Off-shell divergences in perturbative quantum gravity

Quantum Spacetime & the RG 2023

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Leading logs

$$\kappa = \sqrt{32\pi G}$$

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$R^{(1)}_{\mu\alpha\nu\beta} = 2p_{[\mu|}\, p_{[\nu} h_{\beta]\,|\alpha]} \qquad \varphi = \frac{1}{2} h^{\mu}{}_{\mu}$$

$$\frac{1}{2} h_{\mu\nu} \Gamma^{\mu\nu\alpha\beta}(p) h_{\alpha\beta} = p^2 \left(\varphi^2 - \frac{1}{2} h^2_{\mu\nu} \right) + \frac{\kappa^2}{(4\pi)^2} \ln\left(\frac{p^2}{\mu^2}\right) \left(\frac{61}{60} (R^{(1)}_{\mu\nu})^2 - \frac{19}{120} (R^{(1)})^2 \right)$$
$$- \frac{\kappa^4 p^2}{(4\pi)^4} \left[\ln\left(\frac{p^2}{\mu^2}\right) \right]^2 \left(\frac{469}{7200} (R^{(1)})^2 - \frac{79}{400} (R^{(1)}_{\mu\nu})^2 - \frac{31}{1440} p^2 R^{(1)} \varphi \right)$$

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What if we could compute to all orders and resum?

$$\Gamma^{\mu\nu\alpha\beta}(p) \sim p^2 \sum_{\ell \gg 1} \frac{1}{\ell!} \left[a\kappa^2 p^2 \ln\left(\frac{p^2}{\mu^2}\right) \right]^{\ell} = p^2 \left(\frac{p^2}{\mu^2}\right)^{a\kappa^2 p^2} \quad ?$$

UV behaviour could be very different....

- Talk will motivate (proofs in paper):
- many relations due to BRST and RG
- but not enough to bootstrap to higher orders
- A recent proposal for generalised β functions would have made it so, but proposal is wrong

Leading logs

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in exact RG:

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in dim reg:

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RG flow

$g_1(\mu)$ $g_2(\mu)$

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$$\mu\partial_\mu\Gamma = -\frac{\kappa^2}{(4\pi)^2}\left(\frac{61}{30}(R_{\mu\nu}^{(1)})^2 - \frac{19}{60}(R^{(1)})^2\right)$$

$$+\frac{\kappa^4 p^2}{(4\pi)^4}\ln\left(\frac{p^2}{\mu^2}\right)\left(\frac{469}{1800}(R^{(1)})^2 - \frac{79}{100}(R_{\mu\nu}^{(1)})^2 - \frac{31}{360}p^2R^{(1)}\varphi\right)$$

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Two loop flow depends also on one-loop couplings

counterterm diagrams

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Two loop flow depends also on one-loop couplings

counterterm diagrams

For the double logs (double poles):

counterterm diags $\Gamma_{\rm ct} = -2\Gamma_2$ `pure' two loops

Pionic effective lagrangians, Weinberg (1979) Buchler & Colangelo (2003)

BRST invariance

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}$$

$$Qh_{\mu\nu} = \frac{1}{\kappa} Q g_{\mu\nu} = \mathcal{L}_c g_{\mu\nu} = 2\partial_{(\mu} c^\alpha g_{\nu)\alpha} + c^\alpha \partial_\alpha g_{\mu\nu}$$

$$Qc^\mu = \frac{\kappa}{2} \mathcal{L}_c c^\mu = \kappa c^\nu \partial_\nu c^\mu$$

$$S_0 = -\int_x \left\{ \frac{2}{\kappa^2} \sqrt{g} R + (Qh_{\mu\nu})\, h^{*\mu\nu} + (Qc^\mu)\, c^*_\mu \right\}$$

$$Q^2 = 0 \quad \Longrightarrow \quad 0 = QS_0 = Q\phi^A \frac{\partial_l S_0}{\partial \phi^A} = -\frac{\partial_r S_0}{\partial \phi^*_A} \frac{\partial_l S_0}{\partial \phi^A} = \frac{1}{2}(S_0, S_0)$$

CME

$$(X, Y) = \frac{\partial_r X}{\partial \phi^A} \frac{\partial_l Y}{\partial \phi^*_A} - \frac{\partial_r X}{\partial \phi^*_A} \frac{\partial_l Y}{\partial \phi^A}$$

antibracket

Zinn-Justin (1975) Batalin & Vilkovisky (1981)

BRST invariance

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It is not BRST invariance that is preserved

but the CME (ZJ ids)

