

Massless and Partially Massless Limits in Quadratic Gravity

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Motivations

Einstein's General Relativity:

$$S_{EH} = \frac{M_p^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda)$$

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[Stelle, PRD (1977)]

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p^2 (R - 2\Lambda) + \frac{\alpha}{6} R^2 - \frac{\beta}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)$$

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$\alpha \sim 10^{10}$

Natural explanation for inflation!

[Starobinsky, 1980+]

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Spin-2 massive ghost

[Salvio & Strumia 2015+;
Anselmi & Piva 2017+;
Donoghue & Menezes 2018+;
Holdom 2015+, etc...]

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Question:

$$\lim_{\beta \rightarrow \infty} S = ?$$

Tasks

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p^2 (R - 2\Lambda) + \frac{\alpha}{6} R^2 - \frac{\beta}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)$$

$$\lim_{\beta \rightarrow \infty} S = ?$$

1. Identify the canonically normalized fields
2. Identify the interaction couplings
3. Determine the structure of the particle spectrum

Simpler example

$$S = -\frac{1}{4g^2} \int d^4x \hat{F}_{\mu\nu}^a \hat{F}^{a\mu\nu}, \quad \hat{F}_{\mu\nu}^a = \partial_\mu \hat{A}_\nu^a - \partial_\nu \hat{A}_\mu^a + f^{abc} \hat{A}_\mu^b \hat{A}_\nu^c$$

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Canonically normalized field:

$$A_\mu^a = \frac{1}{g} \hat{A}_\mu^a \quad \Rightarrow \quad S = -\frac{1}{4} \int d^4x F_{\mu\nu}^a F^{a\mu\nu},$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$$

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Limit:

$$\lim_{g \rightarrow 0} S = \frac{1}{2} \int d^4x A_\mu^a (\eta^{\mu\nu} \square - \partial^\mu \partial^\nu) A_\nu^a \quad (A_\mu^a = \text{fixed})$$

Tasks

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p^2 (R - 2\Lambda) + \frac{\alpha}{6} R^2 - \frac{\beta}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)$$

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Spoilers

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p^2 (R - 2\Lambda) + \frac{\alpha}{6} R^2 - \frac{\beta}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)$$

$$\lim_{\beta \rightarrow \infty} S = ?$$

- The limit distinguishes $\Lambda = 0$ & $\Lambda \neq 0$
(couplings, particle spectrum, enhanced gauge symmetry)
- Limit is regular only in $D = 4$
- When $\Lambda \neq 0$ the resulting theory is much simpler
- $\beta \rightarrow \infty$ may be related to the high-energy limit

Additional spin-0 field

$$S[g, \phi] = \frac{\bar{M}_p^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda) - \frac{\beta}{4} \int d^4x \sqrt{-g} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + S_0[g, \phi],$$

$$S_0[g, \phi] = \int d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - \frac{m_0^2}{2} \frac{3\bar{M}_p^2}{2} \left(1 - e^{\sqrt{2/3}\phi/\bar{M}_p} \right)^2 \right]$$

$$\bar{M}_p^2 \equiv M_p^2 + \frac{4}{3} \alpha \Lambda$$

Shifted Planck Mass when $\Lambda \neq 0$

$$m_0^2 \equiv \frac{M_p^2}{\alpha}$$

Mass of the spin-0 field

Additional spin-2 field

Spin-2 field $f_{\mu\nu}$:

[Kaku et al. (1977); Hindawi et al. (1996); Tekin (2016); Anselmi & Piva (2018)]

$$S[g, \phi, f] = \frac{\tilde{M}_p^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda) + S_0[g, \phi] \\ - \int d^4x \sqrt{-g} \left[\tilde{M}_p (G_{\mu\nu} + \Lambda g_{\mu\nu}) f^{\mu\nu} - \frac{m_2^2}{2} (f_{\mu\nu} f^{\mu\nu} - f^2) \right]$$

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Diagonalization: $g_{\mu\nu} \rightarrow g_{\mu\nu} - \frac{2}{\tilde{M}_p} f_{\mu\nu}$

$$S[g, \phi, f] = \frac{\tilde{M}_p^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda) + S_0[g - 2f/\tilde{M}_p, \phi] + S_2[g, f],$$

