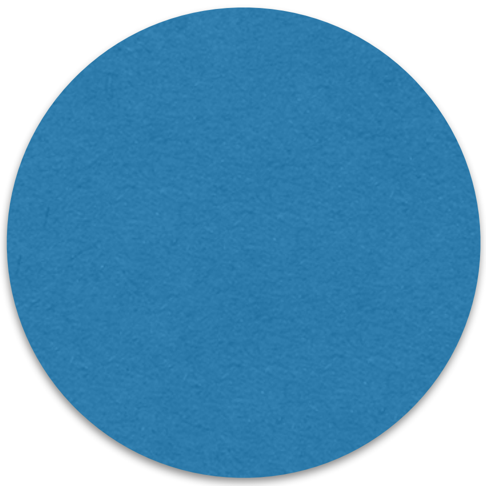


1) the index of the curve (which is the winding number of the tangent vector when one goes along the curve) must be one.
2) the winding number of the curve with respect to any point is equal to the number of layers of the disk covering the point and thus must be positive.

Note that an interesting consequence is that all disks bounded by a given curve must have the same area (even though the corresponding metrics are not related by a diffeomorphism).

Full story: Blank-Poeranu, Astérisque 10 (1966-1968) 473, Bourbaki seminar; Graver and Cargo, SIAM Discrete math 25,1 (2011) 280 (nice review).

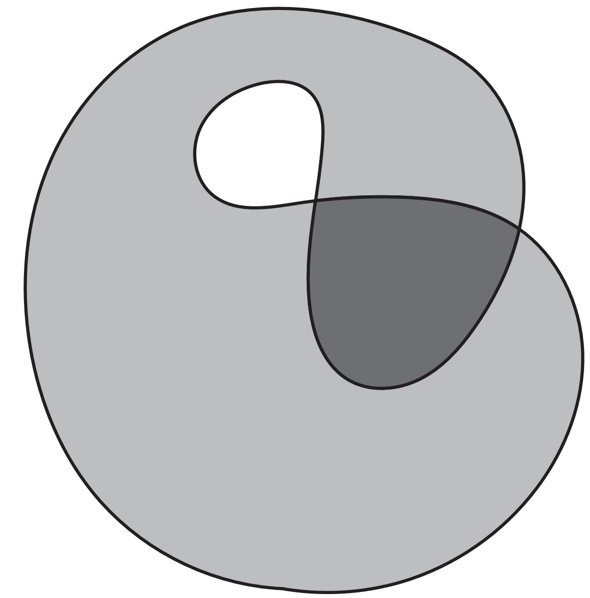
The relation with constant curvature random metrics on the disk



Topological disk

immersion

$$g = F^* G$$



Disk immersed in the Euclidean plane, hyperbolic space or the round sphere

$$\mathfrak{M}(\mathscr{D}) = \begin{cases} \mathrm{ISO}(2)\backslash\mathscr{H}(\mathscr{D})/\mathrm{PSL}(2,\mathbb{R}) \\ \mathrm{SO}(3)\backslash\mathscr{H}(\mathscr{D})/\mathrm{PSL}(2,\mathbb{R}) \\ \mathrm{PSL}(2,\mathbb{R})\backslash\mathscr{H}'(\mathscr{D})/\mathrm{PSL}(2,\mathbb{R}) \end{cases}$$

$$\mathrm{d}s^2 = \begin{cases} |F'(z)|^2 \; |\mathrm{d}z|^2 \\ \dfrac{4|F'(z)|^2}{(1 \pm |F(z)|^2)^2} \; |\mathrm{d}z|^2 \end{cases}$$

<u>The boundary Liouville field</u>

$$\mathrm{d}s^2 = e^{2\sigma}|\mathrm{d}z|^2$$

Liouville equation : $$4\partial_z\partial_{\bar{z}}\sigma = -\eta e^{2\sigma}$$

Theorem 8.2. *Let $\sigma_b : S^1 \to \mathbb{R}$ be a continuous function defined on the boundary of the disk. Then there exists a unique solution $\sigma \in C^\infty(\mathscr{D})$ of the Liouville equation such that $\sigma = \sigma_b$ on the boundary.*

This theorem simple in zero curvature: there is a unique harmonic function with prescribed boundary value on the disk. In this case, an explicit formula expressing σ in terms of σ_{b}, using the Poisson kernel, is well-known. A similar explicit formula for the non-linear Liouville equation does not exist, but the existence and uniqueness of the solution to the Dirichlet problem is still valid.

