Towards understanding the predictivety of asymptotically safe $f(R)$-gravity

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The asymptotic safety scenario

- Goal: predictive quantum field theory of gravity
- The Asymptotic Safety (AS) hypothesis: high-energy completion of gravity is provided by an interacting RG fixed point

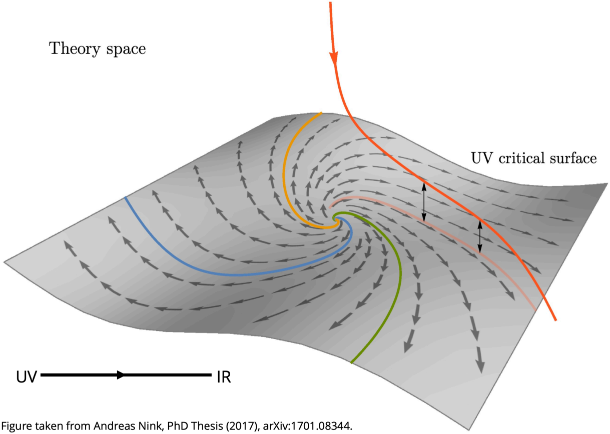

Figure taken from Andreas Nink, PhD Thesis (2017), arXiv:1701.08344.

The AS scenario implies non-trivial quantum corrections to the scaling dimensions of operators (and correlation functions) that are characteristic for the corresponding universality class.

Given an interacting UV fixed point has been identified,

1. How many relevant parameters does the theory have?
2. How do we construct meaningful observables?

Both questions can be probed:

- via the Wetterich equation, a Functional Renormalization Group Equation (FRGE)
- via a composite-operator FRGE

The role of scaling dimensions in the AS approach

- Consider the following scaling argument (cf. Codello, d'Odorico '15):

$$G_{12}(r) \equiv \frac{1}{Z}\int \mathcal{D}g\, e^{-S} \int_{x,y} \frac{1}{\mathrm{Vol}} \sqrt{g_x} O_1(x) \sqrt{g_y} O_2(y)\, \delta\left(d_g(x,y) - r\right)$$

$$\Rightarrow G_{12}(\lambda r) = \lambda^{\frac{\Delta_1^g + \Delta_2^g - \Delta_{\mathrm{Vol}}^g}{\Delta_{d_g}^g} - 1} G_{12}(r)$$

- We need to compute the UV scaling properties of the geometric operators
- The Functional Renomalization Group (FRG) offers two avenues for their computation: the Wetterich equation (for (quasi-)local operators) and via composite operators
- Analogy: scaling dimensions from the KPZ equation

The Asymptotic Safety mechanism

The Wetterich equation:

$$k\partial_k \Gamma_k = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathscr{R}_k\right)^{-1} k\partial_k \mathscr{R}_k\right]$$

- Typically solved with a truncation ansatz of the form $\Gamma_k = \sum_i \bar{u}_i(k) M_i$
- The RG equations take the form $k\partial_k u_i(k) = \beta_i(u(k))$

The UV fixed point

- A UV fixed point u^* is given by $\beta_i(u^*) = 0$
- Solution for the linearized theory: $u_i(k) = u_i^* + \sum_I c_I V_i^I (k_0/k)^{\theta_I}$
- The universal critical exponents θ_I are the eigenvalues of the stability matrix B, given by

$$\sum_j B_{ij} V_j^I = -\theta_I V_I \quad \text{and} \quad B_{ij} = \frac{\partial}{\partial u_j}\beta_i \bigg|_{u=u^*}$$

- The relevant (attractive) directions are those with $\mathrm{Re}\,\theta_I > 0$

The composite-operator FRGE (Cf. Pawlowski '07; Igarashi, Itoh, Sonoda '10; Pagani '16):

$$\partial_t \left[\mathscr{O}_k\right]_i = -\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \left[\mathscr{O}_k\right]_i^{(2)} \left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_t R_k\right]$$