$$\tilde{M}_p^2 \equiv M_p^2 + \frac{2}{3} (2\alpha + \beta) \Lambda$$

Shifted Planck Mass when $\Lambda \neq 0$

Additional spin-2 field

$$\begin{aligned}
 S_2[g, f] = & -S_{PF}[g, f] - \int d^4x \sqrt{-g} \left[(2f_\mu^\rho f_{\rho\nu} - f f_{\mu\nu}) R^{\mu\nu} + \left(\Lambda - \frac{R}{2} \right) \left(f_{\mu\nu} f^{\mu\nu} - \frac{1}{2} f^2 \right) \right] \\
 & - \frac{1}{2} \frac{m_2^2}{\tilde{M}_p} \int d^4x \sqrt{-g} [5f_{\mu\nu} f^{\mu\nu} f - 4f^{\mu\nu} f_\mu^\rho f_{\rho\nu} - f^3] \\
 & + \frac{8}{3} \frac{1}{M_p^2} \frac{1}{\tilde{M}_p} \int d^4x d^4y d^4z \frac{\delta^{(3)} S_{EH}}{\delta g_{\mu\nu}(x) \delta g_{\rho\sigma}(y) \delta g_{\alpha\beta}(z)} f_{\mu\nu}(x) f_{\rho\sigma}(y) f_{\alpha\beta}(z) \\
 & + O(f^4)
 \end{aligned}$$

$S_{PF}[g, f]$ is the covariant Fierz-Pauli action for $f_{\mu\nu}$ with mass m_2^2

$$m_2^2 = \frac{\tilde{M}_p^2}{\beta} = \frac{M_p^2}{\beta} + \frac{2}{3} \left(2 \frac{\alpha}{\beta} + 1 \right) \Lambda$$

spin-2 ghost mass depends on Λ !

$$m_2^2 \geq \frac{2}{3} \Lambda$$

$$(\Lambda \geq 0, \beta > 0)$$

Couplings

n-point interaction couplings for the spin-2 sector:

$$\sim \left(\frac{1}{\tilde{M}_p}\right)^{n-2} = \left(\frac{1}{M_p}\right)^{n-2} \left(\frac{1}{1+2\Lambda(2\alpha+\beta)/3M_p^2}\right)^{\frac{n-2}{2}}$$

Couplings dependence on Λ \Rightarrow additional dependence on β !

Degrees of freedom

Linear analysis:

$$\frac{\delta S}{\delta f^{\mu\nu}} = 0 \quad \Leftrightarrow \quad \tilde{M}_p(G_{\mu\nu} + \Lambda g_{\mu\nu}) = m_2^2(f_{\mu\nu} - g_{\mu\nu}f)$$

4 Constraints:

$$\nabla_\mu f_\nu^\mu = \nabla_\nu f$$

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$$g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Leftrightarrow \quad \tilde{M}_p \left(m_2^2 - \frac{2}{3} \Lambda \right) f = 0$$

1 trace constraint:

$$f = 0$$

$$(m_2^2 - 2\Lambda/3 \neq 0)$$

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dof $f_{\mu\nu}$: $10 - 4 - 1 = 5 \quad \Rightarrow \quad$ massive spin-2 with 5 helicities

Degrees of freedom: remark

$$\tilde{M}_p \left(m_2^2 - \frac{2}{3} \Lambda \right) f = 0, \quad m_2^2 - \frac{2}{3} \Lambda = 0 \quad ?$$

In this case the linear trace equation vanishes identically

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Two possibilities:

1. $\Lambda = 0$: $m_2^2 = 0$ (massless)
2. $\Lambda \neq 0$: $m_2^2 = \frac{2}{3} \Lambda$ (partially massless)

These cases need a separate discussion!

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These cases need a separate discussion!

NB:

$$\beta \rightarrow \infty \quad \Rightarrow \quad m_2^2 - \frac{2\Lambda}{3} = \frac{\bar{M}_p^2}{\beta} \rightarrow 0$$

Tasks

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p^2 (R - 2\Lambda) + \frac{\alpha}{6} R^2 - \frac{\beta}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)$$

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Case $\Lambda = 0$

It is a massless limit:

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NB: typically, the massless limit in theories of Massive Gravity can lead to strong coupling even below M_p . [Reviews by Hinterbichler (2011) and de Rham (2014)]

Digression on Massive Gravity with $\Lambda = 0$

$$S_{MG} = \int d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla_\rho f_{\mu\nu} \nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu} \nabla^\mu f^{\rho\nu} - \nabla_\mu f \nabla_\nu f^{\mu\nu} + \frac{1}{2} \nabla_\rho f \nabla^\rho f \right. \\ \left. - \frac{m_2^2}{2} (f_{\mu\nu} f^{\mu\nu} - f^2) + O(f^3) \right]$$

Naively, the limit $m_2^2 \rightarrow 0$ seems to give a *massless* spin-2 with 2 dofs

$$S_{MG}^{(2)} = \int d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla_\rho f_{\mu\nu} \nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu} \nabla^\mu f^{\rho\nu} - \nabla_\mu f \nabla_\nu f^{\mu\nu} + \frac{1}{2} \nabla_\rho f \nabla^\rho f \right]$$

Gauge symmetry gives 2 dofs

$$\delta f_{\mu\nu}(x) = \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x),$$