The boundary Liouville field is a good degree of freedom, solving all the combinatorics of the boundary curve for free.

Explicit reconstruction in the flat case:

$$F'(z) = e^{H(z)}$$

$$\operatorname{Re} H(e^{i\theta}) = \sigma_{\mathrm{b}}(\theta)$$

$$H(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1+ze^{-i\theta'}}{1-ze^{-i\theta'}} \operatorname{Re} H(e^{i\theta'})\, \mathrm{d}\theta' + i \operatorname{Im} H(0)$$

$$\gamma(\theta) = F(e^{i\theta})$$

Note that, by construction, the closed curve that we obtain in this way does bound a disk (even though it might be far from obvious on first sight, as we shall illustrate later).

Upshot: a probability measure on the space of constant curvature metrics on the disk, or equivalently on the space of closed curves that bound a disk (the problem we started from), is immediately obtained from a measure on the space of continuous periodic function!

This looks like a huge simplification of the problem, since we are quite familiar with this space. We may immediately think of using a Brownian bridge (Euclidean path integral for a free particle) to get a nice model.

But this is overlooking a crucial feature.

The disk automorphisms act with $$h_{\mathscr{D}}(z) = e^{i\alpha}\frac{z - z_0}{1 - \bar{z}_0 z}$$

which yields, for the boundary Liouville field,

$$\mathrm{PSL}(2,\mathbb{R}) \qquad h \cdot \sigma_{\mathrm{b}} = \sigma_{\mathrm{b}} \circ h^{-1} - \ln h' \circ h^{-1}$$

$$h(\theta) = 2\arctan\frac{a\tan\frac{\theta}{2} + b}{c\tan\frac{\theta}{2} + d}\,, \quad ad - bc = 1$$

$$z_0 = re^{i\phi}$$

$$a = \frac{\cos\frac{\alpha}{2} + r\cos(\frac{\alpha}{2} + \phi)}{\sqrt{1-r^2}}\,, \quad b = \frac{\sin\frac{\alpha}{2} - r\sin(\frac{\alpha}{2} + \phi)}{\sqrt{1-r^2}}$$
$$c = -\frac{\sin\frac{\alpha}{2} + r\sin(\frac{\alpha}{2} + \phi)}{\sqrt{1-r^2}}\,, \quad d = \frac{\cos\frac{\alpha}{2} - r\cos(\frac{\alpha}{2} + \phi)}{\sqrt{1-r^2}}\,.$$

The probability measure must be invariant under these transformations.

A very nice rephrasing is obtained by writing

$$e^{\sigma_{\rm b}} = \frac{\ell}{2\pi}(f^{-1})' \qquad\qquad \theta = f(\zeta)\,, \quad \zeta = \frac{2\pi s}{\ell}$$

$$f \in \mathrm{Diff}_+(\mathrm{S}^1)$$

The transformation law for σ is equivalent to the left multiplication on $\mathrm{Diff}_+(\mathrm{S}^1)$

$$h\cdot\sigma_{\rm b} = \sigma_{\rm b}\circ h^{-1} - \ln h'\circ h^{-1}$$

$$\Longleftrightarrow$$

$$h\cdot f = h\circ f$$

We have shown that the space of metrics on the disk is exactly diffeomorphic to a simple quotient of the group of diffeomorphisms of the circle:

$$\mathfrak{M}(\mathscr{D}) = \mathrm{PSL}(2,\mathbb{R})\backslash\mathrm{Diff}_+(\mathrm{S}^1)/\mathrm{S}^1$$

