The UV critical manifolds of AS

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October 2, 2023

Phys.Rev.D 108 (2023) 2, 026008 e-Print: 2304.12011, e-Print: 2304.12011

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Overview

- What is the role of the conformal factor in AS?
- Conformally reduced $R + R^2$ theory
- Beyond the polynomial truncations
- The UV critical manifold: evidence for and against finite dimensionality
- The role of split-symmetry
- Gauge invariance reloaded

$R + R^2$

- First attempt to go beyond EH truncation by Reuter and Lauscher (2002)
- Machado and Percacci (2009): pure quadratic theory
- Demmel, Saueressig and Zanusso (2015): scaling solution from $f(R)$ approximation
- Knorr (2021): $R + R^2$ and $R + R^3$ - flat projection -
- Question: does the projection/background matter? - see η_{kin} vs η_{pot} discussion

Conformal factor in QG

- In a series of papers Reuters and collaborators have clarified that conformal factor plays a central role in the emergence of the NGFP in the ultraviolet region and in the determination of the critical properties of the theory (see Reuter and Weyer)
- There are two issues in particular which make the emergence of AS gravity highly non-trivial, the first one is the use of the background field approach, and the second is the pivotal role played by the conformal mode instability.
- In fact, the central idea of conformal field quantization is to employ the background metric (in the sense of the background field method) in constructing the Wilsonian renormalization group equations.

- On the other hand, as the conformal factor has the wrong kinetic sign in the euclidean theory, either the conformal factor is integrated out before doing any functional integral ('t Hooft), or a special regulator must be employed to cutoff modes with the "wrong" ultraviolet stability properties.
- As first discussed in AB and Guarnieri (2012), the IR evolution of the renormalization trajectory can be problematic and only an ultraviolet evolution can be consistently defined.
- Most probably a new kind of perturbative continuum limit for quantum gravity emerges in the deep UV for the conformally reduced theory. (see Morris)

Conformally reduced EH (CREH)

- In the standard framework a background metric $\bar{g}_{\mu\nu}$ is chosen in order to perform the actual calculations, and the fluctuations $h_{\mu\nu}$ are thus "integrated-out" in momentum shell.
- The background should be dynamically determined by the requirement that the expectation value of the fluctuation field vanishes, $\langle h_{\mu\nu} \rangle \equiv \bar{h}_{\mu\nu} = 0$.
- Any physical length must then be proper with respect to the background metric $\bar{g}_{\mu\nu}$.
- In the conformally reduced theory we assume

$$\bar{g}_{\mu\nu} = \chi_B^2 \hat{g}_{\mu\nu}$$

where $\hat{g}_{\mu\nu}$ is a reference metric which plays no dynamical role but it is instead fixed to perform the actual calculations on the geometry defined by $\bar{g}$.

CREH

- Let us now consider $S[\chi]$ to be the action for the fundamental field $\chi(x)$ that we write as $\chi(x) = \chi_B(x) + f(x)$ where $\chi_B(x)$ is a non-dynamical background field and $f(x)$ a dynamical (fluctuating) field.
- In this formalism χ plays the same role of a microscopic metric $\gamma_{\mu\nu}$ in the full theory.
- the expectation values $\bar{f} \equiv \langle f \rangle$ and $\phi \equiv \langle \chi \rangle = \chi_B + \bar{f}$ are the analogs of $\bar{h}_{\mu\nu} \equiv \langle h_{\mu\nu} \rangle$ and $g_{\mu\nu} = \langle \gamma_{\mu\nu} \rangle = \bar{g}_{\mu\nu} + \bar{h}_{\mu\nu}$ in the full theory.

CREH $R + R^2$

Consider the RG flow equation approach to study the conformal sector of the the following theory:

$$S = \int d^d x \sqrt{g} \left[\frac{1}{16\pi G} (-R + 2\Lambda) + \beta R^2 \right]. \tag{1}$$

A Weyl rescaling $g_{\mu\nu} = \phi(x)^{2\nu} \hat{g}_{\mu\nu}$ is implemented, where $\nu = 2/(d-2)$ and $\hat{g}_{\mu\nu}$ is a reference metric. Weyl rescaling leads to

$$R = \phi^{-2\nu} \left(\hat{R} - \frac{2\nu(d-1)\hat{\Box}\phi}{\phi} + g(d) \frac{\hat{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi}{\phi^2} \right), \tag{2}$$

where $g(d) = \big(2\nu(d-1) - \nu^2(d-1)(d-2)\big) = 0$ as $\nu = 2/(d-2)$ and $\hat{R} = R(\hat{g})$.

