Operator product expansion and the functional renormalization group

C. Pagani[1]

[1] Johannes Gutenberg-Universität Staudingerweg 7 55128 Mainz.

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Quantum spacetime and the renormalization group 2023

Based on work with H. Sonoda (2001.07015), F. Rose and N. Dupuis (2110.13174).

Outline

Operator product expansion (OPE) and the renormalization group (RG)

Two keys idea in quantum and statistical field theory. Both are inherently non-perturbative:

- the operator product expansions (Wilson 1969);
- the Wilsonian functional renormalization group.

OPE definition

Let O_i denote a composite operator (e.g. $O_i = \phi, \phi^2, \Box\phi^2, \cdots$), the OPE states

$$\left\langle \left(O_i\left(x\right) O_j\left(y\right) = \sum_k C_{ijk}\left(x-y\right) O_k\left(\frac{x+y}{2}\right)\right) \Psi\left(z_1, \cdots\right) \right\rangle$$

in *any* correlation function for $|x-y| \to 0$.

Examples of use of the OPE

- Implements normal ordering for free field theory.

$$\phi(x)\,\phi(0) = G(x) + :\phi^2(0): + \cdots .$$

- QCD and deep inelastic scattering

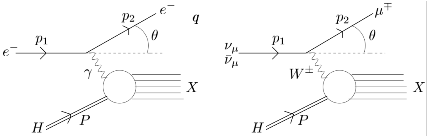


Figure: The cross section is proportional to the OPE of two currents $J^\mu J^\nu$.

- Backbone of CFT. CFT is fully solved when all scaling dimensions Δ_i and the OPE coefficients c_{ijk} are found.
  - $C_{ijk}(x-y) = c_{ijk} \times (x-y)^{\Delta_k - \Delta_i - \Delta_j}$.
  - Example in 2D

$$T(z)\,T(w) = \frac{c}{2(z-w)^4} + \cdots ,$$

where c is the central charge (conformal anomaly).

Perturbation theory and the OPE

- Existence of the OPE in perturbation theory (Zimmermann 1970, Wilson-Zimmermann 1972).
- Existence of the OPE in perturbation theory via FRG (Hughes 1989, Keller and Kopper 1993, ...).
- Under suitable conditions and restrictions, a subset of the OPE coefficients is associated with the second order coupling expansion of the renormalization group (see e.g. Cardy 1996)

$$\check{\beta}^i = y_i \check{g}^i - \sum_{j,k} {c_{jk}}^i \, \check{g}^j \check{g}^k + O\left(\check{g}^3\right) ,$$

where ${c_{jk}}^i$ is the OPE coefficient of suitably normalized operators.

The main purpose of this talk

Describe an approximation scheme that allows to go beyond perturbation theory via the functional renormalization group.

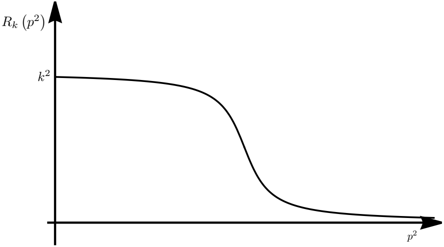
The functional renormalization group (FRG)

We modify the path integral so that the momentum modes below the RG scale k are suppressed:

$$e^{W_k[J]} \equiv \int \mathcal{D}\chi \, \exp\left\{ -S[\chi] - \int d^d x \, \varepsilon_i(x) \, \mathcal{O}_i(x) - \Delta S_k[\chi] + \int J\chi \right\},$$

where

$$\Delta S_k[\chi] \equiv \frac{1}{2} \int_p \chi(-p) \, R_k\left(p^2\right) \chi(p), \text{ with } R_k\left(p^2\right) \approx \begin{cases} k^2 & p^2 \to 0 \\ 0 & p^2 \to \infty \end{cases}.$$



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$\Rightarrow$ A convenient way to consider one or more insertions of composite operators in a correlation function is to introduce a source conjugate to the composite operator of interest.