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Stückelberg formalism:

$$f_{\mu\nu} = \varphi_{\mu\nu} + \frac{1}{m_2} (\nabla_\mu A_\nu + \nabla_\nu A_\mu) + \frac{2}{m_2^2} \nabla_\mu \nabla_\nu \chi,$$

Gauge symmetries:

$$\begin{aligned} \delta\varphi_{\mu\nu} &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \\ \delta A_\mu &= -m_2 \xi_\mu + \nabla_\mu \xi, \\ \delta\chi &= -m_2 \xi, \end{aligned}$$

Massless limit (2+2+1 = 5 dofs):

$$S_{MG}[\varphi, A, \chi] = S_{FP}^{(m_2=0)}[\varphi] + \int d^4x \sqrt{-g} \left(-\frac{1}{2} F^{\mu\nu} F_{\mu\nu} - 3 \nabla_\rho \chi \nabla^\rho \chi \right) + O(f^3)$$

Possible strong coupling from helicity-0 interactions: $O(f^3) \sim \frac{1}{m_2} O(\chi^3) \rightarrow \infty$

Case $\Lambda = 0$

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The limit is regular in $D = 4$:

$$\lim_{\beta \rightarrow \infty} S = \frac{M_p^2}{2} \int d^4x \sqrt{-g} R - M_p \int d^4x \sqrt{-g} G_{\mu\nu} \varphi^{\mu\nu} + \frac{1}{4} \int d^4x \sqrt{-g} F^{\mu\nu} F_{\mu\nu} + S_{\phi\chi}[g, \phi, \chi]$$

$$S_{\phi\chi}[g, \phi, \chi] = \int d^4x \sqrt{-g} \left[\frac{1}{2} e^{-\sqrt{2/3}\chi/M_p} (\nabla_\mu \chi \nabla^\mu \chi - \nabla_\mu \phi \nabla^\mu \phi) - \frac{m_0^2}{2} \frac{3M_p^2}{2} e^{-2\sqrt{2/3}\chi/M_p} \left(1 - e^{\sqrt{2/3}\phi/M_p} \right)^2 \right]$$

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$f_{\mu\nu}$ splits into 5 massless ghost-like dofs ($\pm 2, \pm 1, 0$)

Case $\Lambda > 0$

It is NOT a massless limit:

$$\beta \rightarrow \infty \quad \Rightarrow \quad m_2^2 = \frac{\bar{M}_p^2}{\beta} + \frac{2}{3}\Lambda \rightarrow \frac{2}{3}\Lambda$$

Case $\Lambda > 0$

It is NOT a massless limit:

$$\beta \rightarrow \infty \quad \Rightarrow \quad m_2^2 = \frac{\bar{M}_p^2}{\beta} + \frac{2}{3}\Lambda \rightarrow \frac{2}{3}\Lambda$$

NB: In Massive Gravity theories this limit is known as *partially massless limit* and in general may lead to strong coupling! [de Rham et al. (2018)]

Digression on Massive Gravity with $\Lambda > 0$

$$S_{MG} = \int d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla_\rho f_{\mu\nu} \nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu} \nabla^\mu f^{\rho\nu} - \nabla_\mu f \nabla_\nu f^{\mu\nu} + \frac{1}{2} \nabla_\rho f \nabla^\rho f \right. \\ \left. + \Lambda \left(f_{\mu\nu} f^{\mu\nu} - \frac{1}{2} f^2 \right) - \frac{m_2^2}{2} \left(f_{\mu\nu} f^{\mu\nu} - f^2 \right) + O(f^3) \right]$$

Naively, the limit $m_2^2 \rightarrow \frac{2}{3} \Lambda$ seems to give a *partially massless* spin-2 with 4 dofs

$$S_{MG}^{(2)} = \int d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla_\rho f_{\mu\nu} \nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu} \nabla^\mu f^{\rho\nu} - \nabla_\mu f \nabla_\nu f^{\mu\nu} + \frac{1}{2} \nabla_\rho f \nabla^\rho f \right. \\ \left. + \Lambda \left(f_{\mu\nu} f^{\mu\nu} - \frac{1}{2} f^2 \right) - \frac{\Lambda}{3} \left(f_{\mu\nu} f^{\mu\nu} - f^2 \right) \right]$$

Scalar gauge symmetry (10-4-2=4 dofs)

$$\delta f_{\mu\nu}(x) = \nabla_\mu \nabla_\nu \zeta(x) + \frac{\Lambda}{3} g_{\mu\nu} \zeta(x),$$