$S = S_0 + \hbar S_1 + \hbar^2 S_2 + \cdots$ **s.t.** $(S,S) = 0\,.$

$$\Longrightarrow$$

$\Gamma = S_0 + \hbar \Gamma_1 + \hbar^2 \Gamma_2 + \cdots$ **s.t.** $(\Gamma,\Gamma) = 0\,.$

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$$\Longrightarrow$$

$$\Gamma = S_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \cdots$$ s.t. $$(\Gamma, \Gamma) = 0\,.$$

$(S_0, \Gamma_1) = 0$ **1 loop** $(S_0, S_1) = 0$

$(S_0, \Gamma_2) = -\frac{1}{2}(\Gamma_1, \Gamma_1)$ **2 loops** $(S_0, S_2) = -\frac{1}{2}(S_1, S_1)$

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$$s_0 K = (S_0, K) = \frac{\partial_r S_0}{\partial \phi^A}\frac{\partial_l K}{\partial \phi^*_A} - \frac{\partial_r S_0}{\partial \phi^*_A}\frac{\partial_l K}{\partial \phi^A}$$

Total classical BRST charge $s_0^2 = 0$

$s_0 \phi^A = Q\phi^A$ $s_0 \phi^*_A = \frac{\partial_r S_0}{\partial \phi^A}$ **Koszul-Tate diff^l**

BRST invariance

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$s_0\Gamma_1 = 0$ 1 loop $s_0 S_1 = 0$

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For the double logs (double poles):

$s_0\Gamma_2 = -\frac{1}{2}(S_1, S_1)$ 2 loops $s_0 S_2 = -\frac{1}{2}(S_1, S_1)$

How can this be consistent?

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BRST invariance

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$s_0\Gamma_1 = 0$ 1 loop $s_0 S_1 = 0$

For the double logs (double poles):

$s_0(\Gamma_2 + \Gamma_{\rm ct}) = +\frac{1}{2}(S_1, S_1)$ 2 loops $s_0 S_2 = -\frac{1}{2}(S_1, S_1)$

How can this be consistent?

$\Gamma_{\rm ct} = -2\Gamma_2$

RG ensures BRST invariance consistent!

General s_0-closed solution

1 loop $s_0 S_1 = 0$

2 loops $s_0 S_2 = -\frac{1}{2}(S_1, S_1)$

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$s_0^2 = 0$

Flat space: $s_0 S_\ell = 0 \quad \Longrightarrow \quad S_\ell = S_\ell[g_{\mu\nu}] + s_0 K_\ell[\phi, \phi^*]$

Barnich, Brandt & Henneaux (1994)

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Change of variables: $\delta \phi^A = \frac{\partial_l K}{\partial \phi^*_A} \qquad \delta \phi^*_A = -\frac{\partial_l K}{\partial \phi^A}$

$$\delta_K S_0 = \frac{\partial_r S_0}{\partial \phi^A} \delta \phi^A + \frac{\partial_r S_0}{\partial \phi^*_A} \delta \phi^*_A$$

canonical transformation

Canonical transformation

2 loops $$s_0 S_2 = -\frac{1}{2}(S_1, S_1)$$

$$S_1 = s_0 K_1$$

$$S_2 = \frac{1}{2}(S_1, K_1) +$$ s_0-closed

$$\delta\phi^A = \frac{\partial_l K_1}{\partial\phi^*_A} + \frac{1}{2}\frac{\partial_l}{\partial\phi^*_A}\frac{\partial_r K_1}{\partial\phi^B}\frac{\partial_l K_1}{\partial\phi^*_B} - \frac{1}{2}\frac{\partial_l}{\partial\phi^*_A}\frac{\partial_r K_1}{\partial\phi^*_B}\frac{\partial_l K_1}{\partial\phi^B}$$

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Canonical transformation

2 loops $s_0 S_2 = -\frac{1}{2}(S_1, S_1)$

If $S_1 = s_0 K_1$ then $S_2 = \frac{1}{2}(S_1, K_1) +$ s_0-closed

$$\delta\phi^A = \frac{\partial_l K_1}{\partial\phi^*_A} + \frac{1}{2}\frac{\partial_l}{\partial\phi^*_A}\frac{\partial_r K_1}{\partial\phi^B}\frac{\partial_l K_1}{\partial\phi^*_B} - \frac{1}{2}\frac{\partial_l}{\partial\phi^*_A}\frac{\partial_r K_1}{\partial\phi^*_B}\frac{\partial_l K_1}{\partial\phi^B}$$

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Curved space: $s_0 S_\ell = 0 \quad \Longrightarrow \quad S_\ell = S_\ell[\bar{g}_{\mu\nu}] + \cdots$?

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No separate background metric divergence!

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No separate background metric divergence!