Example

$$\left\langle \mathcal{O}_i\left(x\right) \right\rangle = -\frac{\delta W_k\left[J, \varepsilon\right]}{\delta \varepsilon_i\left(x\right)}\Big|_{J=0, \varepsilon=0}.$$

Composite operators and the EAA

- One defines the effective average action (EAA) as the Legendre transform of W_k in the usual way.

$$\Gamma_k[\varphi,\varepsilon] + \Delta S_k[\varphi] = \int d^d x\, J(x)\,\varphi(x) - W_k[J,\varepsilon]\ ,$$

- A composite operator $[\mathcal{O}_a](x)$ is defined by

$$[\mathcal{O}_a](x) \equiv \left.\frac{\delta\Gamma_k[\varphi,\varepsilon]}{\delta\varepsilon_a(x)}\right|_{\varepsilon=0}.$$

- $[\mathcal{O}_a](x)$ contains the information to deduce any correlation function with a *single* composite operator insertion. For example, the vacuum expectation value is given by

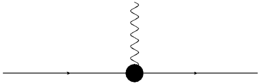
$$\langle\mathcal{O}_a(x)\rangle = \left.\frac{\delta\Gamma_k[\varphi,\varepsilon]}{\delta\varepsilon_a(x)}\right|_{\varepsilon=0,\varphi=0} = \left.[\mathcal{O}_a](x)\right|_{\varphi=0}.$$

Composite operators and the EAA – insertion of single operator

- A composite operator $[\mathcal{O}_a](x)$ generates the 1PI vertices of a composite operator.
- Example, calculation of a three-point function in a $\mathbb{Z}_2$-invariant theory:

$$\left\langle \chi(p_1)\,\chi(p_2)\,\mathcal{O}_a(-p_1-p_2)\right\rangle = G(p_1)\,\frac{\delta\Gamma_k}{\delta\varphi(p_1)\,\delta\varphi(p_2)\,\delta\varepsilon(-p_1-p_2)}\,G(p_2)$$

Diagramatically:



Composite operators and the EAA – insertion of two composite operators

We note that:

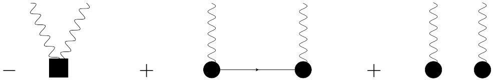
$$\left\langle \mathcal{O}_a(x)\,\mathcal{O}_b(y)\right\rangle = \frac{1}{Z_k[0,0]}\frac{\delta^2 Z_k[J,\varepsilon]}{\delta\varepsilon_a(x)\,\delta\varepsilon_b(y)}\Big|_{J=\varepsilon=0}$$

$$\text{1PI} \qquad = \Big\{ -\frac{\delta^2\Gamma_k[\varphi,\varepsilon]}{\delta\varepsilon_a(x)\,\delta\varepsilon_b(y)}\Big|_{\varepsilon=0}$$

$$\text{conn} \qquad + \int_{z_1,z_2}\frac{\delta^2\Gamma_k[\varphi,\varepsilon]}{\delta\varepsilon_b(x)\,\delta\varphi(z_1)}\Big|_{\varepsilon=0}\cdot G_k(z_1,z_2)\cdot\frac{\delta^2\Gamma_k[\varphi,\varepsilon]}{\delta\varphi(z_2)\,\delta\varepsilon_b(y)}\Big|_{\varepsilon=0}$$

$$\text{disc} \qquad + \frac{\delta\Gamma_k[\varphi,\varepsilon]}{\delta\varepsilon_a(x)}\Big|_{\varepsilon=0}\frac{\delta\Gamma_k[\varphi,\varepsilon]}{\delta\varepsilon_b(y)}\Big|_{\varepsilon=0}\Big\}\Big|_{\varphi=0}.$$