CREH $R + R^2$

$$\Gamma_k = \int d^dx\sqrt{\hat{g}}\left[-\frac{1}{2}Z_k\phi\hat{\Box}\phi + U[\phi] + 16\left(\frac{d-1}{d-2}\right)^2\beta_k(\hat{\Box}\phi)^2\phi^{\frac{2d-8}{d-2}-2} + \right.$$
$$\left. -8\left(\frac{d-1}{d-2}\right)\beta_k\hat{R}(\hat{\Box}\phi)\phi^{\frac{2d-8}{d-2}-1}\right], \qquad (3)$$

where $U[\phi] = AZ_k\hat{R}\phi^2 - 2AZ_k\Lambda_k\phi^{\frac{2d}{d-2}} + \beta_k\hat{R}^2\phi^{\frac{2d-8}{d-2}}$,
$A = (d-2)/8(d-1)$ and

$$Z_k = -\frac{1}{2\pi G_k}\frac{d-1}{d-2}.$$

PT flow-equation

We use the PT FRG

$$\partial_t S_k[f;\chi_B] = -\frac{1}{2}\,\mathrm{Tr}\int_0^\infty \frac{ds}{s}\,\partial_t\,\rho_k\,\exp\Big\{-s\frac{\delta^2 S_k[f;\chi_B]}{\delta f^2}\Big\}\,, \qquad (4)$$

where $t \equiv \log(k)$ is the RG time and $\rho_k = \rho_k[\chi_B]$. For actual calculations we shall use various families of smooth cutoffs $\rho_k \equiv \rho_k^{1,2}(s,n)$

$$\rho_k^1(s,n) = \frac{\Gamma(n, s\,\mathcal{Z}\,\hat{k}^2) - \Gamma(n, s\,\mathcal{Z}\,\Lambda^2)}{\Gamma(n)} \qquad (5)$$

$$\rho_k^2(s,n) = \frac{\Gamma(n, s\,n\,\mathcal{Z}\,\hat{k}^2) - \Gamma(n, s\,n\,\mathcal{Z}\,\Lambda^2)}{\Gamma(n)}. \qquad (6)$$

Here n is an arbitrary real, positive parameter that controls the shape of the $\rho_k^{1,2}(s,n)$ in the interpolating regions, and $\Gamma(\alpha,x) = \int_x^\infty dt\; t^{\alpha-1}e^{-t}$ denotes the incomplete Gamma-function.

FRG

- $\mathcal{Z}$ is a constant which has to be adjusted: being the kinetic terms of the field of type a of the form $-Z_a\hat{\Box}$, we impose exactly $\mathcal{Z} = Z_a$. With this prescription, in (5) the eigenvalues of $\hat{\Box}$ are cut off at $\sim \hat{k}^2$, instead of $\sim \hat{k}^2/Z_a$. Similarly, in (6) the cutoff is located at $\sim \hat{k}^2/n$. These two choices represent two so-called 'spectral adjustments'
- the trace inside the flow equation (4) must be performed on the modes of the background $\bar{g}_{\mu\nu}$. This is concretely performed inside the regularizators through the identification $\hat{k}^2 = \chi_B^{2\nu}\bar{k}^2$.
- Finally, Λ represents the cutoff in the UV. As we are interested only in the Wilson-Kadanoff portion of the RG, the UV cut-off is sent to infinity.

FRG

Overall, this leads to implementing the scaling laws

$$\partial_t \rho_k^1(s,n) = -\frac{2}{\Gamma(n)}(s\,Z\,k^2\,\chi_B^{2\nu})^n e^{-s\,Zk^2\chi_B^{2\nu}}. \tag{7}$$

$$\partial_t \rho_k^2(s,n) = -\frac{2}{\Gamma(n)}(s\,n\,Zk^2\,\chi_B^{2\nu})^n e^{-s\,nZk^2\chi_B^{2\nu}} \tag{8}$$

inside the flow equation. Concretely, the calculations for both cutoff families are performed through a range of values for the smoothness parameter n: $n = \{3, 5, 7, 9, 10, 15, 20, 30, 40, 50\}$. The limiting case $n \to \infty$ is also considered for the second regularizator: this is readily done through

$$\lim_{n\to\infty} \partial_t \rho_k^{(2)}(s,n) = -\frac{2}{Z\,k^2\,\chi_B^{2\nu}}\delta\left(s - \frac{1}{Z\,k^2\,\chi_B^{2\nu}}\right). \tag{9}$$