If S_ℓ vanishes on classical EoM $\qquad s_0 h^{*\mu\nu} = \frac{\delta S_0}{\delta h_{\mu\nu}}$

$$S_\ell[g_{\mu\nu}] = -\frac{2}{\kappa}\int_x \sqrt{g}\, G^{\mu\nu} T_{\mu\nu}[g_{\mu\nu}] = s_0 \int_x h^{*\mu\nu} T_{\mu\nu} = s_0 K_\ell$$

One-loop 2-point vertices

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa \bar{h}_{\mu\nu} + \kappa h_{\mu\nu}$$

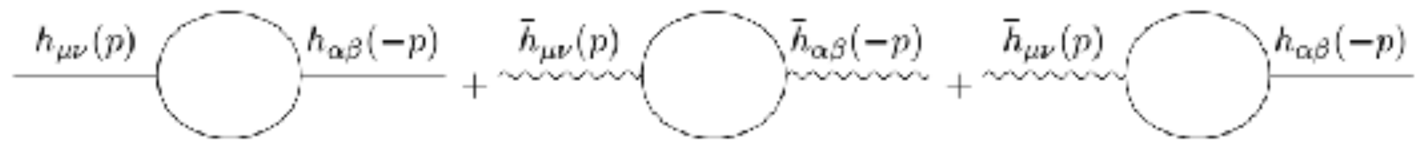

Figure 3.1: Two-point graviton diagrams at one loop. The wavy line represents the background field and the external plain line represents the quantum graviton field. The internal lines represent both a graviton loop and a ghost loop.

$$R^{(1)}_{\mu\nu} = -\partial^2_{\mu\nu}\varphi + \partial_{(\mu}\partial^\alpha h_{\nu)\alpha} - \frac{1}{2}\Box h_{\mu\nu}\,, \qquad R^{(1)} = \partial^2_{\alpha\beta}h_{\alpha\beta} - 2\Box\varphi$$

't Hooft & Veltman (1974)

$$S_1 = \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \left\{ \frac{61}{60}(R^{(1)}_{\mu\nu})^2 - \frac{19}{120}(R^{(1)})^2 + \frac{7}{20}(\bar{R}^{(1)}_{\mu\nu})^2 + \frac{1}{120}(\bar{R}^{(1)})^2 + \frac{41}{30}R^{(1)}_{\mu\nu}\bar{R}^{(1)\mu\nu} - \frac{3}{20}R^{(1)}\bar{R}^{(1)} \right\}$$

Capper (1980)

Kellett, Mitchell & TRM (2021)

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$$+\frac{41}{30}R^{(1)}_{\mu\nu}\bar{R}^{(1)\mu\nu} - \frac{3}{20}R^{(1)}\bar{R}^{(1)}\Big\}$$

$$S_1 = s_0 K_1$$

$$\varphi^* = \frac{1}{2}h^{*\mu}{}_{\mu}$$

$$K_1 = \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \Big\{ \beta h^{*\mu\nu}R^{(1)}_{\mu\nu} + \gamma\varphi^* R^{(1)} + \bar{\beta}h^{*\mu\nu}\bar{R}^{(1)}_{\mu\nu} + \bar{\gamma}\varphi^*\bar{R}^{(1)} \Big\}$$

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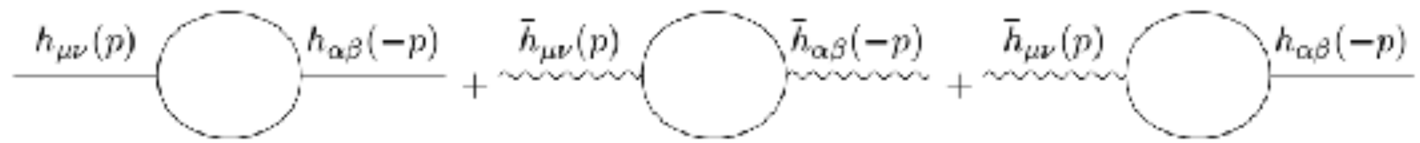

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One-loop 3-point vertices

$$S_1[\phi,\phi^*] = S_1[g_{\mu\nu}] + s_0 K_1[\phi,\phi^*]$$

Gauss-Bonnet

$$S_1[g_{\mu\nu}] = \frac{53}{90}\frac{\mu^{-2\varepsilon}}{(8\pi)^2\varepsilon}\int_x \sqrt{g}\,\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\alpha\beta}R^{\rho\sigma}{}_{\gamma\delta}$$

