Lorentzian asymptotic safety on curved backgrounds

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1 Introduction

2 Flow equations

On the road to QG: where does QFT come in?

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- Some things to consider: Lorentzian signature, background independence, gauge invariant observables, locality vs non-locality.

Algebraic QFT on curved spacetimes

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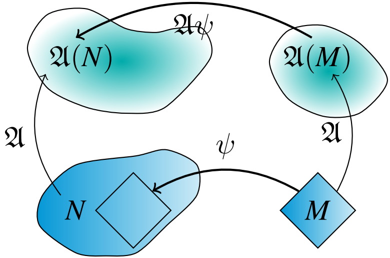

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Main advantage

Construction of observables $\mathfrak{A}(\mathcal{O})$ is independent from the construction of states. Entanglement and superposition are properties of states (always non-local) not of observables (often local).

Perturbative algebraic quantum field theory

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  - Interaction introduced in the causal approach to renormalization due to Epstein and Glaser ([Epstein-Glaser 73]),
  - Generalization to gauge theories using homological algebra ([Hollands 08, Fredenhagen-KR 11]).

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- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M. For the scalar field: $\mathcal{E}(M) \equiv \mathcal{C}^\infty(M, \mathbb{R})$. For perturbative gravity $\mathcal{E}(M) = \Gamma((T^*M)^{\otimes 2})$.

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- The choice of action functional I specifies the dynamics. We use a modification of the Lagrangian formalism (fully covariant).

Building models in pAQFT I

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- On $\mathcal{F}(M)$ we introduce first classical dynamics by means of a Poisson structure (Peierls bracket): $\{F, G\} = \left\langle \frac{\delta F}{\delta\varphi}, \Delta \frac{\delta G}{\delta\varphi} \right\rangle$, where $\Delta = \Delta^{\mathrm{R}} - \Delta^{\mathrm{A}}$ (Green functions for the linearized action).

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- Use the deformation quantization to construct the non-commutative algebra $\mathfrak{A}(M) = (\mathcal{F}(M)[[\hbar]], \star)$, such that

$$F \star G \xrightarrow{\hbar=0} FG \qquad \frac{1}{i\hbar}(F \star G - G \star F) \xrightarrow{\hbar=0} \{F, G\}\,.$$

Building models in pAQFT II

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- For a quadratic action I_0 that induces hyperbolic equations of motion (e.g. $-(\Box + m^2)\varphi = 0$), $\star$ can be constructed directly, starting from Δ and choosing a choice of a 2-point function for a quasifree Hadamard state: $\Delta^+ = \frac{i}{2}\Delta + H$.

$$F \star_H G \doteq m \circ e^{\hbar\left\langle \Delta^+, \frac{\delta}{\delta\varphi}\otimes\frac{\delta}{\delta\varphi}\right\rangle}(F \otimes G)\,,$$

Time-ordered products

- Take an interaction $V \in \mathcal{F}_{\mathrm{loc}}(M)$ and define the formal S-matrix

$$\mathcal{S}(\lambda V) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i\lambda}{\hbar}\right)^n V \cdot_{\mathcal{T}} \ldots \cdot_{\mathcal{T}} V \,,$$

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- We also introduce the time-ordering map $\mathcal{T}$, so that $F \cdot_{\mathcal{T}} G = \mathcal{T}(\mathcal{T}^{-1} F \cdot \mathcal{T}^{-1} G)$. It formally corresponds to path integrating with a Gaussian measure:

$$\mathcal{T} F(0) \sim \int F(\varphi) d\mu(\varphi)$$

Interacting fields and states I

- Define relative S-matrices by: $\mathcal{S}_{\lambda V}(F) \doteq \mathcal{S}(\lambda V)^{-1} \star \mathcal{S}(\lambda V + F)$, where the inverse of $\mathcal{S}$ is the $\star$-inverse.

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- In the algebraic approach, states are functionals $\omega : \mathfrak{A}(M) \to \mathbb{C}$ with $\omega(\mathbb{1}) = 1$ and $\omega(A^*A) \geq 0$. (Relation to Hilbert spaces via GNS theorem).
- A natural state on $\mathcal{F}(M)$ and hence $\mathfrak{A}(M)$ is given by evaluation at a given field configuration. For the scalar field we can take $\omega(F) = F(0)$.