It can be represented diagrammaticaly as



Composite operators and the EAA – insertion of two composite operators

We define the product of two composite operator for generic field configuration as:

$$[\mathcal{O}_a(x)\,\mathcal{O}_b(y)] = -\frac{\delta^2\Gamma_k[\varphi,\varepsilon]}{\delta\varepsilon_a(x)\,\delta\varepsilon_b(y)}\Big|_{\varepsilon=0} + \frac{\delta\Gamma_k[\varphi,\varepsilon]}{\delta\varepsilon_a(x)}\Big|_{\varepsilon=0}\frac{\delta\Gamma_k[\varphi,\varepsilon]}{\delta\varepsilon_b(y)}\Big|_{\varepsilon=0}$$
$$+\int_{z_1,z_2}\frac{\delta^2\Gamma_k[\varphi,\varepsilon]}{\delta\varepsilon_b(x)\,\delta\varphi(z_1)}\Big|_{\varepsilon=0}\cdot G_k(z_1,z_2)\cdot\frac{\delta^2\Gamma_k[\varphi,\varepsilon]}{\delta\varphi(z_2)\,\delta\varepsilon_b(y)}\Big|_{\varepsilon=0}.$$

The OPE amounts to the statement that

$$\Big[\mathcal{O}_a(x)\,\mathcal{O}_b(y)\Big] = c_{abc}(x-y)\Big[\mathcal{O}_c\left(\frac{x+y}{2}\right)\Big]$$

in the short distance limit.

- It involves infinitely many operators, hard to compute in general!
- We usually work in momentum space.

OPE in momentum space

- Let us consider

$$\int_{x,y} e^{-ip_1 x - ip_2 y} c_{ab,c}(x-y)\, O_c\left(\frac{x+y}{2}\right) = c_{ab,c}\left(\frac{p_1-p_2}{2}\right) O_c(p_1+p_2)\ .$$

- $\Rightarrow$ we trade the short distance limit with the large momentum limit: $\frac{p_1-p_2}{2} \equiv P \to \infty$ at fixed $p_1+p_2 \equiv Q$.
- Caveat: one gets only singular OPE terms, example

$$1 \leftrightarrow \delta(p)$$

Application to CFT and the three-point function

Constraints from conformal symmetry:

- Two point functions:

$$\left\langle \mathcal{O}_i(x)\, \mathcal{O}_j(x) \right\rangle \propto \frac{\delta_{ij}}{|x|^{2\Delta_i}}\,.$$

  We can normalize the operator so that

$$\left\langle \mathcal{O}_i(x)\, \mathcal{O}_j(x) \right\rangle = \frac{\delta_{ij}}{|x|^{2\Delta_i}}\,.$$

- The three point function is must take the form

$$\left\langle \mathcal{O}_a(x_1)\, \mathcal{O}_b(x_2)\, \mathcal{O}_c(x_3) \right\rangle = \frac{c_{abc}}{\left(x_{12}^2\right)^{d/2-\nu_c} \left(x_{23}^2\right)^{d/2-\nu_a} \left(x_{12}^2\right)^{d/2-\nu_b}}\,,$$

  where $x_{ij}^2 = |x_i - x_j|^2$, c_{abc} is fully symmetric in its indices, and $\nu_a = \frac{1}{2}\left(d + \Delta_a - \Delta_b - \Delta_c\right)$, etc.

Three-point function in a specific momentum configuration

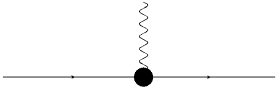
In the limit $p_1 \gg p_2$ the Fourier transform of a conformal three-point function is given by

$$\left\langle \mathcal{O}_a(p_1)\,\mathcal{O}_b(p_2)\,\mathcal{O}_c(-p_1-p_2)\right\rangle \xrightarrow{p_1 \gg p_2} \frac{(4\pi)^d}{4^{\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3)}} \frac{\Gamma\left(\frac{1}{2}(d+\Delta_2-\Delta_1-\Delta_3)\right)}{\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)\right)} \times \frac{\Gamma\left(\frac{d}{2}-\Delta_2\right)}{\Gamma(\Delta_2)} \frac{1}{p_1^{d+\Delta_2-\Delta_1-\Delta_3}} \frac{c_{abc}}{p_2^{d-2\Delta_2}} \,.$$

We will focus on the following three-point function

$$\left\langle \phi_1(p_1)\,\phi_1(p_2)\,\phi_2(-p_1-p_2)\right\rangle ,$$

where $\phi_1 \sim \varphi$ and $\phi_2 \sim \varphi^2$ (scaling operators properly normalized).