Goroff & Sagnotti (1986)
Gibbons, Hawking & Perry (1978)

$$\begin{aligned} K^1_{1/1} = {} & \frac{\kappa\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \left(\bar\beta h^{*\mu\nu}\bar R_{\mu\nu} + \bar\gamma\varphi^*\bar R\right) + \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \Big\{\beta h^{*\mu\nu}\left(\bar\nabla_\mu\bar\nabla^\alpha h_{\alpha\nu} - \tfrac{1}{2}\Box h_{\mu\nu}\right) \\ & + (c_1-\beta)\,h^{*\mu\nu}\bar\nabla_\mu\bar\nabla_\nu\varphi + \gamma\varphi^*\left(\bar\nabla_\alpha\bar\nabla_\beta h^{\alpha\beta} - 2\Box\varphi\right) + \alpha_3 h^{*\mu\nu}\bar R_\mu{}^\alpha h_{\alpha\nu} + \alpha_4 h^{*\mu\nu}\bar R_{\alpha\mu\nu\beta}h^{\alpha\beta} \\ & + \alpha_5 h^{*\mu\nu}\bar R_{\mu\nu}\varphi + \alpha_6\varphi^*\bar R^{\alpha\beta}h_{\alpha\beta} + \alpha_7\bar R\varphi^*\varphi\Big\} + \frac{\kappa^3\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x\sum_{i=1}^{27} b_i\left(h^*h^2\partial^2\right)_i\,, \qquad (3.16) \end{aligned}$$

$$\begin{aligned} K^2_{1/1} = {} & \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x \left(c_2 c^*_\mu\Box c^\mu + \alpha_1 c^*_\mu c^\mu\bar R + \alpha_2 c^*_\mu c^\nu\bar R^\mu{}_\nu\right) + \frac{\kappa^3\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x\frac{1}{\sqrt{\bar g}}\Big(\alpha_8\varphi^*\bar\nabla_\mu\varphi^* c^\mu \\ & + \alpha_9 h^{*\alpha\beta}\bar\nabla_\mu h^*_{\alpha\beta}c^\mu + \alpha_{10}\varphi^* h^*_{\mu\nu}\bar\nabla^\mu c^\nu + \alpha_{11}h^*_{\alpha\mu}h^{*\alpha}{}_\nu\bar\nabla^\mu c^\nu\Big) + \frac{\kappa^3\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\int_x\sum_{i=1}^{21} d_i\left(c^*ch\partial^2\right)_i\,. \end{aligned}$$

One-loop 3-point vertices

$$S_1[\phi, \phi^*] = S_1[g_{\mu\nu}] + s_0 K_1[\phi, \phi^*]$$

Gauss-Bonnet

$$S_1[g_{\mu\nu}] = \frac{53}{90} \frac{\mu^{-2\varepsilon}}{(8\pi)^2 \varepsilon} \int_x \sqrt{g}\, \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\alpha\beta} R^{\rho\sigma}{}_{\gamma\delta}$$

Goroff & Sagnotti (1986)
Gibbons, Hawking & Perry (1978)

$$\begin{aligned} K^1_{1/1} &= \frac{\kappa \mu^{-2\varepsilon}}{(4\pi)^2 \varepsilon} \int_x \left(\bar\beta h^{*\mu\nu} \bar R_{\mu\nu} + \bar\gamma \varphi^* \bar R\right) + \frac{\kappa^2 \mu^{-2\varepsilon}}{(4\pi)^2 \varepsilon} \int_x \Big\{\beta h^{*\mu\nu} \left(\bar\nabla_\mu \bar\nabla^\alpha h_{\alpha\nu} - \tfrac{1}{2} \Box h_{\mu\nu}\right) \\ &+ (c_1 - \beta)\, h^{*\mu\nu} \bar\nabla_\mu \bar\nabla_\nu \varphi + \gamma \varphi^* \left(\bar\nabla_\alpha \bar\nabla_\beta h^{\alpha\beta} - 2\bar\Box \varphi\right) + \alpha_3 h^{*\mu\nu} \bar R_\mu{}^\alpha h_{\alpha\nu} + \alpha_4 h^{*\mu\nu} \bar R_{\alpha\mu\nu\beta} h^{\alpha\beta} \\ &+ \alpha_5 h^{*\mu\nu} \bar R_{\mu\nu} \varphi + \alpha_6 \varphi^* \bar R^{\alpha\beta} h_{\alpha\beta} + \alpha_7 \bar R \varphi^* \varphi\Big\} + \frac{\kappa^3 \mu^{-2\varepsilon}}{(4\pi)^2 \varepsilon} \int_x \sum_{i=1}^{27} b_i \left(h^* h^2 \partial^2\right)_i , \qquad (3.16) \end{aligned}$$