- A set of composite (geometric) operators $[\mathscr{O}_k]_1, \ldots, [\mathscr{O}_k]_n$ can be incorporated into the FRG framework via the simple substitution $\Gamma_k \to \Gamma_k + \sum_i \varepsilon_i \cdot [\mathscr{O}_k]_i$
- Expand the renormalized composite operators in terms of the basis of bare composite operators, $[\mathscr{O}_k]_i[g,\bar{g}] = \sum_j Z_{ij}(k)\, \mathscr{O}_j[g,\bar{g}]$

Then,

$$\sum_{j=1}^{n} \bar{\gamma}_{ij}(k)\mathscr{O}_j = -\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \mathscr{O}_i^{(2)} \left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_t R_k\right]$$

- Anomalous dimension matrix $\bar{\gamma}_{ij}(k) = \sum_l (Z^{-1})_{il}(k)\partial_t Z_{lj}(k)$
- Renormalization behavior: The full scaling dimensions of the family of operators $\{[\mathscr{O}_k]_i\}$ are given by the eigenvalues of $-D_{ij} + \gamma_{ij}$

Flow equation for composite operators

Two approximations are required to solve the composite-operator flow equation,

$$\sum_{j=1}^{n} \bar{\gamma}_{ij}(k)\mathscr{O}_j = -\frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + R_k\right)^{-1} \mathscr{O}_i^{(2)} \left(\Gamma_k^{(2)} + R_k\right)^{-1} \partial_t R_k\right]$$

- Solution strategy:
  - The first truncation is the usual one for the EAA:

$$\Gamma_k = \sum_i \bar{u}_i(k) M_i$$

  - The second truncation is the one for the basis of composite operators:

$$[\mathscr{O}_k]_i = \sum_j Z_{ij}(k)\, \mathscr{O}_j$$

- In general, the size of the anomalous dimension matrix depends on the second, while its arguments depend on the first truncation:

$$\gamma_{ij}(k) \equiv \gamma_{ij}(\bar{u}(k))$$

A comparison of the two approaches in the $f(R)$-truncation

The Wetterich equation:

$$k\partial_k\Gamma_k = \frac{1}{2}\,\mathrm{STr}\left[\left(\Gamma_k^{(2)} + \mathscr{R}_k\right)^{-1} k\partial_k\mathscr{R}_k\right]$$

with $\Gamma_k = \Gamma_k[h_{\mu\nu}, \text{ghosts}; \bar{g}_{\mu\nu}]$. Standard ansatz:

$$\Gamma_k = \Gamma_k[h_{\mu\nu}, \text{ghosts}; \bar{g}_{\mu\nu}] = \bar{\Gamma}_k[g] + \underbrace{\hat{\Gamma}[g, \bar{g}]}_{=0} + S_{\mathrm{gf}}[g - \bar{g}; \bar{g}] + S_{\mathrm{gh}}[g - \bar{g}, \text{ghosts}; \bar{g}]\,.$$

York decomposition, and rescale the field to be orthonormal on Einstein spaces:

$$h_{\mu\nu} \to (h_{\mu\nu}^{\mathrm{TT}}, \hat{\xi}_\mu, \hat{\sigma}, \hat{h})$$

With $S_{\text{gf}} = \frac{1}{2\alpha}\int \mathrm{d}^d x\sqrt{\bar{g}}\bar{g}^{\mu\nu}F_\mu F_\nu$ on has in the limit $\alpha \to 0$ (cf. Machado, Saueressig '15, Codello et al. '08):

$$\partial_t \bar{\Gamma}_k = \frac{1}{2}\mathrm{Tr}_{\mathrm{TT}}\left[\left(\bar{\Gamma}_{k\,\mathrm{TT}}^{(2)} + \mathscr{R}_{k\,\mathrm{TT}}\right)^{-1}\partial_t \mathscr{R}_{k\,\mathrm{TT}}\right] + \frac{1}{2}\mathrm{Tr}_{\hat{h}\hat{h}}\left[\left(\bar{\Gamma}_{k\,\hat{h}\hat{h}}^{(2)} + \mathscr{R}_{k\,\hat{h}\hat{h}}\right)^{-1}\partial_t \mathscr{R}_{k\,\hat{h}\hat{h}}\right] + \Gamma_k\text{-independent terms from ghost terms/Jacobians}$$