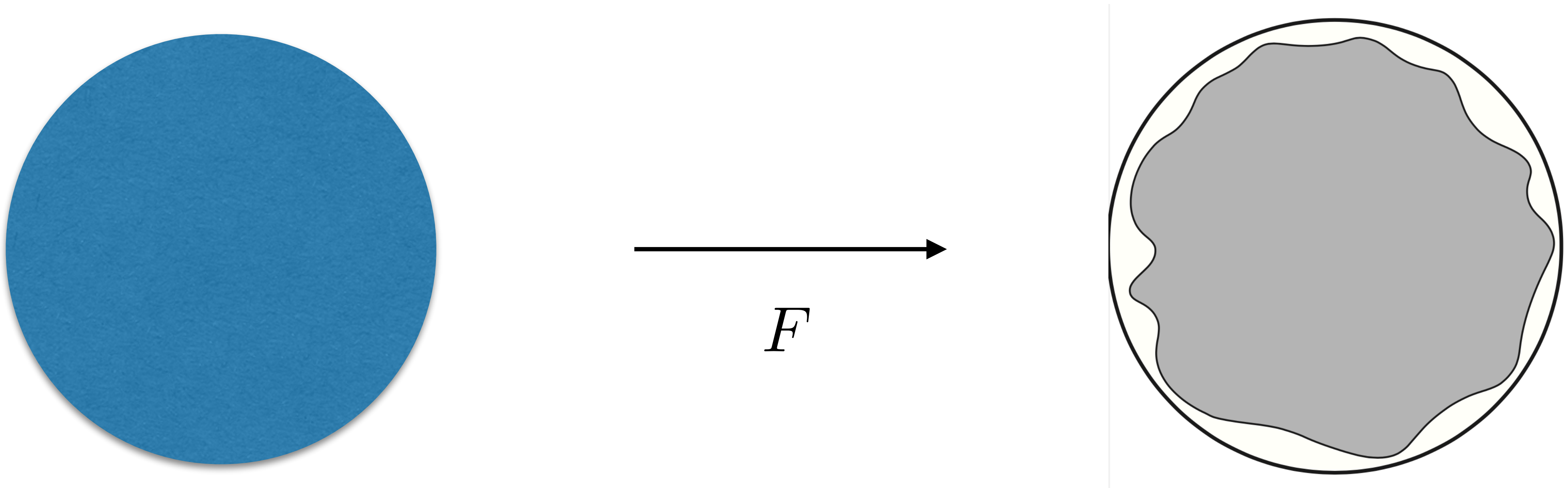
This is valid in zero, positive or negative curvature!

$$\mathfrak{M}(\mathscr{D}) = \mathrm{PSL}(2,\mathbb{R})\backslash \mathrm{Diff}_+(\mathrm{S}^1)/\mathrm{S}^1$$

A model is thus defined as soon as we have a measure on $\mathrm{Diff}_+(\mathrm{S}^1)$ that is invariant under left multiplication by elements of the $\mathrm{PSL}(2,\mathbb{R})$ subgroup.

We shall see that there are two natural models. One is known in mathematics and is nicely related to the Brownian bridge! The other, which is completely new, is conjectured to solve the problem we started from (i.e., to represent the continuum limit of the discretized model we started with).

Note on the reparameterization ansatz in negative curvature



$$z = \rho e^{i\theta}$$

$$e^{\Sigma} = \frac{\ell}{2\pi}(f^{-1})'$$

$$\theta = f(\zeta)$$

$$\mathrm{PSL}(2,\mathbb{R})\backslash\mathrm{Diff}_+(\mathrm{S}^1)$$

$$w = re^{i\phi}$$

$$r = 1 - \epsilon g(\zeta)$$

$$\phi = \mathsf{f}(\zeta)$$

$$\mathrm{PSL}(2,\mathbb{R})'\backslash\mathrm{Diff}_+(\mathrm{S}^1)$$

$$\mathsf{f} = f + \sum_{n\geq 1}\left(\frac{2\pi\epsilon}{\beta}\right)^n f_n$$

$$\mathsf{f}(\vartheta) = f(\vartheta) + \frac{2\pi\epsilon}{\beta}f_1(\vartheta) + O(\epsilon^2) = f(\vartheta) - \frac{\epsilon}{\beta}\mathrm{P}\int_0^{2\pi} \frac{f'(\tilde\vartheta)^2}{\tan\frac{f(\vartheta)-f(\tilde\vartheta)}{2}}\,\mathrm{d}\tilde\vartheta + O(\epsilon^2)$$

$$k = 1 + \left(\frac{2\pi\epsilon}{\beta}\right)^2 \mathrm{Sch}\left[\tan\frac{f}{2}\right] + \sum_{n\geq 3}\left(\frac{2\pi\epsilon}{\beta}\right)^n k_n[f]$$