$$\begin{aligned} K^2_{1/1} &= \frac{\kappa^2 \mu^{-2\varepsilon}}{(4\pi)^2 \varepsilon} \int_x \left(c_2 c^*_\mu \bar\Box c^\mu + \alpha_1 c^*_\mu c^\mu \bar R + \alpha_2 c^*_\mu c^\nu \bar R^\mu{}_\nu\right) + \frac{\kappa^3 \mu^{-2\varepsilon}}{(4\pi)^2 \varepsilon} \int_x \frac{1}{\sqrt{\bar g}} \Big(\alpha_8 \varphi^* \bar\nabla_\mu \varphi^* c^\mu \\ &+ \alpha_9 h^{*\alpha\beta} \bar\nabla_\mu h^*_{\alpha\beta} c^\mu + \alpha_{10} \varphi^* h^*_{\mu\nu} \bar\nabla^\mu c^\nu + \alpha_{11} h^*_{\alpha\mu} h^{*\alpha}{}_\nu \bar\nabla^\mu c^\nu\Big) + \frac{\kappa^3 \mu^{-2\varepsilon}}{(4\pi)^2 \varepsilon} \int_x \sum_{i=1}^{21} d_i \left(c^* c h \partial^2\right)_i . \end{aligned}$$

$$c_1 = -\frac{1}{2}, \qquad \text{and} \qquad c_2 = -\frac{3}{8}.$$

$$\begin{aligned} \sum_{i=1}^{27} b_i \left(h^* h^2 \partial^2\right)_i &= \frac{5}{12} h^{*\mu\nu} \varphi \partial^2_{\mu\nu} \varphi - \frac{13}{160} h^{*\mu\nu} \partial^2_{\mu\nu} h^\beta{}_\alpha h^\alpha{}_\beta + \frac{1}{4} h^{*\mu\alpha} \left(\partial^\nu h_{\alpha\nu} \partial_\mu \varphi - \partial_\mu \partial^\nu h_{\alpha\nu} \varphi\right) \\ &+ \frac{61}{240} h^{*\mu\alpha} \left(\partial_\mu h_{\alpha\nu} \partial^\nu \varphi - h_{\alpha\nu} \partial^\nu \partial_\mu \varphi\right) + \frac{7}{80} h^{*\mu\alpha} \left(\partial_\mu h_{\beta\nu} \partial^\nu h^\beta{}_\alpha - h_{\beta\nu} \partial^\nu \partial_\mu h^\beta{}_\alpha\right) \\ &- \frac{61}{240} h^{*\mu\alpha} \left(\partial_\nu h_\beta{}^\nu \partial_\mu h^\beta{}_\alpha - \partial^2_{\mu\nu} h_\beta{}^\nu h^\beta{}_\alpha\right) + \frac{13}{60} \varphi^* \partial^\alpha h_{\beta\nu} \partial^\nu h^\beta{}_\alpha + \frac{43}{60} \varphi^* h_{\beta\nu} \partial^\nu \partial^\alpha h^\beta{}_\alpha \\ &+ \frac{77}{120} \varphi^* \partial^\nu h_{\beta\nu} \partial^\alpha h^\beta{}_\alpha - \frac{53}{60} \varphi^* h_{\alpha\nu} \partial^\nu \partial^\alpha \varphi - \frac{17}{10} \varphi^* \partial^\nu h_{\alpha\nu} \partial^\alpha \varphi - \frac{3}{10} \varphi^* \varphi \Box \varphi - \frac{11}{60} \varphi^* \varphi \partial^2_{\alpha\nu} h^{\alpha\nu} \\ &+ \frac{9}{40} \varphi^* h^\alpha{}_\beta \Box h^\beta{}_\alpha + \frac{14}{15} \varphi^* \partial_\nu \varphi \partial^\nu \varphi - \frac{11}{80} \varphi^* \partial_\nu h^\alpha{}_\beta \partial^\nu h^\beta{}_\alpha - \frac{131}{240} h^{*\mu\nu} \partial^\alpha h_{\mu\nu} \partial_\alpha \varphi \\ &- \frac{1}{4} h^{*\mu\nu} h^\alpha{}_\mu \partial^2_{\alpha\beta} h^\beta{}_\nu - \frac{1}{12} h^{*\mu\nu} \partial_\alpha h^\alpha{}_\mu \partial_\beta h^\beta{}_\nu - \frac{27}{80} h^{*\mu\nu} \partial_\beta h^\alpha{}_\mu \partial_\alpha h^\beta{}_\nu + \frac{17}{80} h^{*\mu\nu} \partial_\alpha h_{\mu\beta} \partial^\alpha h^\beta{}_\nu \\ &+ \frac{7}{80} h^{*\mu\nu} \partial^2_{\alpha\beta} h_{\mu\nu} h^{\alpha\beta} - \frac{1}{2} h^{*\mu\nu} \Box h_{\mu\beta} h^\beta{}_\nu + \frac{37}{80} h^{*\mu\nu} \partial^\alpha h_{\mu\nu} \partial^\beta h_{\alpha\beta} - \frac{1}{3} h^{*\mu\nu} h_{\mu\nu} \Box \varphi \\ &+ \frac{1}{3} h^{*\mu\nu} h_{\mu\nu} \partial^2_{\alpha\beta} h^{\alpha\beta} + \frac{11}{24} h^{*\mu\nu} \Box h_{\mu\nu} \varphi . \qquad (3.22) \end{aligned}$$