Ansatz:

$$\bar{\Gamma}_k = \int \mathrm{d}^d x\sqrt{g}\, f_k(R) = \int \mathrm{d}^d x\sqrt{g}\, k^d \mathcal{F}_k(\rho)$$

with the dimensionless function/radius

$$\mathcal{F}_k(\rho) = k^{-d} f_k(R) \quad , \quad \rho = k^{-2} R\,.$$

Then, the LHS of the Wetterich equation has the structure

$$\mathrm{LHS} = \partial_t \bar{\Gamma}_k = \tilde{V}\left\{(\partial_t \mathcal{F}_k)(\rho) + d\mathcal{F}_k(\rho) - 2\rho \mathcal{F}_k'(\rho)\right\} \ ,$$

with $\int \mathrm{d}^d x \sqrt{\bar{g}} = V$ and $\tilde{V} = V k^{+d} \propto \rho^{-d/2}$. Since $\mathscr{R}_k$ in generell depends affinely on f_k' and f_k'', the RHS has the structure

$$\mathrm{RHS} = \tilde{V}\left\{ I_0[\mathcal{F}_k](\rho) + (k^{-d+2}\partial_t f_k') I_1[\mathcal{F}_k](\rho) + (k^{-d+4}\partial_t f_k'') I_2[\mathcal{F}_k](\rho) \right\} \ ,$$

with

$$\begin{aligned} I_0[\mathcal{F}_k](\rho) &= I_0^{\mathrm{ind}}(\rho) + I_0^{\mathrm{TT}}[\mathcal{F}_k](\rho) + I_0^{\hat{h}\hat{h}}[\mathcal{F}_k](\rho) \\ I_1[\mathcal{F}_k](\rho) &= I_1^{\mathrm{TT}}[\mathcal{F}_k](\rho) + I_1^{\hat{h}\hat{h}}[\mathcal{F}_k] \\ I_2[\mathcal{F}_k] &= I_2^{\hat{h}\hat{h}}[\mathcal{F}_k] \,. \end{aligned}$$

Next, the further ansatz is to expand $\mathcal{F}_k$ in ρ, i.e., $\mathcal{F}_k(\rho) = \sum_{i=0} u_i(k)\rho^i$.
Then the LHS becomes

$$\mathrm{LHS}/\tilde{V} = \sum_{i=0} \{\partial_t u_i + (d-2i)u_i\}\, \rho^i \,,$$

while the ingredients of the RHS become

$$I_0[\mathcal{F}_k](\rho) = \sum_{i=0} \omega_i(u)\rho^i$$

$$(k^{-d+2}\partial_t f_k')I_1[\mathcal{F}_k](\rho) = \sum_{i,j=0} [(d-2j+\partial_t)u_j]\, \tilde{\omega}_{ji}(u)\rho^i$$

$$(k^{-d+4}\partial_t f_k'')I_2[\mathcal{F}_k] = \sum_{i,j=0} [(d-2j+\partial_t)u_j]\, \check{\omega}_{ji}(u)\rho^i \,.$$

Note: $\tilde{\omega}_{0i} = \check{\omega}_{0i} = \check{\omega}_{1i} \equiv 0$.

With $\bar{\omega} = \tilde{\omega} + \check{\omega}$ and $c_i(u) := -(d-2i)u_i$, the RHS is thus

$$\mathrm{RHS}/\tilde{V} = \sum_{i=0} \left\{ \omega_i(u) - \sum_{j=0} c_j(u)\bar{\omega}_{ji}(u) \right\} \rho^i + \sum_{i,j=0} (\partial_t u_j)\bar{\omega}_{ji}(u)\rho^i \,.$$

Setting LHS=RHS, we have in matrix notation

$$\boxed{\partial_t u - c(u) = \omega(u) - c(u)\bar{\omega}(u) + (\partial_t u)\bar{\omega}(u)}\,.$$

Solving for $\partial_t u$, we arrive at the flow equation (cf. Falls et al. '14)

$$\boxed{\partial_t u = \beta_u = c(u) + \omega(u)(1-\bar{\omega}(u))^{-1} = \text{classical } \beta + \text{quantum corrections}}\,.$$