$$k_3[f](\vartheta) = -\frac{1}{2\pi} f'^2(\vartheta) \mathrm{P}\!\int_0^{2\pi} \left(\frac{f^{(4)}}{f'^2} - \frac{4f''f'''}{f'^3} + \frac{3f''^3}{f'^4} + f''\right)(\tilde{\vartheta}) \frac{\mathrm{d}\tilde{\vartheta}}{\tan\frac{f(\vartheta)-f(\tilde{\vartheta})}{2}}$$

$$\mathscr{O}_k(\vartheta,\tilde{\vartheta}) = \left[\frac{f'(\vartheta)f'(\tilde{\vartheta})}{4\sin^2\frac{f(\vartheta)-f(\tilde{\vartheta})}{2}}\right]^k$$

$$k_3[f]\left(s = \frac{\ell\vartheta}{2\pi}\right) = \frac{6}{\pi}\lim_{\varepsilon\to 0}\left[\left(\int_0^{\vartheta-\varepsilon} + \int_{\vartheta+\varepsilon}^{2\pi}\right)\mathscr{O}_2(\vartheta,\tilde{\vartheta})\,\mathrm{d}\tilde{\vartheta} - \frac{2}{3\varepsilon^3} - \frac{2}{3\varepsilon}\,\mathrm{Sch}\left[\tan\frac{f}{2}\right](\vartheta)\right]$$

manifestly $\mathrm{PSL}(2,\mathbb{R})$-invariant

$\mathrm{PSL}(2,\mathbb{R})_{\mathscr{D}}$ and $\mathrm{PSL}(2,\mathbb{R})_{\mathrm{H}^2}$ coincide in the Schwarzian limit

Model 1: Malliavin-Shavgulidze measure

$$e^{\sigma_{\mathrm{b}}} = \frac{\ell}{2\pi}(f^{-1})'$$

$$e^{\varphi} = \frac{e^{q}}{\frac{1}{2\pi}\int_0^{2\pi} e^{q(\theta')}\mathrm{d}\theta'} = f'$$

$$q(0) = q(2\pi) = 0 \qquad \varphi(0) = \varphi(2\pi)\,, \quad \frac{1}{2\pi}\int_0^{2\pi} e^{\varphi}\mathrm{d}\theta = 1$$

$$f(\theta) = \int_0^{2\pi} e^{\varphi}\mathrm{d}\theta + f_0$$

The measure $\mathrm{d}f_0 Dq\, e^{-\frac{1}{2g^2}\int_0^{2\pi}\dot{q}^2\mathrm{d}\theta} = \mathrm{d}f_0 D\phi\,\delta\Big(\frac{1}{2\pi}\int_0^{2\pi} e^{\varphi}\mathrm{d}\theta\Big)e^{-\frac{1}{2g^2}\int_0^{2\pi}\dot{\varphi}^2\mathrm{d}\theta}$

formally coincides with $D_{\mathrm{L}} f e^{-\frac{1}{2g^2}\int_0^{2\pi}(\frac{f''}{f'})^2\mathrm{d}\theta}$

Slight improvement: the measure

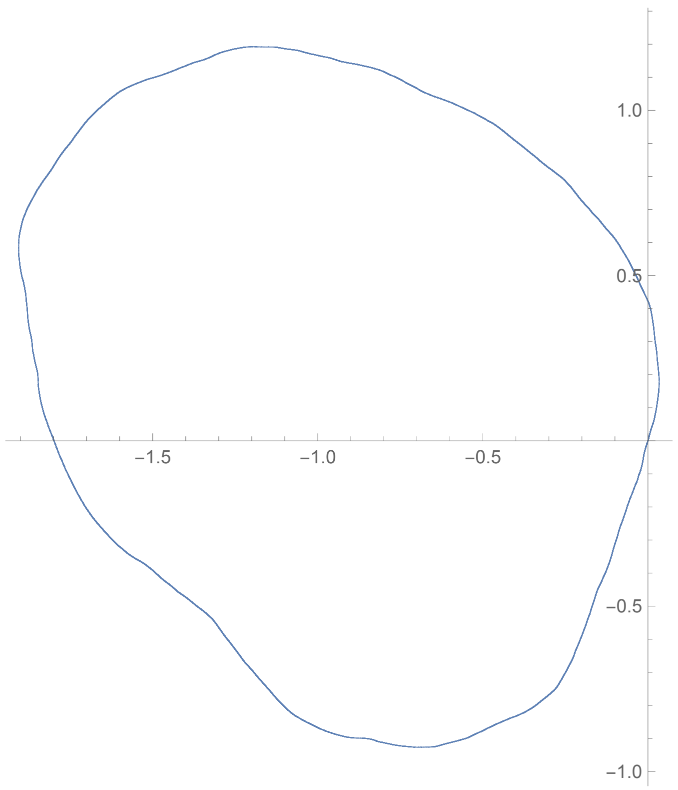
$$\mathrm{d}f_0 D\phi\,\delta\Big(\frac{1}{2\pi}\int_0^{2\pi} e^{\varphi}\mathrm{d}\theta\Big)e^{-\frac{1}{2g^2}\int_0^{2\pi}(\dot{\varphi}^2-e^{2\varphi})\mathrm{d}\theta}$$