Three-point function in a specific momentum configuration

- The three-point function in momentum space is constrained by conformal invariance but it takes a complicated form.
- We limit ourserselves to simple momentum configuration, i.e., $p_2 \gg p_1$ and try to make contact with the OPE

$$\left\langle \mathcal{O}_a(p_1)\,\mathcal{O}_b(p_2)\,\mathcal{O}_c(-p_1-p_2)\right\rangle =$$

$$\sum_k \left\langle \mathcal{O}_a(p_1)\, C_{bck}\left(\frac{p_2-(-p_1-p_2)}{2}\right)\mathcal{O}_k(p_2-p_1-p_2)\right\rangle \overset{p_2\gg p_1}{=}$$

by CFT symm. $$c_{abc} \propto \sum_k \underbrace{C_{bck}(p_2)}_{\propto c_{bck}/p_2^{\alpha}} \underbrace{\left\langle \mathcal{O}_a(p_1)\,\mathcal{O}_k(-p_1)\right\rangle}_{\propto \delta_{ak}/p_1^{\beta}}$$

Purpose of the talk

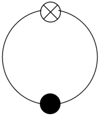
Show that c_{abc} can be computed in a non-perturbative approximation via FRG.

Flow equation for composite operators

The flow equation for a composite operator takes the following form:

$$\partial_t \left(\frac{\delta \Gamma_k}{\delta \varepsilon} = [\mathcal{O}_k] \right) = -\frac{1}{2} \mathrm{Tr} \left[\left(\Gamma_k^{(2)} + R_k \right)^{-1} \cdot [\mathcal{O}_k]^{(2)} \cdot \left(\Gamma_k^{(2)} + R_k \right)^{-1} \partial_t R_k \right]$$

Diagrammatically, this flow equation can be displayed as below:

$$\partial_t \left[O_k \right] = -\tfrac{1}{2}$$ 

The equation requires two independent approximations:

- An approximation for the EAA Γ_k itself.
- An approximation for the composite operator $[\mathcal{O}_k] = \frac{\delta \Gamma_k}{\delta \varepsilon} [\varphi]$ itself.

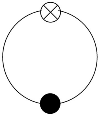
OPE and the ϵ-expansion – sketch

Let's start by considering the Gaussian approximation.

$$\left[\frac{\varphi^2}{2}\right] = \frac{\varphi^2}{2} + c_0^{(0)}\,,$$

where $c_0^{(0)}$ is determined by

$$\partial_t \left[\frac{\varphi^2}{2}\right] = \int_q G\left(q\right)^2 \partial_t R_k\,.$$

$$\partial_t \left[O_k\right] = -\tfrac{1}{2}$$ 

OPE and the ϵ-expansion – sketch

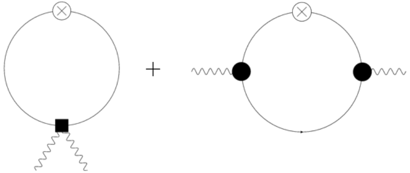
It is crucial to determine the normalization $\mathcal{N}_2$:

$$\mathcal{N}_2^2 \left\langle \frac{\varphi^2}{2}(p)\,\frac{\varphi^2}{2}(-p) \right\rangle \overset{!}{=} \int_x e^{ipx} \frac{1}{x^{2\Delta_2}}\,.$$

For a $\mathbb{Z}_2$-invariant theory one has where $c_0^{(0)}$ is determined by

$$\left\langle \frac{\varphi^2}{2}(p)\,\frac{\varphi^2}{2}(-p) \right\rangle = -\frac{\delta^2 \Gamma_k}{\delta\varepsilon_2(p)\,\delta\varepsilon_2(-p)}\,.$$

$$\partial_t \left(\frac{\delta^2 \Gamma}{\delta\varepsilon_2(x)\delta\varepsilon_2(0)} \right) = -\frac{1}{2}$$



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For a $\mathbb{Z}_2$-invariant theory one has where $c_0^{(0)}$ is determined by

$$\begin{aligned} \left\langle \frac{\varphi^2}{2}(p)\, \frac{\varphi^2}{2}(-p) \right\rangle &= -\frac{\delta^2 \Gamma_k}{\delta\varepsilon_2(p)\, \delta\varepsilon_2(-p)} \\ &= \frac{1}{2} \int_q G(q)\, G(q+p) \, . \end{aligned}$$

By inserting the above result into the defining equation for $\mathcal{N}_2$ one can compute such normalization factor.