Digression on Massive Gravity with $\Lambda > 0$

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Stückelberg trick

$$f_{\mu\nu} = \varphi_{\mu\nu} + \sqrt{\frac{3}{\Lambda}} \frac{1}{\Delta} \left(\nabla_\mu \nabla_\nu \chi + g_{\mu\nu} \frac{\Lambda}{3} \chi \right) \\ \Delta \equiv m_2^2 - \frac{2}{3} \Lambda$$

Gauge symmetries:

$$\delta \varphi_{\mu\nu} = \nabla_\mu \nabla_\nu \zeta + \frac{\Lambda}{3} g_{\mu\nu} \zeta, \\ \delta \chi = -\sqrt{\frac{\Lambda}{3}} \Delta \zeta,$$

Partially massless limit $\Delta \rightarrow 0$ (4+1=5 dofs):

$$S_{MG}[\varphi, \chi] = S_{FP}^{(\Delta=0)}[\varphi] + 3 \int d^4x \sqrt{-g} \left(-\frac{1}{2} \nabla_\rho \chi \nabla^\rho \chi - \frac{m_\chi^2}{2} \chi^2 \right) + O(f^3), \quad m_\chi^2 \equiv -\frac{4}{3} \Lambda$$

Possible strong coupling from χ interactions: $O(f^3) \sim \frac{1}{\Delta} O(\chi^3) \rightarrow \infty$

Case $\Lambda > 0$

Stückelberg formalism:

$$f_{\mu\nu} = \varphi_{\mu\nu} + \frac{1}{\sqrt{\Lambda}\sqrt{m_2^2 - 2\Lambda/3}} \left(\nabla_\mu \nabla_\nu \chi + \frac{\Lambda}{3} g_{\mu\nu} \chi \right)$$

$$\begin{aligned} \lim_{\beta \rightarrow \infty} S &= \frac{\tilde{M}_p^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda) \\ &+ \int d^4x \sqrt{-g} \left[-\tilde{M}_p (G_{\mu\nu} + \Lambda g_{\mu\nu}) \varphi^{\mu\nu} + \frac{\Lambda}{3} (\varphi_{\mu\nu} \varphi^{\mu\nu} - \varphi^2) \right] + S_{\phi\chi} \end{aligned}$$

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$$\begin{aligned} S_{\phi\chi}[g, \phi, \chi] &= \int d^4x \sqrt{-g} \left[\frac{1}{2} e^{-\sqrt{2/3}\chi/\bar{M}_p} (\nabla_\mu \chi \nabla^\mu \chi - \nabla_\mu \phi \nabla^\mu \phi) \right. \\ &\quad \left. - \Lambda \bar{M}_p^2 \left(1 - e^{\sqrt{2/3}\phi/\bar{M}_p} \right)^2 - \frac{m_0^2}{2} \frac{3\bar{M}_p^2}{2} e^{-2\sqrt{2/3}\chi/\bar{M}_p} \left(1 - e^{\sqrt{2/3}\phi/\bar{M}_p} \right)^2 \right] \end{aligned}$$

$f_{\mu\nu}$ splits into 1 partially massless graviton (4 dof) + 1 scalar dof

Case $\Lambda > 0$

Interaction couplings: $\frac{1}{\tilde{M}_p} \sim \frac{1}{\sqrt{\Lambda\beta}}$ for spin-2 sector & $\frac{1}{\bar{M}_p}$ for spin-0 sector

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- Diagonalize kinetic term for $h_{\mu\nu}$ and $\varphi_{\mu\nu}$

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$$\lim_{\beta \rightarrow \infty} S = S_{EH}^{(2)}[\bar{g}, h] - S_2^{(2)}[\bar{g}, \varphi] \Big|_{m_2^2 = \frac{2}{3}\Lambda} + S_{\phi\chi}[\bar{g}, \phi, \chi]$$

Spin-2 sector completely decouples!

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Spin-2 sector completely decouples!

The only compatible metric background is

$$\bar{R}_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}$$

Case $\Lambda > 0$

Resulting interacting theory:

$$S_{\phi\chi}[g, \phi, \chi] = \int d^4x \sqrt{-g} \left[\frac{1}{2} e^{-\sqrt{2/3}\chi/\bar{M}_p} (\nabla_\mu \chi \nabla^\mu \chi - \nabla_\mu \phi \nabla^\mu \phi) \right. \\ \left. - \Lambda \bar{M}_p^2 \left(1 - e^{\sqrt{2/3}\phi/\bar{M}_p} \right)^2 - \frac{m_0^2}{2} \frac{3\bar{M}_p^2}{2} e^{-2\sqrt{2/3}\chi/\bar{M}_p} \left(1 - e^{\sqrt{2/3}\phi/\bar{M}_p} \right)^2 \right]$$

