Some thoughts on gauge invariance and functional identities within functional renormalization

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Introduction

- In computations in the continuum, gauge invariance is typically dealt with by gauge fixing;
- This replaces gauge invariance by the powerful BRST symmetry which essentially reflects the fact that the gauge-fixing procedure amounts to introduce an identity in the path integral;
- Preserving BRST invariance allows for an easy control of spurious dependences such as gauge-parameter dependence;
- However, introducing regulators typically deform BRST invariance thanks to the mass-like behavior of such terms;
- Such a deformation is encoded in the so-called modified Ward identities (mWI) and modified Slavnov-Taylor identity (mSTI);

- In order to avoid such complications, several different gauge-invariant flow equations were proposed along the history of the FRG;
- In fact, we shall argue that at least some of those gauge-invariant formulations can be nearly recovered by dressed gauge fields;

Dealing with gauge invariance in the path integral

Warming up: Abelian gauge theories
(Euclidean Path Integral)

$$Z[J] = \int \mathscr{D}A \, \mathrm{e}^{-S_{\mathrm{M}}[A] + \int \mathrm{d}^d x \, J_\mu(x) A_\mu(x)}$$

generating functional of correlation functions

$$S_{\mathrm{M}}[A] = \frac{1}{4} \int \mathrm{d}^4 x \; F_{\mu\nu} F_{\mu\nu}$$

Maxwell's Action

Maxwell's action is invariant under Abelian gauge transformations written as

$$A'_\mu = A_\mu - \partial_\mu \xi$$

Such invariance prevents the definition of the propagator of the photon field

GAUGE FIXING!

Faddeev-Popov Gauge-Fixing Procedure

We choose the Landau gauge for concreteness

$$Z[J] = \int \mathscr{D}A \, \delta(\partial_\mu A_\mu) \, \det \mathscr{M}_{\mathrm{FP}} \, \mathrm{e}^{-S_{\mathrm{M}}[A] + \int \mathrm{d}^d x \, J_\mu(x) A_\mu(x)}$$

Faddeev-Popov Identity

Faddeev-Popov Operator: $\mathscr{M}_{\mathrm{FP}} = -\partial^2$ (field independent)

Introduction of the so-called FP ghosts and the LN field:

$$Z[J] = \int [\mathscr{D}\mu]_{\mathrm{FP}} \, \mathrm{e}^{-S[A,b,\bar{c},c] + S_{\mathrm{sources}}}$$

$$[\mathscr{D}\mu]_{\mathrm{FP}} = [\mathscr{D}A][\mathscr{D}b][\mathscr{D}\bar{c}][\mathscr{D}c]$$

$$S[A,b,\bar{c},c] = S_{\mathrm{M}}[A] + S_{\mathrm{FP}}[A,b,\bar{c},c]$$

$$S_{\mathrm{FP}}[A,b,\bar{c},c] = \int \mathrm{d}^4 x \; b \, \partial_\mu A_\mu + \int \mathrm{d}^4 x \; \bar{c} \partial^2 c$$

BRST Symmetry

An important outcome of the FP quantization: BRST symmetry

$$sA_\mu = -\partial_\mu c$$
$$sc = 0$$
$$s\bar{c} = b$$
$$sb = 0$$

$$s^2 = 0$$

$$S_{\rm FP}[A, b, \bar{c}, c] = s\int \mathrm{d}^4x \; \bar{c}\,\partial_\mu A_\mu$$

FP action is written as a BRST variation: BRST exact

Formally the same

$$\delta A_\mu = -\partial_\mu \xi$$

Gauge transformation

$$sA_\mu = -\partial_\mu c$$

BRST transformation

Maxwell's Action is thus invariant under BRST transformations but it is not BRST exact: BRST closed

Slavnov-Taylor Identity

The effective action satisfies the Slavnov-Taylor identity that encodes BRST symmetry

$$\mathcal{S}(\Gamma) = \int \mathrm{d}^4x \left(c\, \partial_\mu \frac{\delta \Gamma}{\delta A_\mu} + b\, \frac{\delta \Gamma}{\delta \bar{c}} \right) = 0$$