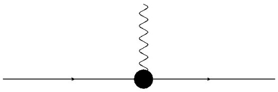
OPE and the ϵ-expansion – sketch

We go back to the three-point function:

$$\left\langle \phi_1(p_1)\,\phi_1(p_2)\,\phi_2(-p_1-p_2)\right\rangle \overset{p_1\gg p_2}{\longrightarrow} \frac{(4\pi)^d}{4^{\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3)}}\frac{\Gamma\left(\frac{1}{2}(d+\Delta_2-\Delta_1-\Delta_3)\right)}{\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)\right)} \times \frac{\Gamma\left(\frac{d}{2}-\Delta_2\right)}{\Gamma(\Delta_2)}\frac{1}{p_1^{d+\Delta_2-\Delta_1-\Delta_3}}\frac{c_{112}}{p_2^{d-2\Delta_2}}$$

and compare to

$$\mathcal{N}_1^2\mathcal{N}_2\left\langle \varphi(p_1)\,\varphi(p_2)\left[\frac{\varphi^2}{2}\right](-p_1-p_2)\right\rangle \overset{p_1\gg p_2}{\longrightarrow} \mathcal{N}_1^2\mathcal{N}_2\frac{1}{p_1^2}\left(\Gamma^{(2,1)}=1\right)\frac{1}{p_2^2}$$



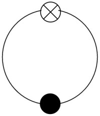
$\Rightarrow c_{112} = \sqrt{2}.$

Composite operator – $O(\epsilon)$

We consider the flow equation for the composite operators.

$$\partial_t \left(\varepsilon \cdot \left[\frac{\varphi^2}{2} \right] \right) = -\frac{1}{2} \mathrm{Tr} \left[\left(\Gamma_k^{(2)} + R_k \right)^{-1} \left(\varepsilon \cdot \left[\frac{\varphi^2}{2} \right]^{(2)} \right) \left(\Gamma_k^{(2)} + R_k \right)^{-1} \partial_t R_k \right]$$

$$\partial_t \left[O_k \right] = -\frac{1}{2}$$



We introduce the anomalous dimension $\gamma_2 = Z_2^{-1} \partial_t Z_2$, where Z_2 is defined by

$$Z_2 \equiv \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \left[\frac{\varphi^2}{2} \right] \Big|_{\varphi=0} (p = 0) \,,$$

where $\phi = \sqrt{Z}\varphi$. The scaling dimension is then given by $\Delta_2 = d - 2 + \gamma_2$.

Scaling composite operator – $O(\epsilon)$

Let us solve the dimensionless, fixed point (and diagonalized) flow equation for the *scaling* composite operator equation. Symbolically:

$$\Delta_2 \left[\frac{\varphi^2}{2}\right] - \Delta_\partial \left[\frac{\varphi^2}{2}\right] - \frac{d-2+\eta}{2}\Delta_\varphi \left[\frac{\varphi^2}{2}\right] = -\frac{1}{2} G \cdot \left[\frac{\varphi^2}{2}\right]^{(2)} \cdot G \cdot \dot{R} \,,$$

where $\Delta_\partial = \sum_i p_i \frac{d}{dp_i}$ counts the derivatives appearing in a vertex and $\Delta_\partial = \varphi \frac{\delta}{\delta\varphi}$ counts the number of fields.