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Constraint:

$$g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} = 0 \quad \Leftrightarrow \quad \tilde{M}_p \left(m_2^2 - \frac{2}{3} \Lambda \right) \varphi = -3T^{(\phi\chi)}, \quad T_{\mu\nu}^{(\phi\chi)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\phi\chi}}{\delta g^{\mu\nu}}$$

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$$\beta \rightarrow \infty \quad \Rightarrow \quad m_2^2 \rightarrow \frac{2}{3} \Lambda \quad \Rightarrow \quad T^{(\phi\chi)} = 0$$

Case $\Lambda > 0$

Resulting interacting theory:

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$$\bar{M}_p^2 = M_p^2 + \frac{4}{3} \alpha \Lambda$$

Field redefinitions:

$$\phi = -\sqrt{\frac{3}{2}} \bar{M}_p \log \left(\frac{\tilde{\chi} + \tilde{\phi}}{\tilde{\chi} - \tilde{\phi}} \right), \quad \chi = -\sqrt{\frac{3}{2}} \bar{M}_p \log \left(\frac{\tilde{\chi}^2 - \tilde{\phi}^2}{6\bar{M}_p^2} \right)$$

$$S_{\phi\chi}[g, \tilde{\phi}, \tilde{\chi}] = \int d^4x \sqrt{-g} \left[\frac{1}{2} (\partial_\mu \tilde{\chi} \partial^\mu \tilde{\chi} - \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi}) - V(\tilde{\phi}, \tilde{\chi}) \right]$$

$$V(\tilde{\phi}, \tilde{\chi}) = \frac{\Lambda}{36\bar{M}_p^2} (\tilde{\chi}^2 - \tilde{\phi}^2 - 6\bar{M}_p^2)^2 + \frac{m_0^2}{36\bar{M}_p^2} \tilde{\phi}^2 (\tilde{\chi} + \tilde{\phi})^2$$

Physical implications?

$$g_2^2 \equiv \frac{2}{\beta}$$

$$\frac{dg_2^2}{d\tau} = -\frac{133}{10} g_2^4,$$

$$g_2^2(\tau) = \frac{g_2^2(0)}{1 + \frac{133}{10} g_2^2(0)\tau},$$

$$\tau = \frac{1}{(4\pi)^2} \log \mu/\mu_0$$

g_2^2 (β) decreases (grows) with the energy

[Julve & Tonin (1978); Fradkin & Tseytlin (1982); Avramidi & Barvinsky (1985); etc]

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[Julve & Tonin (1978); Fradkin & Tseytlin (1982); Avramidi & Barvinsky (1985); etc]

$\beta \rightarrow \infty$ (partial masslessness) as high (infinite)-energy limit ?

Ghost instability?

An exercise: Hamiltonian for purely time-dependent fields in de Sitter

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2, \quad a(t) = e^{\sqrt{\frac{\Lambda}{3}}t}$$

$$H = a(t)^3 \left[(\partial_0 \tilde{\phi})^2 - (\partial_0 \tilde{\chi})^2 + V(\tilde{\phi}, \tilde{\chi}) \right]$$

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$$H = a(t)^3 \left[(\partial_0 \tilde{\phi})^2 - (\partial_0 \tilde{\chi})^2 + V(\tilde{\phi}, \tilde{\chi}) \right]$$

Contraint:

$$T^{(\phi\chi)} = 0 \quad \Leftrightarrow \quad (\partial_0 \tilde{\phi})^2 - (\partial_0 \tilde{\chi})^2 = 2V(\tilde{\phi}, \tilde{\chi})$$

$$\Rightarrow \quad H = 3a(t)^3 V(\tilde{\phi}, \tilde{\chi}) \geq 0$$

Summary

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p^2 (R - 2\Lambda) + \frac{\alpha}{6} R^2 - \frac{\beta}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)$$

- Renormalizability + ghost-like nature of spin-2 \Rightarrow limit $\beta \rightarrow \infty$ is regular
- The limit $\beta \rightarrow \infty$ depends non-trivially on Λ
- When $\Lambda \neq 0$: structure of degrees of freedom is different (but same number); spin-2 sector decouples; helicity-zero acquires a self-potential

Discussion

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p^2 (R - 2\Lambda) + \frac{\alpha}{6} R^2 - \frac{\beta}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)$$

- Can the limit $\beta \rightarrow \infty$ help understand the high-energy behavior of the spin-2 ghost?
- Current quantization approaches to Quadratic Gravity are formulated in flat spacetime. Are they still valid when $\Lambda \neq 0$?
- Role of the cosmological constant in quantum gravity?

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- Can the limit $\beta \rightarrow \infty$ help understand the high-energy behavior of the spin-2 ghost?
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- Role of the cosmological constant in quantum gravity?

A nice formula:

$$\Lambda = \frac{3}{2} m_2^2 \frac{\beta - M_p^2/m_2^2}{\beta + 2M_p^2/m_0^2}$$

...Extra Slides...