FRG

Cutoffs (7) and (8) are built through the regularization on $\bar{k}^2$, which are the modes of the $-\bar{\Box}$ operator. Because our model contains both $-\bar{\Box}$ and $\bar{\Box}^2$ operators, we also apply a regularization on the quadratic operator in order to check possible differences in the physics of the results. Higher-derivative cutoffs are defined:

$$\tilde{\rho}_k^1(s,n) = \frac{\Gamma(n, s\,(Z\hat{k}^2 + Z^2\hat{k}^4)) - \Gamma(n, s\,(Z\Lambda^2 + Z^2\Lambda^4))}{\Gamma(n)} \tag{10}$$

$$\tilde{\rho}_k^2(s,n) = \frac{\Gamma(n, s\,n\,(Z\hat{k}^2 + Z^2\hat{k}^4)) - \Gamma(n, s\,n\,(Z\Lambda^2 + Z^2\Lambda^4))}{\Gamma(n)}, \tag{11}$$

see also (Buccio and Percacci recent works).

FRG

In particular in the $\Lambda \to \infty$ limit we obtain,

$$\partial_t \tilde{\rho}^1_k(s,n) = -\frac{2}{\Gamma(n)}(s\,(Zk^2\chi_B^{2\nu} + Z^2k^4\chi_B^{4\nu}))^n \times \\ \times\, e^{-s\,(Zk^2\chi_B^{2\nu}+Z^2k^4\chi_B^{4\nu})}. \tag{12}$$

$$\partial_t \tilde{\rho}^2_k(s,n) = -\frac{2}{\Gamma(n)}(s\,n\,(Zk^2\chi_B^{2\nu} + Z^2k^4\chi_B^{4\nu}))^n \times \\ \times\, e^{-s\,n\,(Zk^2\chi_B^{2\nu}+Z^2k^4\chi_B^{4\nu})}. \tag{13}$$

S^4 projection

Projecting on S^4 we arrive at:

$$\partial_t \Gamma_k = \frac{1}{2}\int_0^\infty \frac{ds}{s}\left(-\frac{2}{\Gamma(n)}(s\,n\,Z_k\chi_B^{\frac{4}{d-2}}k^2)^n e^{-s\,n\,Z_k\chi_B^{\frac{4}{d-2}}k^2}\right)\times$$
$$\times\, e^{-s\left[Z_k\left(2A\hat{R}-2AB\Lambda_k\chi_B^{\frac{2d}{d-2}-2}\right)+C\beta_k\hat{R}^2\chi_B^{\frac{2d-8}{d-2}-2}\right]}\overline{\mathrm{Tr}}W(-\bar{\Box}), \qquad (14)$$

where

$$W(-\bar{\Box}) = e^{-s\left[-\left(Z_k\chi_B^{\frac{4}{d-2}}+D\beta_k\hat{R}\chi_B^{\frac{2d-4}{d-2}-2}\right)\bar{\Box}+E\beta_k\chi_B^{\frac{2d}{d-2}-2}\bar{\Box}^2\right]}, \qquad (15)$$

with $D = 16\left(\frac{d-1}{d-2}\right)\left(\frac{2d-8}{d-2}-1\right)$ and $E = 32\left(\frac{d-1}{d-2}\right)^2$.

Local expansion

The last trace can be performed through a Seeley-Gilkey-deWitt heat kernel expansion at quadratic order in curvature:

$$\overline{\mathrm{Tr}}[W(-\bar{\Box})] = \frac{1}{(4\pi)^{d/2}} \int d^d x \sqrt{\bar{g}} \sum_{k\geq 0} [a_k] Q_{\frac{d}{2}-k}, \qquad (16)$$

where $Q_n = \frac{1}{\Gamma[n]} \int_0^\infty dz W(z) z^{n-1}$ and where the first coefficients of the expansion in the spherical projection are $[a_0] = 1$, $[a_1] = \hat{R}/6$, $[a_2] = f(d)\hat{R}^2$ with

$$f(d) = \frac{1}{18}\left(\frac{1}{4} + \frac{1}{5d(d-1)} - \frac{1}{10d}\right). \qquad (17)$$