coincides with $D_{\mathrm{L}} f\, e^{\frac{1}{g^2}\int_0^{2\pi}\mathrm{Sch}[\tan\frac{f}{2}]\mathrm{d}\theta}$

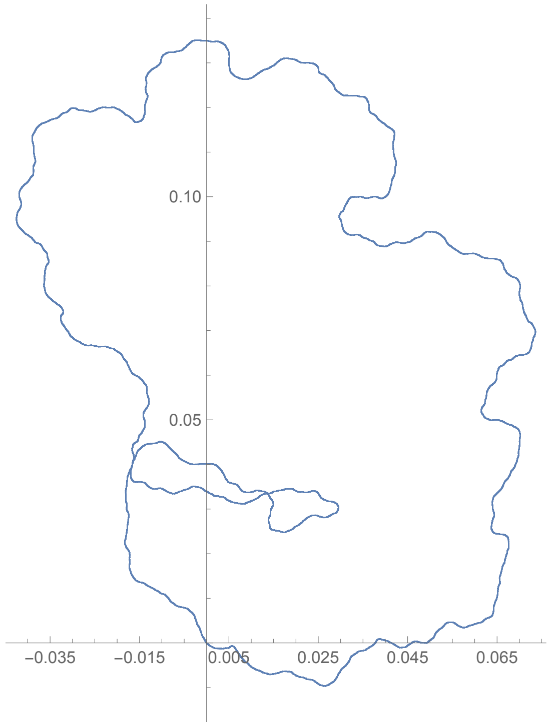
$$\mathrm{Sch}[f] = \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2 = \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2$$

This can be the starting point for investigating many interesting properties of the models

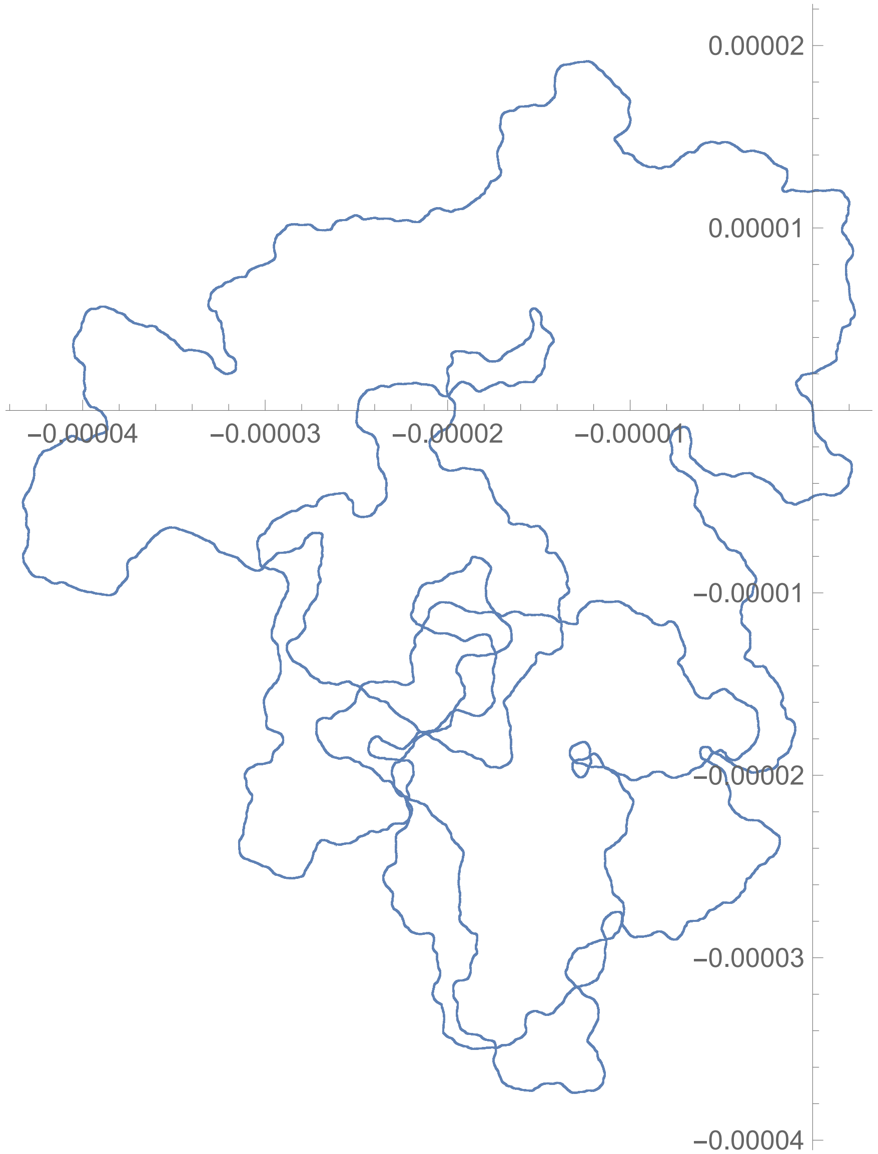
Numerics : Nicolas Delporte, Romain Pascalie