Renormalizable Quantum Gravity

Quadratic gravity is 'strictly' renormalizable (in $D = 4$):

[Stelle, PRD (1977)]

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p^2 (R - 2\Lambda) + \frac{\alpha}{6} R^2 - \frac{\beta}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)$$

Usual story: assume $\Lambda \simeq 0$, expand $g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_p} h_{\mu\nu}$, the propagator is

$$\Pi(k) = \Pi_{GR} + \frac{1}{2} \frac{\mathcal{P}^{(0)}}{k^2 + m_0^2} - \frac{\mathcal{P}^{(2)}}{k^2 + m_2^2}, \quad m_0^2 = \frac{M_p^2}{\alpha}, \quad m_2^2 = \frac{M_p^2}{\beta}$$

Spin-2 massive ghost

Reason why such a QFT of gravity is usually claimed to be pathological when quantized with conventional methods!

Motivations

General Relativity (GR): $S_{GR} = \frac{M_p^2}{2} \int d^4x \sqrt{-g} R$

Physics beyond GR:

1. Current accelerated expansion of the Universe (*late time*)
2. CMB anisotropies (*early time*)

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General Relativity (GR): $S_{GR} = \frac{M_p^2}{2} \int d^4x \sqrt{-g} R$

Physics beyond GR:

1. Current accelerated expansion of the Universe (*late time*)
2. CMB anisotropies (*early time*)

Simplest phenomenological model:

$$S = \frac{M_p^2}{2} \int d^4x \sqrt{-g} R + S_\Lambda + S_\phi + \dots,$$

$$S_\Lambda \equiv M_p^2 \int d^4x \sqrt{-g} \Lambda,$$

$S_\phi \equiv$ inflaton action

Motivations

How to select fundamental theories?

1. Guidance from Nature: experiments!
2. 'Guiding principles': perturbative QFT (renormalizability criterion!)

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How to select fundamental theories?

1. Guidance from Nature: experiments!
2. 'Guiding principles': perturbative QFT (renormalizability criterion!)

The criterion of (strict) renormalizability gives: [Stelle, PRD (1977)]

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p^2 (R - 2\Lambda) + \frac{\alpha}{6} R^2 - \frac{\beta}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)$$

$\Lambda \sim 10^{-122} M_p^2$

$\alpha \sim 10^{10}$

Natural explanation for inflation!

[Starobinsky, 1980+]

Quadratic Gravity as Quantum Gravity

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p^2 (R - 2\Lambda) + \frac{\alpha}{6} R^2 - \frac{\beta}{2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right)$$

Cosmological constant: $\Lambda \sim 10^{-122} M_p^2$

Natural candidate for inflaton: $\alpha \sim 10^{10}$

[Starobinsky, 1980+]

In my opinion, if we accept these facts very important implications follow:

1. The framework of perturbative QFT and the criterion of renormalizability (as a tool to select theories) are quite successful also when applied to gravity!
2. CMB observations have provided for the first time a test of higher-curvature gravity and an 'indirect' proof of quantized gravity (the scalar field is a gravitational dof)!!
3. Contrary to some beliefs, Starobinsky inflation is not just a model!

Motivations

Obvious question: What about the spin-2 massive ghost?

1. Throw the entire theory away just because maybe we don't know how to deal with the spin-2 ghost?
2. Or, instead, after appreciating the achievements described before, should we feel very motivated to understand the role of the ghost at a deeper level?

I opt for the 2nd option!

Recent proposals to recover unitarity with ghost

S-matrix unitarity and optical theorem:

$$S^+S = 1, \quad S = 1 + iT, \\ 1 = \sum_{\{n\}} c_n |n\rangle\langle n| \quad \Rightarrow \quad 2\text{Im}\{\langle a|T|a\rangle\} = \sum_{\{n\}} c_n |\langle n|T|a\rangle|^2$$

Interesting approaches

- Quantize the ghost with negative norms ($c_n < 0$ for ghost states but positive energies) [Salvio, Strumia (2014+); Holdom (2021+); etc]
- Loop corrections make the ghost decay after times of order $\tau \sim M_p^2/m_2^3$: treat the ghost as an unstable particle, unitarity restored for $t > \tau$ [Donoghue, Menezes (2018+)]
- Replace the Feynman $i\epsilon$ with the *Fakeon* prescription and convert the ghost into a purely virtual particle (LHS=0 for ghost cuts and $c_n = 0$ for ghost states) [Anselmi & Piva 2017+]

Additional spin-0 field

1. Auxiliary scalar field φ

$$S[g, \varphi] = \frac{\bar{M}_p^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda) + S_W[g] + \frac{\alpha}{12} \int d^4x \sqrt{-g} (2R - 8\Lambda - \varphi) \varphi$$