The fixed points are given by

$$\beta_u = 0 \quad \Leftrightarrow \quad \omega(\mathbb{1} - \bar{\omega})^{-1} = -c \quad (\text{cf. } \eta = D)\,.$$

The stability matrix is given by

$$\boxed{B = \partial\beta\big|_{\mathrm{FP}} = (\partial\omega - c\partial\bar{\omega})(\mathbb{1}-\bar{\omega}){-}1\big|_{u=u^*} + \partial c}\,,$$

where $\partial c = -D \overset{d=4}{=} \mathrm{diag}(-4, -2, 0, +2, \dots)$.

Relationship with the composite operator formalism

Take the u-derivative of the RHS of the Wetterich equation:

$$\frac{\partial}{\partial u_i}\mathrm{RHS} = \sum_{l=\mathrm{TT},\hat{h}\hat{h}} \frac{1}{2}\mathrm{Tr}_l\left[\frac{\partial}{\partial u_i}\frac{\partial_t \mathscr{R}_{k\,l}}{\left(\bar{\Gamma}^{(2)}_{k}{}_l + \mathscr{R}_{k\,l}\right)}\right]$$

$$= \sum_l \Bigg\{ -\frac{1}{2}\mathrm{Tr}_l\left[\left(\bar{\Gamma}^{(2)}_{k}{}_l + \mathscr{R}_{k\,l}\right)^{-1}\left(\frac{\partial}{\partial u_i}\underbrace{\bar{\Gamma}^{(2)}_{k}{}_l}_{=\mathscr{O}^{(2)}_{k}{}_l=\left(\int \mathrm{d}^d x\sqrt{g}\,k^d \rho^i\right)^{(2)}}\right)\left(\bar{\Gamma}^{(2)}_{k}{}_l + \mathscr{R}_{k\,l}\right)^{-1}\right]$$

$$-\frac{1}{2}\mathrm{Tr}_l\left[\left(\bar{\Gamma}^{(2)}_{k}{}_l + \mathscr{R}_{k\,l}\right)^{-1}\left(\frac{\partial}{\partial u_i}\mathscr{R}_{k\,l}\right)\left(\bar{\Gamma}^{(2)}_{k}{}_l + \mathscr{R}_{k\,l}\right)^{-1}\right]$$

$$+\frac{1}{2}\mathrm{Tr}_l\left[\frac{\frac{\partial}{\partial u_i}\partial_t \mathscr{R}_{k\,l}}{\left(\bar{\Gamma}^{(2)}_{k}{}_l + \mathscr{R}_{k\,l}\right)}\right]\Bigg\}.$$

Evaluate at the fixed point and expand around $\rho = 0$:

$$\frac{\partial}{\partial u_i}\mathrm{RHS}\Big|_{\mathrm{FP}} = \tilde{V}\left\{\sum_j \gamma_{ij}(u^*) + \delta\gamma_{ij}(u^*)\right\}\rho^j\,.$$

Relationship with the composite operator formalism

Thus we have,

$$\frac{1}{\tilde{V}} \frac{\partial}{\partial u_i} \mathrm{RHS}\Big|_{\mathrm{FP}} = \left\{ \sum_j \gamma_{ij}(u^*) + \delta\gamma_{ij}(u^*) \right\} \rho^j ,$$

and on the other hand,

$$\frac{1}{\tilde{V}} \frac{\partial}{\partial u_i} \mathrm{RHS}\Big|_{\mathrm{FP}} = \left\{ \partial_i \omega_j \Big|_{u^*} - \sum_k \partial_i (c_k \bar{\omega}_{kj}) \Big|_{u^*} \right\} \rho^j .$$

Consequently, with $\omega' := \omega - c\bar{\omega}$, we have the relations

$$\boxed{\partial\omega' \equiv \gamma + \delta\gamma}$$

and (note: $\partial c = -D$),

$$\boxed{\begin{aligned} B := \partial\beta &= (\partial\omega - c\partial\bar{\omega})\,(\mathbb{1} - \bar{\omega})^{-1} + \partial c \\ &= (\partial c + \partial\omega')\,(\mathbb{1} - \bar{\omega})^{-1} \\ &= (\partial c + \gamma + \delta\gamma)\,(\mathbb{1} - \bar{\omega})^{-1} \end{aligned}}$$