$$\alpha_1 = -\frac{1}{8}, \quad \alpha_2 = -\frac{1}{24}, \quad \alpha_3 = \frac{161}{120}, \quad \alpha_4 = \frac{1}{120}, \quad \alpha_5 = -\frac{3}{4}, \quad \alpha_6 = -\frac{7}{15},$$
$$\alpha_7 = \frac{19}{60}, \quad \alpha_8 = -\frac{1}{6}, \quad \alpha_9 = -\frac{1}{12}, \quad \alpha_{10} = -\frac{4}{15}, \quad \alpha_{11} = -\frac{1}{6},$$

$$\begin{aligned} \sum_{i=1}^{21} d_i \left(c^* c h \partial^2\right)_i &= \frac{1}{12} c^{*\mu} \partial^2_{\mu\nu} c^\nu \varphi - \frac{121}{480} c^*_\mu \partial^\mu c^\nu \partial_\nu \varphi + \frac{61}{480} c^*_\mu \partial^\mu c^\nu \partial_\alpha h^\alpha{}_\nu - \frac{11}{24} c^{*\mu} \partial^2_{\mu\alpha} c^\nu h^\alpha{}_\nu \\ &- \frac{1}{3} c^{*\mu} c^\nu \partial^2_{\mu\nu} \varphi + \frac{1}{6} c^{*\mu} c^\nu \partial^2_{\alpha\mu} h^\alpha{}_\nu - \frac{1}{24} c^*_\mu \partial_\nu c^\nu \partial^\mu \varphi - \frac{101}{480} c^*_\mu \partial_\alpha c^\nu \partial^\mu h^\alpha{}_\nu - \frac{1}{8} c^*_\alpha \partial^2_{\mu\nu} c^\nu h^{\alpha\mu} \\ &- \frac{119}{480} c^*_\alpha \partial^\mu c^\nu \partial_\nu h^\alpha{}_\mu + \frac{1}{12} c^*_\alpha c^\nu \partial^2_{\mu\nu} h^{\alpha\mu} + \frac{1}{8} c^*_\alpha \partial_\nu c^\nu \partial^\mu h^\alpha{}_\mu - \frac{301}{480} c^*_\alpha \partial^\nu c^\alpha \partial_\nu \varphi + \frac{1}{4} c^*_\alpha c^\alpha \Box \varphi \\ &+ \frac{1}{3} c^*_\alpha \Box c^\alpha \varphi - \frac{1}{12} c^*_\alpha c^\alpha \partial^2_{\mu\nu} h^{\mu\nu} + \frac{27}{160} c^*_\alpha \Box c^\mu h^\alpha{}_\mu - \frac{239}{480} c^*_\alpha \partial_\nu c^\mu \partial^\nu h^\alpha{}_\mu - \frac{1}{4} c^*_\alpha c^\mu \Box h^\alpha{}_\mu \\ &+ \frac{7}{160} c^*_\alpha \partial^2_{\mu\nu} c^\alpha h^{\mu\nu} + \frac{241}{480} c^*_\alpha \partial^\nu c^\alpha \partial^\mu h_{\mu\nu} . \qquad (3.23) \end{aligned}$$