2. Conformal transformation: $g_{\mu\nu} \rightarrow \frac{1}{1 + \alpha\varphi/3\bar{M}_p} g_{\mu\nu}$

3. Canonical normalization: $\phi = \sqrt{\frac{3}{2}} \frac{\alpha}{3\bar{M}_p} \varphi$

$$S[g, \phi] = \frac{\bar{M}_p^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda) + S_W[g] + S_0[g, \phi],$$

$$S_0[g, \phi] = \int d^4x \sqrt{-g} \frac{1}{(1 + \sqrt{2/3}\phi/\bar{M}_p)^2} \left(-\frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - \frac{m_0^2}{2} \phi^2 \right)$$

$$\bar{M}_p^2 \equiv M_p^2 + \frac{4}{3} \alpha \Lambda$$

Rescaled Planck Mass when $\Lambda \neq 0$

$$m_0^2 \equiv \frac{M_p^2}{\alpha}$$

Mass of the scalar field

Additional spin-0 field: couplings

$$S_0[g, \phi] = \int d^4x \sqrt{-g} \left(1 - 2 \sqrt{\frac{2}{3}} \frac{1}{\bar{M}_p} \phi + \dots + (n-1)(-1)^{n-2} \left(\sqrt{\frac{2}{3}} \frac{1}{\bar{M}_p} \right)^{n-2} \phi^{n-2} + \dots \right) \\ \times \left(-\frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - \frac{m_0^2}{2} \phi^2 \right)$$

Derivative interaction couplings $\sim \left(\frac{1}{\bar{M}_p} \right)^{n-2} = \left(\frac{1}{M_p} \right)^{n-2} \left(\frac{1}{1+4\alpha\Lambda/3M_p^2} \right)^{\frac{n-2}{2}}$

Non-derivative interaction couplings $\sim m_0^2 \left(\frac{1}{\bar{M}_p} \right)^{n-2} = \frac{1}{\alpha} \left(\frac{1}{M_p} \right)^{n-4} \left(\frac{1}{1+4\alpha\Lambda/3M_p^2} \right)^{\frac{n-2}{2}}$

Couplings for the massless spin-2

When $\Lambda \neq 0$ the couplings of the massless spin-2 also change:

$$\frac{\tilde{M}_p^2}{2} \int d^4x \sqrt{-g} (R - 2\Lambda), \quad g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{2}{\tilde{M}_p} h_{\mu\nu},$$

Self-interaction couplings

$$\sim \left(\frac{1}{\tilde{M}_p}\right)^{n-2} = \left(\frac{1}{M_p}\right)^{n-2} \left(\frac{1}{1+2\Lambda(2\alpha+\beta)/3M_p^2}\right)^{\frac{n-2}{2}}$$

When $\Lambda \neq 0 \Rightarrow$ additional dependences on Λ, α, β !

Brief digression on Massive Gravity with $\Lambda = 0$

Massive Gravity:

$$S_{MG} = \int d^4x \sqrt{-g} \left[-\frac{1}{2} \nabla_\rho f_{\mu\nu} \nabla^\rho f^{\mu\nu} + \nabla_\rho f_{\mu\nu} \nabla^\mu f^{\rho\nu} - \nabla_\mu f \nabla_\nu f^{\mu\nu} + \frac{1}{2} \nabla_\rho f \nabla^\rho f - \frac{m_2^2}{2} (f_{\mu\nu} f^{\mu\nu} - f^2) \right] + O(f^3)$$

Naively, the limit $m_2^2 \rightarrow 0$ seems to give a massless spin-2 with 2 dofs...

But, the number of degrees of freedom must be preserved: **Stückelberg trick**

$$f_{\mu\nu} = f'_{\mu\nu} + \frac{1}{m_2} (\nabla_\mu A_\nu + \nabla_\nu A_\mu) + \frac{2}{m_2^2} \nabla_\mu \nabla_\nu \varphi,$$

Gauge symmetries:

$$\delta f'_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu,$$

$$\delta A_\mu = -m_2 \xi_\mu + \nabla_\mu \xi,$$

$$\delta \varphi = -m_2 \xi,$$

Massless limit (2+2+1 = 5 dofs):

$$S_{MG}[f', A, \varphi] = S_{FP}^{(m_2=0)}[f'] + \int d^4x \sqrt{-g} \left(-\frac{1}{2} F^{\mu\nu} F_{\mu\nu} - 3 \nabla_\rho \varphi \nabla^\rho \varphi \right) + O(f^3)$$

Possible strong coupling from helicity-0 interactions: $O(f^3) \sim \frac{1}{m_2} O(\varphi^3) \rightarrow \infty$

Limit $\beta \rightarrow \infty$ with $\Lambda = 0$: massless limit

Does a strong coupling (below M_p) arise in quadratic gravity?