Hence, there are two different ways of obtaining the theory's critical exponents, depending on whether the couplings' anomalous dimensions $\eta(u)$ on the RHS is differentiated or not:

1. Eigenvalues of $B \equiv \partial\beta(u^*)$ – here, $\eta(u)$ on the RHS is differentiated
2. Eigenvalues of $-D + \gamma(u^*)$ – here, $\eta(u) \equiv \eta^*$ on the RHS is fixed

Here, with an $f(R)$-type first and second truncation (with $\rho = k^{-2}R$):

- $\Gamma_k = \int \mathrm{d}^d x \sqrt{g} f_k(R) = \int \mathrm{d}^d x \sqrt{g} k^d \sum_{n=0}^{N_{\mathrm{prop}}} u_n(k)\rho^n$
- $\mathscr{O}_n = \int \mathrm{d}^d x \sqrt{g} R^n$ with $n = 0, 1, \ldots, N_{\mathrm{scal}}$

An important feature is that the negative eigenvalues θ_n of B behave as (Falls et al. '14)

$$\theta_n = \theta_n^{\mathrm{G}} + \Delta\theta_n \quad \text{with} \quad \Delta\theta_n \to 0$$

as $n, N \to \infty$ and with $\theta_n^{\mathrm{G}} = d - 2n$. Then, there are 3 relevant directions in the $f(R)$-truncation: $\theta_1 = \theta' \pm \theta''\mathrm{i}$ and θ_2.

Results

$$B := \partial\beta = (\partial\omega - c\partial\bar{\omega})\,(\mathbb{1} - \bar{\omega})^{-1} + \partial c$$
$$= (\partial c + \gamma + \delta\gamma)\,(\mathbb{1} - \bar{\omega})^{-1}$$

The results differ vastly – e.g., for $N_{\mathrm{scal}} = N_{\mathrm{prop}} = 6$ we have

$$B = \begin{pmatrix} -0.567615 & -7.42554 & 0.0225945 & -3.50096 & 9.57729 & 10.5702 & 5.27183 \\ 1.78037 & -5.23758 & -0.94019 & -4.94022 & 5.66781 & 9.55175 & 8.17227 \\ -0.149968 & 4.02842 & -4.00389 & -8.85596 & -4.79984 & 11.4452 & 18.4268 \\ -0.567917 & 1.77611 & 0.75612 & 2.0677 & -9.68653 & -0.174769 & 4.20415 \\ -0.0406642 & -0.874394 & 0.754532 & 3.05935 & 5.79338 & -8.28503 & -4.63791 \\ 0.14956 & -0.512764 & -0.167635 & -0.33753 & 2.60899 & 7.08364 & -5.49057 \\ -0.0464313 & 0.374747 & -0.0986957 & -0.526183 & -1.70664 & 0.424809 & 10.7684 \end{pmatrix}$$

Eigenvalues: $\{-2.391 \pm 2.384i, -1.512, 4.161, 4.681 \pm 6.085i, 8.677\}$

$$-D + \gamma(u^*) = \begin{pmatrix} -2.03863 & -6.10938 & 24.0098 & -249.086 & 2590.25 & -26300.5 & 267992. \\ 1.28713 & -4.98407 & 14.2126 & -141.618 & 1139.79 & -8161.99 & 48955.9 \\ 0.798176 & -5.69927 & 55.7601 & -364.884 & 1188.24 & 13484.7 & -397450. \\ 0. & 2.39453 & -20.9592 & 180.194 & -1143.4 & 4013.58 & 36785.8 \\ 0. & 0. & 4.78906 & -44.4927 & 368.318 & -2321.34 & 8334.4 \\ 0. & 0. & 0. & 7.98176 & -76.2996 & 620.131 & -3898.69 \\ 0. & 0. & 0. & 0. & 11.9726 & -116.38 & 935.633 \end{pmatrix}$$

Eigenvalues: $\{-41.065 \pm 21.204i, -6.824, -0.084 \pm 3.964i, 588.102, 1654.03\}$

Negative eigenvalues of B and $-D+\gamma$ at the UV fixed point, sorted by their real part, for selected values of N_{scal} and N_{prop}. Relevant directions are those with $\mathrm{Re}\,\theta > 0$. Results have been obtained in the physical gauge (for $d=4$).