Results

Fixed point parameters for $\rho_k^1(s,n)$					Fixed point parameters for $\rho_k^2(s,n)$				
n	$\mathbf{z}^*$	$\mathbf{g}^*$	λ^*	$\mathbf{b}^*$	**n**	$\mathbf{z}^*$	$\mathbf{g}^*$	λ^*	$\mathbf{b}^*$
3	3.276×10^{-3}	-72.870	0.169	-3.695×10^{-4}	**3**	9.828×10^{-3}	-24.290	0.509	-3.695×10^{-4}
5	2.027×10^{-3}	-117.731	0.099	-3.223×10^{-4}	**5**	1.013×10^{-2}	-23.546	0.496	-3.223×10^{-4}
7	1.460×10^{-3}	-163.484	0.069	-3.061×10^{-4}	**7**	1.022×10^{-2}	-23.354	0.488	-3.061×10^{-4}
9	1.139×10^{-3}	-209.495	0.053	-2.979×10^{-4}	**9**	1.025×10^{-2}	-23.277	0.484	-2.979×10^{-4}
10	1.026×10^{-3}	-232.547	0.048	-2.951×10^{-4}	**10**	1.026×10^{-2}	-23.254	0.482	-2.951×10^{-4}
15	6.859×10^{-4}	-348.009	0.031	-2.872×10^{-4}	**15**	1.028×10^{-2}	-23.200	0.478	-2.872×10^{-4}
20	5.149×10^{-4}	-463.619	0.023	-2.835×10^{-4}	**20**	1.029×10^{-2}	-23.180	0.475	-2.835×10^{-4}
30	3.435×10^{-4}	-694.983	0.015	-2.798×10^{-4}	**30**	1.030×10^{-2}	-23.166	0.472	-2.798×10^{-4}
40	2.576×10^{-4}	-926.417	0.011	-2.781×10^{-4}	**40**	1.030×10^{-2}	-23.160	0.471	-2.781×10^{-4}
50	2.061×10^{-4}	-1157.879	0.009	-2.770×10^{-4}	**50**	1.030×10^{-2}	-23.157	0.470	-2.770×10^{-4}
$+\infty$	$-$	$-$	$-$	$-$	$+\infty$	1.031×10^{-2}	-23.150	0.467	-2.729×10^{-4}

Reuter-Lauscher (full, S^4): $(g_*, \lambda_*, \beta_*) = (0.292, 0.330, 0.005)$

Crit

Fixed point parameters for $\rho_k^1(s,n)$				Fixed point parameters for $\rho_k^2(s,n)$			
n	λ_1	θ'	θ''	**n**	λ_1	θ'	θ''
3	-3.035	-0.584	1.488	**3**	-3.035	-0.584	1.488
5	-2.546	-0.586	1.494	**5**	-2.546	-0.586	1.494
7	-2.382	-0.588	1.500	**7**	-2.382	-0.588	1.500
9	-2.300	-0.589	1.504	**9**	-2.300	-0.589	1.504
10	-2.272	-0.589	1.505	**10**	-2.272	-0.589	1.505
15	-2.193	-0.590	1.510	**15**	-2.193	-0.590	1.510
20	-2.156	-0.591	1.512	**20**	-2.156	-0.591	1.512
30	-2.120	-0.592	1.515	**30**	-2.120	-0.592	1.515
40	-2.103	-0.592	1.516	**40**	-2.103	-0.592	1.516
50	-2.092	-0.592	1.517	**50**	-2.092	-0.592	1.517
$+\infty$	$-$	$-$	$-$	$+\infty$	-2.503	-0.709	1.966

For additional details see Maria's poster. - open questions -

Beyond polynomial approximations

- Let us consider CR gravity theory of the type

$$S = \int d^d x \sqrt{g} \left[f(g) R + h(g) \right]. \tag{18}$$

  where $f(g)$ and $h(g)$ are completely arbitrary functions. What is the flow of f_k and g_k in the CR theory?
- First attempts: Reuter and Weyer (2009), AB and F. Guarnieri (2012) - only flow for $h_k(g)$ (LPA approximation).
- Complete flow for f_k and g_k described for the first time by Morris and collaborators in a series of papers: Dietz, Morris and Slade (2019), Dietz, Morris (2015), Bridle, Dietz and Morris (2014), Labus, Morris and Slade (2016)
- Key strategy: solve flow equations + msWI using a special class of cutoff functions.

Beyond polynomial approximations

- Key results: infinitely many relevant directions
- Is the theory predictive?