- The $O\left(g^0\right)$ is solved by the Gaussian result.
- We consider an ansatz:

$$\left[\frac{\varphi^2}{2}\right](p) = \int_{q_1,q_2} \left(1 + g\gamma_1^{(2)}(p_1, p_2)\right) \frac{\varphi_{p_1}\varphi_{p_2}}{2} \delta(p_1 + p_2 - p) + (a_0 + g a_1) \,.$$

Scaling composite operator – $O(\epsilon)$

Flow equation for fixed point scaling operator

$$\Delta_2\left[\frac{\varphi^2}{2}\right]-\Delta_\partial\left[\frac{\varphi^2}{2}\right]-\frac{d-2+\eta}{2}\Delta_\varphi\left[\frac{\varphi^2}{2}\right]=-\frac{1}{2}G\cdot\left[\frac{\varphi^2}{2}\right]^{(2)}\cdot G\cdot\dot{R}\,,$$

By taking $\Delta_2=d-2+\gamma_2=d-2+g\gamma_{2,1}+\cdots$ and two functional derivatives one finds:

$$\left(\sum_{i=1,2}p_i\cdot\partial_{p_i}\right)\gamma_1^{(2)}(p_1,p_2)=\gamma_{2,1}-\int_q G(q)^2G(q+p)\dot{R}\left(q\right)\,,$$

At $p_i=0$ we find

$$\gamma_{2,1}\;=\;\int_q G(q)^3\dot{R}\left(q\right)$$

$$\Delta_2\;\approx\;d-2+g\gamma_{2,1}=2-\frac{2}{3}\epsilon\,.$$

Scaling composite operator – $O(\epsilon)$

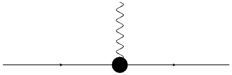
With $p = p_1 + p_2$, the flow equation for fixed point scaling operator reads

$$\begin{aligned} p\frac{d}{dp}\gamma_1^{(2)}(p) &= \gamma_{2,1} - \int_q G(q)^2 G(q+p)\dot{R}(q) \\ &= -\int_q G(q)^2 \left[G(q+p) - G(q)\right]\dot{R}(q)\ , \end{aligned}$$

The solution is

$$\begin{aligned} \gamma_1^{(2)}(p) &= -\frac{1}{2}\int_q G(q)\left[G(q+p) - G(q)\right] \\ &\overset{\text{large p}}{=} \frac{1}{2(4\pi)^2}\log p^2 + \cdots + \text{cutoff dependent terms}\,, \end{aligned}$$

which enters in



OPE and the ϵ-expansion – sketch

- A similar computation allows one to compute $\langle \varphi^2/2 \varphi^2/2 \rangle$. As a by product one can calculate $\mathcal{N}_2$ at $O(\epsilon)$.
- The normalization constant $\mathcal{N}_2$ is scheme, i.e., cutoff, dependent.

$$\left\langle \phi_1(p_1)\, \phi_1(p_2)\, \phi_2(-p_1-p_2) \right\rangle \xrightarrow{p_1 \gg p_2} \frac{(4\pi)^d}{4^{\frac{1}{2}(\Delta_1+\Delta_2+\Delta_3)}} \frac{\Gamma\left(\frac{1}{2}(d+\Delta_2-\Delta_1-\Delta_3)\right)}{\Gamma\left(\frac{1}{2}(\Delta_1+\Delta_3-\Delta_2)\right)} \times \frac{\Gamma\left(\frac{d}{2}-\Delta_2\right)}{\Gamma(\Delta_2)} \frac{1}{p_1^{d+\Delta_2-\Delta_1-\Delta_3}} \frac{c_{112}}{p_2^{d-2\Delta_2}}$$

and compare to

$$\mathcal{N}_1^2 \mathcal{N}_2 \left\langle \varphi_{p_1} \varphi_{p_2} \left[\frac{\varphi^2}{2}\right]_{-p_1-p_2} \right\rangle \xrightarrow{p_1 \gg p_2} \mathcal{N}_1^2 \mathcal{N}_2 \frac{1}{p_1^2} \left(\Gamma^{(2,1)} = 1 + \epsilon \frac{\log p^2}{6} + \cdots \right) \frac{1}{p_2^2}$$

$\Rightarrow c_{112} = \sqrt{2}\left(1 - \frac{\epsilon}{6}\right)$ independently from the cutoff kernel R_k.