	B	$-D+\gamma$	B	$-D+\gamma$	B	$-D+\gamma$	B	$-D+\gamma$	$-D+\gamma$
$(N_{\text{scal}}, N_{\text{prop}})$	(2,2)	(2,2)	(3,3)	(3,3)	(4,4)	(4,4)	(6,6)	(6,6)	(6,4)
$\mathrm{Re}\,\theta_1$	1.26	4.03	2.67	1.96	2.83	3.17	2.39	0.084	1.06
$\mathrm{Im}\,\theta_1$	-2.44	-1.40	-2.26	-1.61	-2.42	-3.14	-2.38	-3.96	-4.13
θ_2	27.02	0.89	2.07	-6.39	1.54	-5.09	1.51	6.82	16.83
θ_3			-4.42	-305.82	-4.28	-64.03	-4.16	-588.10	24.06
$\mathrm{Re}\,\theta_4$					-5.09	-534.47	-4.68	41.06	41.44
$\mathrm{Im}\,\theta_4$					0	0	6.08	-21.20	0
θ_5							–	–	-651.07
θ_6							-8.68	-1654.03	-1586.38

Results – Observations

- Both methods agree qualitatively for small $N_{\mathrm{scal}} = N_{\mathrm{prop}} \lesssim 2$
- The results are sensitive to the full information carried by the propagator

Solving the Wetterich equation:

- The critical exponents derived from B become Gaussian, $\theta_n \xrightarrow{n\to\infty} \theta_n^{\text{Gaussian}} = 4 - 2n$ (cf. Falls et al. '14)

Solving the composite-operator flow equation:

- The critical exponents derived from $-D + \gamma$ become (unacceptably) large
  - The two points above have a technical origin: For $N \geq 3$, the 2-point function evaluated at the fixed point has a pole in R inside the unit circle, symbolically:

$$\left.\frac{1}{\Gamma_k^{(2)} + \mathscr{R}_k}\right|_{\mathrm{FP}} \sim \frac{1}{1 - 10\rho} \overset{\text{expand around } \rho = 0}{=} \sum_{n=0} 10^n \rho^n = \sum_{n=0} (\gamma, \omega, \bar{\omega})_n \rho^n$$

  - In B, the "ratio" of these larges coefficients drives the critical exponents into a Gaussian regime, whereas for $-D + \gamma$ these coefficients are taken at face value

Results – Observations

At this stage, we can ask some

- Is the Gaussian scaling limit in the $f(R)$-truncation a truncation artefact?
- Or, on the other hand, is it a more general property of a wider class of theories? (cf. design of the Wetterich equation)
- A comprehensive study of the $f(R)$-truncation with

$$f_k(R) = \sum_i \bar{u}_i(k)(R - R_0)^i$$

could be insightful (How to choose R_0?)

- What is the meaning of the unexpectedly large values of γ?

Scaling of geometric operators

- The eigenvalues of $-D+\gamma(u^*)$ can also be interpreted as the full geometric scaling dimensions of the operators $[\mathscr{O}_k]_1(r),\dots,[\mathscr{O}_k]_n(r)$ in the fixed point regime (given that these depend on some length scale r). (Cf. Pagani '16)
- In particular, for a single composite operator one has:

$$[\mathscr{O}_k]_{k\to\infty}(r) \sim r^{d-\gamma(u^*)}$$

- Applied to the volume operator, $\int \mathrm{d}^d x\sqrt{g}$, i.e., $N_{\mathrm{scal}}=0$, we obtain a stable result in the physical gauge for $N_{\mathrm{prop}}\geq 3$ of

$$\gamma = 1.9614$$

- Thus for the full spacetime volume, we observe a dimensional reduction from $d=4$ down to $4-\gamma\approx 2$

Thanks for your attention.

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