Beyond LPA

let us consider

$$S^{EH}[g_{\mu\nu}] = -\frac{1}{16\pi}\int d^dx\sqrt{g}\,G^{-1}\,(R-2\Lambda), \tag{19}$$

which, by Weyl rescaling $g_{\mu\nu} = \phi^{2\nu}\hat{g}_{\mu\nu}$ can be written as

$$S_k^{EH}[\phi] = \int d^dx\,\frac{\sqrt{\hat{g}}\,Z_k}{2}\,\left(\hat{g}^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi + A\,\hat{R}\,\phi^2 - 4A\,\Lambda_k\phi^{\frac{2d}{d-2}}\right), \tag{20}$$

where $\hat{R} \equiv R(\hat{g})$, $A = A(d) = \frac{d-2}{8(d-1)}$ and

$$Z_k = -\frac{1}{2\pi\,G_k}\,\frac{d-1}{d-2} \tag{21}$$

beyond LPA

It makes sense to consider a general action of the type

$$S_k[\phi] = \int d^dx\sqrt{\hat{g}}\,\left(\frac{1}{2}Z_k[\phi]\,\hat{g}^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + V_k[\phi]\right), \qquad (22)$$

so that the LHS of the flow equation reads

$$\partial_t\Gamma_k = \int d^dx\sqrt{\hat{g}}\left(\frac{1}{2}(k\partial_k Z_k)[\chi_B]\bar{f}\hat{\Box}\bar{f} + k\partial_k V_k[\chi_B]\right). \qquad (23)$$

The RHS can instead be expressed as

$$-\frac{1}{2}\int\frac{ds}{s}\int d^dx\sqrt{\bar{g}}\int\frac{d^d\bar{p}}{(2\pi)^d}(k\partial_k\rho_k)\langle x|\bar{p}\rangle\langle\bar{p}|e^{-s\Gamma_k^{(2)}}|x\rangle, \qquad (24)$$

where the trace is performed on the modes of $-\bar{\Box}$.

Beyond LPA

This expression can be expanded through Baker-Campbell-Hausdorff expansion up to quadratic terms in $\bar{f}$, leading to

$$-\frac{1}{2}\int\frac{ds}{s}\int d^d x\,\chi_B^{d\nu}\sqrt{\hat{g}}\,(k\partial_k\rho_k)\int\frac{d^d\bar{p}}{(2\pi)^d}e^{-s(Z_k\chi_B^{2\nu}\bar{p}^{2-\eta\nu}+V_k^{(2)})}\times$$
$$\times\left(1-sB_1+\frac{s^2}{2!}B_2-\frac{s^3}{3!}B_3+\frac{s^4}{4!}B_4\right) \tag{25}$$

where the identity $\sqrt{\bar{g}}=\chi_B^{d\nu}\sqrt{\hat{g}}$ was inserted.

The anomalous dimension

We consider an anomalous scaling for the χ_B and we write

$$\chi_B = \psi\, k^{\frac{\eta}{2}}. \tag{26}$$

being ψ_B a dimensionless field and η the anomalous dimension. While the identity $\hat{\Box} = \chi_B^{2\nu}\,\bar{\Box}$ always holds true, in order for $\hat{\Box}$ to show the correct dimensionality of k^2 ($[\hat{\Box}] = k^2$), it must imply: $[\bar{\Box}] = k^{2-\eta\nu}$. Hence, the relation between momenta built with the reference metric and the background metric must be of the following form:

$$\hat{k}^2 = \chi_B^{2\nu}\bar{k}^{2-\eta\nu}, \tag{27}$$

which for $d = 4$, i.e. $\nu = 1$, simply reduces to $\hat{k}^2 = \chi_B^2\bar{k}^{2-\eta}$.

FRG for V and Z

The dimensional equations (with general dimension d and shaping parameter n) for V_k and Z_k are:

$$\partial_t V_k = n^{\frac{d}{f_0}+n} \psi^{d\nu} k^{-f_0 n} W_1^{-\frac{d}{f_0}-n} Z_k^{\frac{d}{f_0}+n} \chi_B^{2\nu\left(\frac{d}{f_0}+n\right)} \tag{28}$$