OPE and FRG beyond perturbation theory

We note:

- The recipe requires the *momentum dependence* of the EAA vertices.
- The BMW scheme offers such an approximation.

Basic idea behind BMW approximation.

- The loop momentum q is regulated by the cutoff derivative $\dot{R}$.
- For large $p \gg k$ one approximates, e.g.,

$$\frac{\delta^3 \Gamma_k [\bar{\varphi}]}{\delta\varphi(q)\,\delta\varphi(p)\,\delta\varphi(-q-p)} \approx \frac{d}{d\bar{\varphi}} \frac{\delta^2 \Gamma_k [\bar{\varphi}]}{\delta\varphi(p)\,\delta\varphi(-p)}\,,$$

  with $\bar{\varphi}$ a constant field configuration.
- Similar approximation for the four-point vertices.
- $\Rightarrow$ closed flow equation for $\Gamma^{(2)}(p, \bar{\varphi})$.

OPE and FRG beyond perturbation theory

- The main vertex to construct the three-point function is

$$\frac{\delta^3\Gamma_k}{\delta\varphi(p_1)\,\delta\varphi(p_2)\,\delta\varepsilon(-p_1-p_2)} \overset{p_1\gg p_2}{\approx} \frac{d}{d\bar{\varphi}}\frac{\delta^2\Gamma_k}{\delta\varphi(p_1)\,\delta\varepsilon(-p_1)}$$

- Closure in terms of the following vertices

$$\frac{\delta^2\Gamma_k}{\delta\varphi(p)\,\delta\varphi(-p)}(\bar{\varphi}),\ \frac{\delta^2\Gamma_k}{\delta\varphi(p)\,\delta\varepsilon(-p)}(\bar{\varphi}),\ \frac{\delta^2\Gamma_k}{\delta\varepsilon(p)\,\delta\varepsilon(-p_1)}(\bar{\varphi})$$

- The three-point vertex in the limit $p_1 \gg p_2$ is approximated as

$$\mathcal{N}_1^2\mathcal{N}_2\left\langle\varphi_{p_1}\varphi_{p_2}\left[\frac{\varphi^2}{2}\right]_{-p_1-p_2}\right\rangle \overset{p_1\gg p_2}{\longrightarrow} \frac{\mathcal{N}_1^2\mathcal{N}_2}{p_1^{2-\eta}}\left(\Gamma^{(2,1)}\approx\frac{d}{d\bar{\varphi}}\Gamma^{(1,1)}(p_1)\right)\frac{1}{p_2^{2-\eta}}$$

- Within this approximation schemes we are able to estimate the following *CFT data*: Δ_1 , Δ_2 , c_{112}.

OPE and FRG beyond perturbation theory – Ising model

d	2	3	4
FRG	0.484	1.039	1.413 , $\sqrt{2}$
CB	0.5	1.0518537(41)	$\sqrt{2}$
ϵ exp. $(3,1)$	0.4259	1.0432	$\sqrt{2}$
ϵ exp. $(2,2)$	-0.0698	1.0200	$\sqrt{2}$
ϵ exp. $(1,3)$	0.7442	1.0805	$\sqrt{2}$
ϵ exp. $+ 2d$ (3,2)		1.0507	$\sqrt{2}$
ϵ exp. $+ 2d$ (2,3)		1.0464	$\sqrt{2}$

Table: The ϵ-expansion results have been obtained via the so-called "analytic bootstrap" approach (Gopakumar, Kaviraj, Sen, and Sinha). In the table we present possible resummations of the $O(\epsilon^4)$-results via different Padé approximants.

Estimation of the uncertainty

The (exact) result does not depend on the RG scheme used. However, when approximations are made some dependence remains. This latter can be used to have an estimate of the uncertainty of the results.

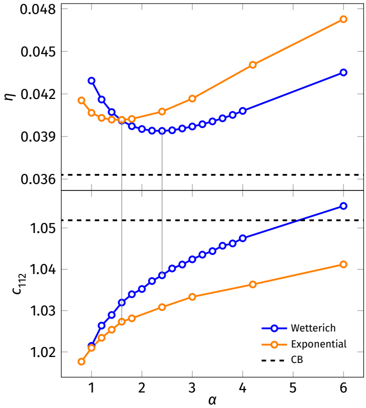


Figure: OPE coefficient c_{112} as a function of the parameter α introduced in the RG cutoff scheme.