Leading two-loop 2-point vertices

$$S_2 = -\frac{1}{2}\frac{\kappa^4\mu^{4\varepsilon}}{(4\pi)^4\varepsilon^2}\int_x \Bigg\{ \frac{11}{36}\bar{R}^{(1)}\Box\bar{R}^{(1)} + \frac{5}{72}\bar{R}^{(1)\mu\nu}\Box\bar{R}^{(1)}_{\mu\nu} - \frac{469}{3600}R^{(1)}\Box R^{(1)} + \frac{79}{200}R^{(1)\mu\nu}\Box R^{(1)}_{\mu\nu}$$

$$+\frac{781}{3600}\bar{R}^{(1)}\Box R^{(1)} + \frac{53}{150}\bar{R}^{(1)\mu\nu}\Box R^{(1)}_{\mu\nu} - \frac{31}{720}\left(\bar{R}^{(1)} + R^{(1)}\right)\Box^2\varphi\Bigg\}$$

$$S_2 = \frac{1}{2}(S_1, K_1) + s_0 K_2$$

$$K_2 = \frac{\kappa^4\mu^{4\varepsilon}}{(4\pi)^2\varepsilon^2}\int_x \Big\{ \frac{877}{28800}h^{*\mu\nu}\Box R^{(1)}_{\mu\nu} + \frac{71}{1800}\varphi^*\Box R^{(1)}$$

$$+\frac{361}{28800}h^{*\mu\nu}\Box\bar{R}^{(1)}_{\mu\nu} + \frac{2719}{14400}\varphi^*\Box\bar{R}^{(1)} - \frac{31}{1440}\varphi^*\Box^2\varphi\Big\}$$

Leading two-loop 2-point vertices

$$S_2 = -\frac{1}{2}\frac{\kappa^4\mu^{4\varepsilon}}{(4\pi)^4\varepsilon^2}\int_x \left\{ \frac{11}{36}\bar{R}^{(1)}\Box\bar{R}^{(1)} + \frac{5}{72}\bar{R}^{(1)\mu\nu}\Box\bar{R}^{(1)}_{\mu\nu} - \frac{469}{3600}R^{(1)}\Box R^{(1)} + \frac{79}{200}R^{(1)\mu\nu}\Box R^{(1)}_{\mu\nu} \right.$$

$$\left. + \frac{781}{3600}\bar{R}^{(1)}\Box R^{(1)} + \frac{53}{150}\bar{R}^{(1)\mu\nu}\Box R^{(1)}_{\mu\nu} - \frac{31}{720}\left(\bar{R}^{(1)} + R^{(1)}\right)\Box^2\varphi \right\}$$

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$$\left. + \frac{361}{28800}h^{*\mu\nu}\Box\bar{R}^{(1)}_{\mu\nu} + \frac{2719}{14400}\varphi^*\Box\bar{R}^{(1)} - \frac{31}{1440}\varphi^*\Box^2\varphi \right\}$$

Generalised β functions

Solodhukin (2021) <- Kazakov (1988)

$$\bar{g}^{0}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \sum_{k=1} \frac{1}{\varepsilon^{k}}\, \bar{\mathfrak{g}}^{k}_{\mu\nu}(x)$$

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't Hooft & Veltman (1974)

$$S_1 = \frac{\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon} \int_x \sqrt{\bar{g}} \left(\frac{1}{120}\bar{R}^2 + \frac{7}{20}\bar{R}^2_{\mu\nu} \right)$$

$$\bar{g}^0_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\, \bar{\mathfrak{g}}^1_{\mu\nu}\,, \qquad \text{where} \qquad \bar{\mathfrak{g}}^1_{\mu\nu} = \frac{7}{40}\bar{R}_{\mu\nu} + \frac{11}{120}\bar{g}_{\mu\nu}\bar{R}$$

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$$\bar{g}^0_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \sum_{k=1} \frac{1}{\varepsilon^k} \bar{\mathfrak{g}}^k_{\mu\nu}(x)$$

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$$\bar{\beta}_{\mu\nu} = \mu \partial_\mu \bar{g}_{\mu\nu} = 2 \frac{\kappa^2 \mu^{-2\varepsilon}}{(4\pi)^2} \bar{\mathfrak{g}}^1_{\mu\nu} + \cdots$$

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$$\bar{g}^0_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\,\bar{\mathfrak{g}}^1_{\mu\nu} + \frac{\kappa^4\mu^{-4\varepsilon}}{(4\pi)^4\varepsilon^2}\,\bar{\mathfrak{g}}^2_{\mu\nu}$$

+ finiteness of $\bar{\beta}_{\mu\nu}$ $\Longrightarrow$ $\bar{\mathfrak{g}}^2_{\alpha\beta} = \frac{4\pi^2\mu^{2\varepsilon}}{\kappa^2}\,\mu\partial_\mu \bar{\mathfrak{g}}^1_{\alpha\beta}[\bar{g}]$

Generalised β functions

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$$\bar{\beta}_{\mu\nu} = \mu\partial_\mu \bar{g}_{\mu\nu} = 2\,\frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2}\,\bar{\mathfrak{g}}^1_{\mu\nu} + \cdots$$