and

$$\begin{aligned}
\partial_t Z_k = &-\frac{1}{3f_0^4} q_0\, n^{\frac{d}{f_0}+n}\, Z_k^{\frac{d}{f_0}+n-1} k^{\frac{\eta\nu(d+f_0)}{f_0}+2n} \chi_B^{\frac{2\nu(d+nf_0-4)}{f}} \psi^{d\nu}\, W_2^{-\frac{d+f_0(n+3)}{f_0}} \times \\
&\times \Bigg(\chi_B^{\frac{8\nu}{f_0}} k^{2\eta\nu} \left(V_k''^2 \left(3f_0^3 Z_k Z_k'' + u_0 {Z_k'}^2\right) + r_0 V_k^{(3)} Z((f_0+1) s_0 V_k^{(3)} Z_k - 2t_0 V_k'' Z_k') \right) \\
&\quad - 2n\, Z_k\, k^{l_0}\, \chi_B^{\frac{2l_0\nu}{f_0}} \left(r_0\, t_0\, V_k^{(3)} Z_k Z_k' - V_k'' \left(3f_0^3 Z_k Z_k'' + u_0\, {Z_k'}^2 \right) \right) + \\
&\quad + k^4\, n^2\, Z_k^2\, \chi_B^{\frac{4\eta\nu^2}{f_0}} \left(3f_0^3 Z_k Z_k'' + u_0 {Z_k'}^2 \right) \Bigg) ,
\end{aligned} \tag{29}$$

where the derivatives are taken with respect to the field ϕ.

Scaling

- We now introduce Y_k, z_k, X as the dimensionless counterparts of the potential V_k, the renormalization function Z_k and the field ϕ:

$$V_k[\phi] = Y_k[X] k^d \tag{30}$$

$$Z_k[\phi] = z_k[X] k^{d-2-\eta} \tag{31}$$

$$\phi = X\, k^{\frac{\eta}{2}}, \tag{32}$$

- Following the single metric approximation the expectation value of the dimensionless field X is identified with the the background ψ_B, which, for the sake of simplicity, from now on is indicated as x:

$$X = \psi_B \equiv x\,. \tag{33}$$

Flow equations

$$\partial_t Y_k(x) = -4Y_k + \frac{\eta\, x\, Y_k'}{2} + n^{\frac{4}{\eta-2}+n} x^{\frac{10\eta}{\eta-2}+2n} z_k^{\frac{4}{\eta-2}+n}\, Q_1^{-\frac{4}{\eta-2}-n-3}\, Q_2^3$$

$$\partial_t z_k(x) = (\eta-2)\, z_k + \frac{\eta x z_k'}{2} - \frac{n^{\frac{4}{\eta-2}+n}((\eta-2)n+4)}{3(\eta-2)^4} x^{\frac{4\eta}{\eta-2}+2n} \times$$
$$\times\, z_k^{\frac{4}{\eta-2}+n-1}\, Q_1^{-\frac{4}{\eta-2}-n-3} \Big\{ (\eta-1)(Y_k^{(3)})^2 z_k^2 \Big((\eta-2)(n+1)+4\Big) \times$$
$$\times \Big((\eta-2)(n+2)+4\Big) - \Big(4(\eta-1)+(\eta-2)(3\eta-5)\Big) \times$$
$$\times \Big((\eta-2)(n+1)+4\Big) 2\, Y_k^{(3)} z_k z_k'\, Q_1 + \Big[3(\eta-2)^3 z_k z_k'' +$$
$$+ \Big((\eta-2)^2 + 4(5\eta-9)(\eta-2) + 16(\eta-1)\Big)(z_k')^2\Big]\, Q_1^2 \Big\}$$

$n \to \infty$

In this limit, the two flow equations read:

$$\partial_t Y_k(x) = -4Y_k + \frac{1}{2}\eta x Y_k' + x^4 e^{-\frac{Y_k''}{x^2 z_k}} \qquad (34)$$

and

$$\begin{aligned}
\partial_t z_k(x) = (\eta - 2) z_k + \frac{1}{2}\eta x z_k' - \frac{e^{-\frac{Y_k''}{x^2 z_k}}}{3(\eta-2)^3 x^2 z_k^2} \times \\
\times \Big[(\eta-2)^2(\eta-1)(Y_k^{(3)})^2 + \\
+ 2\left(-3\eta^3 + 13\eta^2 - 20\eta + 12\right) x^2 Y_k^{(3)} z_k' + \\
+ x^4 \left(3(\eta-2)^3 z_k z_k'' + \left(21\eta^2 - 64\eta + 60\right)(z_k')^2 \right) \Big]
\end{aligned} \qquad (35)$$