Interacting fields and states II

- Wightman n-point functions of the free theory are

$$W_n(f_1,\dots,f_n) = (\Phi(f_1)\star\dots\star\Phi(f_n))(0)\,,$$

where $\Phi(f)(\varphi) = \int \varphi(x)f(x)d\mu(x)$.

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- Interacting correlation functions are obtained as:

$$(\Phi_{\mathrm{int}}(f_1)\star\cdots\star\Phi_{\mathrm{int}}(f_n))(0)\,,$$

similarly for other observables in the theory.

RG flow and pAQFT

- *Wetterich equation on Lorentzian manifolds*, Edoardo D'Angelo, Nicolò Drago, Nicola Pinamonti, KR [arXiv:2202.07580]

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- We propose new flow equations that can be realized on arbitrary globally hyperbolic manifolds in any Hadamard state (examples: deSitter, thermal states).

Generating functions

- For an arbitrary but fixed Hadamard state ω, define:

$$Z(j) := \omega(\mathcal{S}_V(J)) = \omega[\mathcal{S}(V)^{-1} \star S(V+J)] = \omega[R_V \mathcal{S}(J)]\,,$$

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- It is a generating function for time-ordered interacting correlators:

$$\frac{\delta^n Z}{i^n \delta j(x_1)\dots\delta j(x_n)}\Big|_{j=0} = \omega \circ R_V\left(\chi(x_1) \cdot_T \dots \cdot_{\mathcal{T}} \chi(x_n)\right)$$
$$= \omega_V(\chi(x_1) \cdot_{\mathcal{T}} \cdots \cdot_T \chi(x_n)) = \omega \circ R_V(\chi(x_1) \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} \chi(x_n))\,,$$

where $\omega_V \doteq \omega \circ R_V$ is the interacting state.

Effective action

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- The effective action $\tilde{\Gamma}$ is $\tilde{\Gamma}(\phi) = W(j_\phi) - J_\phi(\phi)$, where $j_\phi \in C_c^\infty(M)$ is the current defined by

$$\left.\frac{\delta W}{\delta j}\right|_{j=j_\phi} = \phi ,$$

for $\phi \in \mathcal{E}$.

Choice of the regulator

- We use a local regulator

$$Q_k = -\frac{1}{2}\int dx\, q_k(x)\chi(x)^2\,,$$

and chose $q_k(x) = k^2 f(x)$, where f is a compactly supported smooth function (to be taken to 1). Compare: *Spectral functions of gauge theories with Banks-Zaks fixed points*, Yannick Kluth, Daniel F. Litim, Manuel Reichert, Phys.Rev.D 2023.

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- Modify the free theory: $I_{0k} = I_0 + Q_k$. The regularised generating functional Z_k is

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- We also have $W_k(j) = -i\log Z_k(j)$, $\tilde{\Gamma}_k(\phi) = W_k(j_\phi) - J_\phi(\phi)$ and finally we can translate $\tilde{\Gamma}_k$ to get the *average effective action*,

$$\Gamma_k(\phi) = \tilde{\Gamma}_k(\phi) - Q_k(\phi)\,.$$

Flow equations I

- By definition:

$$\partial_k W_k(j) = -\frac{1}{2}\int dx \partial_k q_k(x) \frac{1}{Z_k(j)} \omega(S(V)^{-1} \star [S(V+J+Q_k) \cdot_{\mathcal{T}} \mathcal{T}\chi^2(x)]) \,.$$

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- After a short computation:

$$\begin{aligned}&\partial_k \Gamma_k(\phi) \\ &= -\frac{1}{2}\int dx \partial_k q_k(x) \left[\frac{1}{Z_k(j_\phi)} \omega\big(R_V(S(J_\phi+Q_k) \cdot_{\mathcal{T}} \mathcal{T}\chi^2(x))\big) - \phi^2(x)\right] \\ &\qquad = \lim_{y\to x} \frac{i}{2}\int dx \partial_k q_k(x) \left[\frac{\delta^2 W_k(j)}{\delta j(x)\delta j(y)} - i\widetilde{H}_F(x,y)\right] ,\end{aligned}$$

where we use an appropriate distribution $\tilde{H}_F$. This corresponds to a choice of normal ordering. Hence...

Flow equations II

Wetterich-form equation

$$\partial_k \Gamma_k = -\frac{i}{2} \int dx \partial_k q_k(x) : \left[\Gamma_k^{(2)} - q_k \right]^{-1} :_{\widetilde{H}_F} (x) \,,$$

Thank you very much for your attention!