Dependence on d

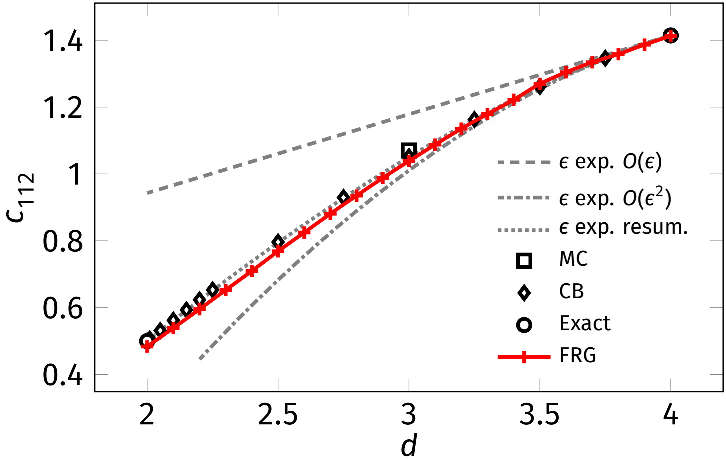


Figure: OPE coefficient across d for the Ising universality class.

The OPE coefficient c_{112} for the $O(N)$ model

N	FRG	ϵ-exp.	MC	CB
1	1.039	1.075	1.07(3)	1.0518537(41)
2	0.943	1.002	0.9731(14)	0.971743(38)
3	0.884	0.943		0.908047(102)
4	0.829	0.895		
5	0.792	0.855		
6	0.766	0.823		
8	0.731	0.773		
10	0.709	0.737		
100	0.643	0.607		
1000	0.638	0.603		

Table: Rescaled OPE coefficient $\sqrt{N}c_{112}$ of the three-dimensional O(N) model. We compare the FRG results to the (2,1) resummation of the ϵ expansion to order $O(\epsilon^3)$, CB and MC estimates.

The exact large-N result is $\lim_{N\to\infty}\sqrt{N}c_{112} = 2/\pi \simeq 0.637$.

The OPE coefficient c_{112} for the $O(N)$ model

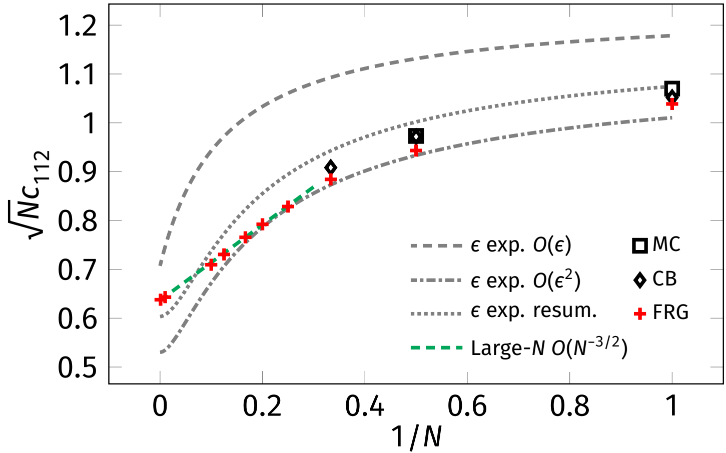


Figure: OPE coefficient across N for the $O(N)$ universality class. Large N calculation by Lang and W. Rühl (1992,1994).

Summary

- We show how to compute OPE coefficients by introducing sources conjugate to the composite operators of interests and by computing a suitable three-point function.
- We retrieved the results known from ϵ-expansion.
- We computed the OPE coefficient c_{112} for the $O(N)$ universality class by employing a non-perturbative approximation scheme and obtained quantitatively corrected results.
- Several possible extensions: inclusion of new operators, extension to curved space, etc.

THANK YOU!!