Fixed points

It is easy to check that the functions

$$z = -\frac{1}{w} \tag{36}$$

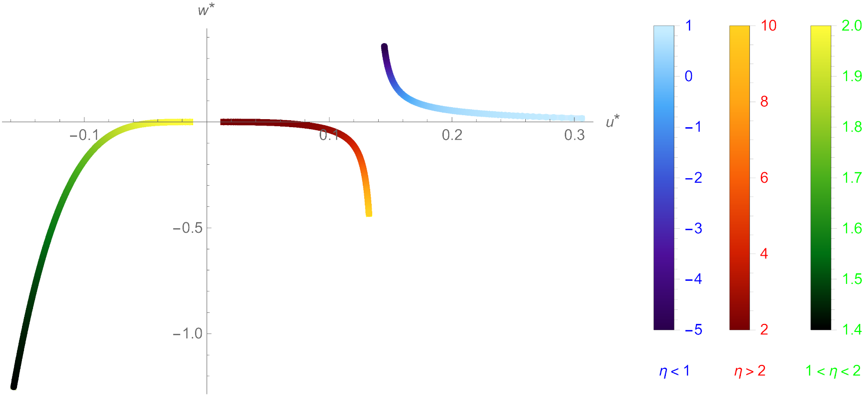
$$Y = \frac{u}{w}x^4 \tag{37}$$

are such that all dependence on x gets canceled both in Eqs. (34) and (35), so that the two differential equations reduce to simple algebraic equation:

$$e^{12\,u} + \frac{2u\,(-2+\eta)}{w} = 0 \tag{38}$$

$$\frac{2-\eta}{w} - \frac{192\,e^{12\,u}\,u^2\,(-1+\eta)}{-2+\eta} = 0 \tag{39}$$

phase diagram



Eigendirections and critical exponents

The stability of the FP is studied by assuming:

$$Y(x,t) = Y^*(x) + \delta\, e^{-t\,\theta} h(x)\,, \tag{40}$$

$$z(x,t) = z^*(x) + \delta\, e^{-t\,\theta} f(x)\,. \tag{41}$$

We consider $f(x) = s\, x^p$ and $h(x) = r\, x^q$, with non-negative integers p and q and with constant s and r.

First order equations

The linearized equations become

$$4 - \frac{\eta q}{2} - e^{12u^*}\, w^* \left(\frac{12\, s\, u^*\, x^{-q+p+4}}{r} + (q-1)\, q \right) = \theta \qquad (42)$$

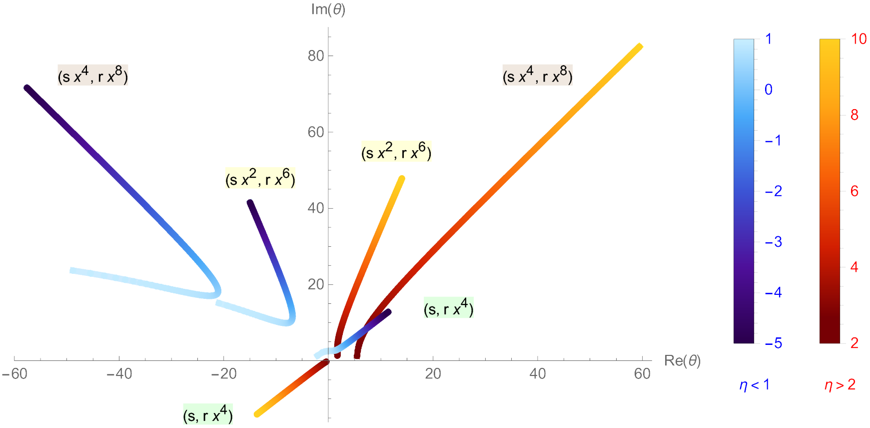
$$\frac{16\,(\eta-1)\,(q-1)\,q\,r\,e^{12u^*}\,u^*\,w^*\,(q+12u^*-2)\,x^{q-p-4}}{(\eta-2)s}$$

$$+ \Big[384(\eta-2)(\eta-1)\,(u^*)^2\,(6u^*+1) + 16(\eta\,(3\eta-7)+6)\,p\,u^*$$

$$+ (\eta-2)^2 p - (\eta-2)^2 p^2 \Big] \frac{e^{12u^*}\, w^*}{(\eta-2)^2} - 2 - \frac{\eta}{2}\,(p+2) = \theta \qquad (43)$$