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+ finiteness of $\bar{\beta}_{\mu\nu}$ $\implies$ $\bar{\mathfrak{g}}^2_{\alpha\beta} = \frac{4\pi^2\mu^{2\varepsilon}}{\kappa^2}\,\mu\partial_\mu \bar{\mathfrak{g}}^1_{\alpha\beta}[\bar{g}]$ ✗

Generalised β functions

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$$\bar{g}^{0}_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \sum_{k=1} \frac{1}{\varepsilon^k}\,\bar{\mathfrak{g}}^{k}_{\mu\nu}(x)$$

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$$\bar{\beta}_{\mu\nu} = \mu\partial_\mu \bar{g}_{\mu\nu} = 2\,\frac{\kappa^2\mu^{-2\varepsilon}}{(4\pi)^2}\,\bar{\mathfrak{g}}^{1}_{\mu\nu} + \cdots$$

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+ finiteness of $\bar{\beta}_{\mu\nu}$ $\Longrightarrow$ $\bar{\mathfrak{g}}^{2}_{\alpha\beta} = \frac{4\pi^2\mu^{2\varepsilon}}{\kappa^2}\,\mu\partial_\mu\bar{\mathfrak{g}}^{1}_{\alpha\beta}[\bar{g}]$

Generalised β functions

$$\delta\phi^A = \frac{\partial_l K_1}{\partial\phi^*_A} + \frac{1}{2}\frac{\partial_l}{\partial\phi^*_A}\frac{\partial_r K_1}{\partial\phi^B}\frac{\partial_l K_1}{\partial\phi^*_B} - \frac{1}{2}\frac{\partial_l}{\partial\phi^*_A}\frac{\partial_r K_1}{\partial\phi^*_B}\frac{\partial_l K_1}{\partial\phi^B}$$

$$\delta\phi^*_A = -\frac{\partial_l K_1}{\partial\phi^A} + \frac{1}{2}\frac{\partial_l}{\partial\phi^A}\frac{\partial_r K_1}{\partial\phi^*_B}\frac{\partial_l K_1}{\partial\phi^B} - \frac{1}{2}\frac{\partial_l}{\partial\phi^A}\frac{\partial_r K_1}{\partial\phi^B}\frac{\partial_l K_1}{\partial\phi^*_B}$$

$$\beta^A[\phi,\phi^*] = \mu\partial_\mu\phi^A$$

$$\phi_0^{(*)} = \phi^{(*)} + \delta\phi^{(*)}$$

$$\beta^*_A[\phi,\phi^*] = \mu\partial_\mu\phi^*_A$$

Generalised β functions

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$$\phi_0^{(*)} = \phi^{(*)} + \delta\phi^{(*)}$$

$$\beta^A[\phi,\phi^*] = \mu\partial_\mu\phi^A$$

$$\beta^*_A[\phi,\phi^*] = \mu\partial_\mu\phi^*_A$$

Not finite

Generalised β functions

$$\delta\phi^A = \frac{\partial_l K_1}{\partial \phi^*_A} + \frac{1}{2}\frac{\partial_l}{\partial \phi^*_A}\frac{\partial_r K_1}{\partial \phi^B}\frac{\partial_l K_1}{\partial \phi^*_B} - \frac{1}{2}\frac{\partial_l}{\partial \phi^*_A}\frac{\partial_r K_1}{\partial \phi^*_B}\frac{\partial_l K_1}{\partial \phi^B}$$

$$\delta\phi^*_A = -\frac{\partial_l K_1}{\partial \phi^A} + \frac{1}{2}\frac{\partial_l}{\partial \phi^A}\frac{\partial_r K_1}{\partial \phi^*_B}\frac{\partial_l K_1}{\partial \phi^B} - \frac{1}{2}\frac{\partial_l}{\partial \phi^A}\frac{\partial_r K_1}{\partial \phi^B}\frac{\partial_l K_1}{\partial \phi^*_B}$$

$$\phi_0^{(*)} = \phi^{(*)} + \delta\phi^{(*)}$$

$$\beta^A[\phi,\phi^*] = \mu\partial_\mu \phi^A$$

$$\beta^*_A[\phi,\phi^*] = \mu\partial_\mu \phi^*_A$$

Not finite

$$\mu\partial_\mu \mathcal{Z}[J,\phi^*] = \int \mathcal{D}\phi\ \beta^A[\phi,\phi^*] J_A\ \mathrm{e}^{-S[\phi_0,\phi_0^*]+\phi^A J_A}$$

Summary

- Leading divergences important even if vanish on EoM?
- RG required for BRST invariance to be consistent.
- Divergences: diff inv fnl of total metric + canon transf
- No separate background metric divergence.
- Computed off-shell divergences up to 3pt & two loops
- Generalised β functions are not finite