[first asked by Hinterbichler & Saravani (2016)]

A strong coupling in the limit $m_2^2 \rightarrow 0$ (i.e., $\beta \rightarrow \infty$) can be avoided *only* in $D = 4$!

Stückelberg decomposition for $\Lambda = 0$ and in D dimensions:

$$f_{\mu\nu} = \tilde{f}_{\mu\nu} + \frac{1}{m_2} (\nabla_\mu \tilde{A}_\nu + \nabla_\nu \tilde{A}_\mu), \quad \tilde{A}_\mu = A_\mu + \frac{1}{m_2} \nabla_\mu \chi$$

$$\begin{aligned} \Rightarrow S'_2[g, f] &= \frac{M_p^{D-2}}{2} \int d^D x \sqrt{-g} R + \int d^D x \sqrt{-g} \left[-M_p^{\frac{D-2}{2}} G_{\mu\nu} f^{\mu\nu} + \frac{m_2^2}{2} (f_{\mu\nu} f^{\mu\nu} - f^2) \right] \\ &= \frac{M_p^{D-2}}{2} \int d^D x \sqrt{-g} R + \int d^D x \sqrt{-g} \left[-M_p^{\frac{D-2}{2}} G_{\mu\nu} \tilde{f}^{\mu\nu} + \frac{m_2^2}{2} (\tilde{f}_{\mu\nu} \tilde{f}^{\mu\nu} - \tilde{f}^2) \right. \\ &\quad \left. + \frac{1}{2} F^{\mu\nu} F_{\mu\nu} + 2m_2 \tilde{f}^{\mu\nu} (\nabla_\mu \tilde{A}_\nu - g_{\mu\nu} \nabla^\rho \tilde{A}_\rho) - 2R^{\mu\nu} \tilde{A}_\mu \tilde{A}_\nu \right], \end{aligned}$$

Possible strong coupling from $R^{\mu\nu} \tilde{A}_\mu \tilde{A}_\nu \sim \frac{1}{m_2^2} R^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi$???

Limit $\beta \rightarrow \infty$ with $\Lambda = 0$: massless limit

Make a field redefinition:

$$\tilde{f}_{\mu\nu} \rightarrow \tilde{f}_{\mu\nu} + a \tilde{A}_\mu \tilde{A}_\nu + b g_{\mu\nu} \tilde{A}_\rho \tilde{A}^\rho$$

In the massless limit $m_2^2 \rightarrow 0$ ($\beta \rightarrow \infty$, $\Lambda = 0$) we get

$$\begin{aligned} \Rightarrow S'_2[g, \tilde{f}, \tilde{A}] = & \frac{M_p^{D-2}}{2} \int d^D x \sqrt{-g} R + \int d^D x \sqrt{-g} \left[-M_p^{\frac{D-2}{2}} G_{\mu\nu} \tilde{f}^{\mu\nu} + \frac{m_2^2}{2} (\tilde{f}_{\mu\nu} \tilde{f}^{\mu\nu} - \tilde{f}^2) + \frac{1}{2} F^{\mu\nu} F_{\mu\nu} \right. \\ & + 2m_2 \tilde{f}^{\mu\nu} (\nabla_\mu \tilde{A}_\nu - g_{\mu\nu} \nabla^\rho \tilde{A}_\rho) + m_2^2 a \tilde{f}^{\mu\nu} \tilde{A}_\mu \tilde{A}_\nu + m_2^2 [b(1-D) - a] \tilde{f} \tilde{A}_\rho \tilde{A}^\rho \\ & - \left(a M_p^{\frac{D-2}{2}} + 2 \right) R^{\mu\nu} \tilde{A}_\mu \tilde{A}_\nu + M_p^{\frac{D-2}{2}} \left(\left(1 - \frac{D}{2} \right) b - \frac{a}{2} \right) R \tilde{A}_\rho \tilde{A}^\rho \\ & \left. - m_2 (2b(1-D) - 3a) \tilde{A}_\mu \tilde{A}_\nu \nabla^\mu \tilde{A}^\nu - \frac{m_2^2}{2} (b^2 D(1-D) + 2ab(1-D)) (\tilde{A}_\rho \tilde{A}^\rho)^2 \right] \end{aligned}$$

4 conditions to avoid strong coupling in the massless limit:

$$a M_p^{\frac{D-2}{2}} + 2 = 0, \quad 2b(1-D) - 3a = 0,$$

$$\left(1 - \frac{D}{2} \right) b - \frac{a}{2} = 0, \quad b^2 D(1-D) + 2ab(1-D) = 0$$

can be simultaneously satisfied
only in $D = 4$!!!

$$a = -\frac{2}{M_p} = -2b,$$