$g = 0.5$



$g = 4$



$g = 8$

Model 2: from first principles

Start from the standard quantum gravity path integral, do the gauge fixing etc.

$$e^{\sigma_{\mathrm{b}}} = \frac{\ell}{2\pi}(f^{-1})'$$

The measure is $D_{\mathrm{L}} f\, e^{-S}$ for an action

$$S = \frac{24-c}{24\pi}\left[\int_{\mathscr{D}} \mathrm{d}^2 x \partial_a \sigma \partial_a \sigma + 2\int_0^{2\pi} \mathrm{d}\theta\, \sigma\right] = \frac{24-c}{24\pi}\int_0^{2\pi} \mathrm{d}\theta\, \left(\sigma\partial_n\sigma + 2\sigma\right)$$

$$\partial_n \sigma = \mathrm{Hilb}[\sigma'] = \frac{1}{2\pi}\mathrm{P}\int_0^{2\pi} \mathrm{d}\theta'\, \frac{\sigma_{\mathrm{b}}'(\theta')}{\tan\frac{\theta-\theta'}{2}}$$

Massaging, we can express the action in terms of the diffeomorphism,

$$S = \frac{24-c}{24\pi} S_{\mathrm{L}} \qquad\qquad \mathscr{O}_{\mathrm{L}}(\theta_1,\theta_2) = \frac{f'\ln f'(\theta_1) f' \ln f'(\theta_2)}{4\sin^2\frac{f(\theta_1)-f(\theta_2)}{2}}$$

$$S_{\mathrm{L}} = 4\pi \ln\frac{\ell}{2\pi} - 2\int \mathrm{d}\theta\, f'\ln f' - \frac{1}{\pi}\mathrm{P}_{\varepsilon}\int \mathrm{d}\theta_1 \mathrm{d}\theta 2\, \mathscr{O}_{\mathrm{L}} + \frac{2}{\varepsilon\pi}\int \mathrm{d}\theta\, (\ln f')^2$$

$$\mathscr{O}_{\mathrm{L}}(\theta_1, \theta_2) = \frac{f' \ln f'(\theta_1) f' \ln f'(\theta_2)}{4 \sin^2 \frac{f(\theta_1) - f(\theta_2)}{2}}$$

$$S_{\mathrm{L}} = 4\pi \ln \frac{\ell}{2\pi} - 2 \int \mathrm{d}\theta \, f' \ln f' - \frac{1}{\pi} \mathrm{P}_{\varepsilon} \int \mathrm{d}\theta_1 \mathrm{d}\theta 2 \, \mathscr{O}_{\mathrm{L}} + \frac{2}{\varepsilon \pi} \int \mathrm{d}\theta \, (\ln f')^2$$

Remarkably, this is PSL(2,R)-invariant.

We can start to calculate…

$$S = \frac{24 - c}{24\pi} S_{\mathrm{L}}$$

At least in a semi-classical expansion $c \to -\infty$

Thank you for your attention !