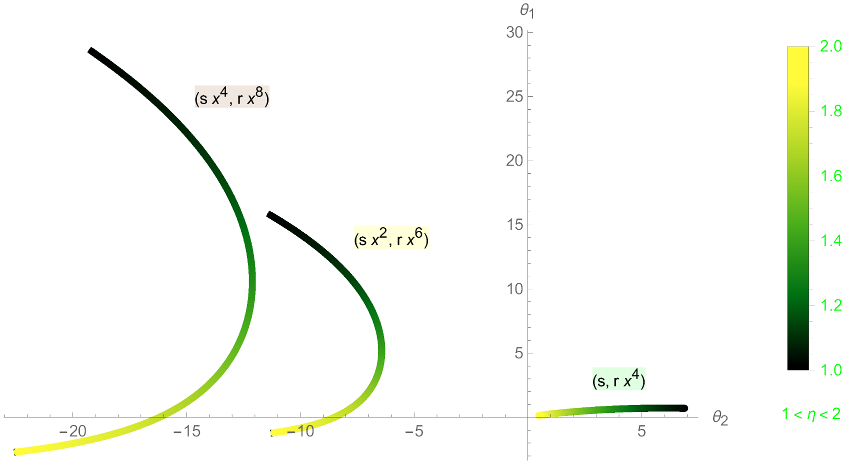
if $q = p + 4$ these two equations become independent of x and we have:

Stability



Im(θ)
Re(θ)
80
60
40
20
−60
−40
−20
20
40
60
(s x^4, r x^8)
(s x^4, r x^8)
(s x^2, r x^6)
(s x^2, r x^6)
(s, r x^4)
(s, r x^4)
1
0
−1
−2
−3
−4
−5
10
8
6
4
2
$\eta < 1$
$\eta > 2$

Stability



θ_1
30
25
20
15
10
5
θ_2
−20
−15
−10
−5
5
$(s\,x^4, r\,x^8)$
$(s\,x^2, r\,x^6)$
$(s, r\,x^4)$
2.0
1.8
1.6
1.4
1.2
1.0
$1 < \eta < 2$

UV critical manifold

- For $-5 < \eta < \eta_c = 0.96$ (the blue strip) the only relevant operator is the couple (s, rx^4), all higher powers are irrelevant
- For $1 < \eta < 2$ the critical exponents are both reals
- For $2 < \eta < 10$ infinitely many relevant operators are generated
- The theory is predictive, despite the existence of a continuous line of fixed points.
- There does exist a phase of unbroken diffeomorphism invariance

Gauge invariance reloaded

- When delving into the realm of gauge transformations, it's customary not to explicitly specify whether they should be interpreted as active transformations, where the system undergoes modification, or passive transformations. This ambiguity arises from the inherent difficulty of distinguishing between active and passive gauge transformations in many cases.
- In the context of General Relativity, a similar conundrum exists with respect to differentiating between diffeomorphisms (active point transformations) and coordinate transformations (passive point transformations). Initially, there is no clear-cut method for making this distinction.

GI

- In the realm of physics, it's common practice to perceive gauge transformations as passive alterations. This perspective leads physicists to consider gauge symmetry as a mere redundancy within the mathematical description of a physical system.
- Conversely, mathematicians tend to view gauge transformations as active changes. From this viewpoint, gauge symmetry is seen as a genuine physical property of the system, not merely a mathematical artifact. This difference in perspective highlights the diverse ways in which physicists and mathematicians approach the concept of gauge symmetry and its role in understanding physical systems.
- This is true as long as one deals only with dynamical fields. **But when one introduces a background field into the game this equivalence does not hold anymore!**

GI

- **Weak Gauge Invariance**: invariance under passive transformation
- **Strong Gauge Invariace**: invariance under Strong and Weak gauge transformation

Measurable quantities, i.e. quantities that can be extracted from experimental data, must be gauge invariant, but WGI or SGI? Cross sections are clearly observables and must be SGI, but we can introduce a class of "pseudo-observables" which can only be observed via a controlled expansion on a BF. (see parton distribution in QCD).
BF as it stands represents somehow a reference configuration. The choice of a particular background is essentially a matter of convenience. In Cosmology this amounts to the choice of a preferred foliation (see also S-W lemma).

Conclusions

- CR $R + R^2$ theory is AS -
- Not clear if this is due to the choice of the background or to the search domain in the parameter space
- The structure of the UV critical mfd is finite dimensional in a controlled BF expansion.
- SGI can be problematic in QG