- **The imposition of the Slavnov-Taylor identity brings a powerful and elegant framework to control gauge-parameter dependence of correlation functions.**
- **Let us have a look at gauge conditions involving free gauge parameters:**

$$\partial_\mu A_\mu = \alpha\, b$$

non-negative gauge parameter

$$S_{\mathrm{FP}}[A, b, \bar{c}, c] = s \int \mathrm{d}^4x\; \bar{c} \left(\partial_\mu A_\mu - \frac{\alpha}{2} b \right)$$

FP gauge-fixing action

- **The gauge parameter enters only in the gauge-fixing action and thus in a BRST-exact term.**
- **Due to cohomological techniques, the gauge parameter does not enter correlation functions of gauge-invariant operators.**
- **This can be controlled by an extended Slavnov-Taylor.**

Extended Slavnov-Taylor Identity

We can introduce the gauge parameter in a BRST-doublet structure, i.e.,

$$s\alpha = \chi \qquad s\chi = 0$$

$$S_{\text{FP}}[A, b, \bar{c}, c] = \int \mathrm{d}^4x \; b \left(\partial_\mu A_\mu - \frac{\alpha}{2} b \right) + \int \mathrm{d}^4x \left(\bar{c}\, \partial^2 c - \frac{\chi}{2} \bar{c} b \right)$$

Extended Slavnov-Taylor Identity

$$\mathcal{S}_{\text{ext}}(\Gamma) = \int \mathrm{d}^4x \left(c\, \partial_\mu \frac{\delta \Gamma}{\delta A_\mu} + b\, \frac{\delta \Gamma}{\delta \bar{c}} \right) + \chi \frac{\partial \Gamma}{\partial \alpha} = 0$$

With this identity, we can easily prove that correlation functions of gauge-invariant operators are gauge-parameter independent. Take a gauge invariant operator $\mathcal{O}(x)$, i.e., $s\mathcal{O}(x) = 0$ with $\mathcal{O} \neq s\hat{\mathcal{O}}$.

Correlation functions of gauge-invariant operators can be computed by coupling sources to $\mathcal{O}(x)$, i.e.,

$$\int \mathrm{d}^4x \; J^{\mathcal{O}}(x)\, \mathcal{O}(x)$$

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle = \frac{\delta}{\delta J^{\mathcal{O}}(x_1)} \cdots \frac{\delta}{\delta J^{\mathcal{O}}(x_n)} W[J] \bigg|_{J=0}$$

Extended Slavnov-Taylor Identity

Applying the test operator:

$$\frac{\partial}{\partial \chi}\frac{\delta}{\delta J^{\mathcal{O}}(x_1)}\cdots\frac{\delta}{\delta J^{\mathcal{O}}(x_n)}$$

and turning off sources and χ

$$\frac{\partial}{\partial \alpha}\langle \mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\rangle_c = 0$$

gauge-parameter independence

Hence, the extended Slavnov-Taylor identity ensures gauge-parameter independence of gauge-invariant correlation functions.

Coarse-graining and the fate of BRST invariance

Our interest is to apply Functional Renormalization Group (FRG) techniques and hence we introduce quadratic regulators on the elementary fields:

$$Z_k[J] = \int [\mathscr{D}\mu]_{\mathrm{FP}}\, \mathrm{e}^{-S[A,b,\bar{c},c]-\Delta S_k^{(A)}-\Delta S_k^{(\bar{c}c)}+S_{\mathrm{sources}}}$$

$$\Delta S_k^{(A)} = \frac{1}{2}\int \mathrm{d}^4x\; A_\mu\, \mathscr{R}^{\mu\nu}_{k,(A)}(-\partial^2)\, A_\nu$$

$$\Delta S_k^{(\bar{c}c)} = \int \mathrm{d}^4x\; \bar{c}\, \mathscr{R}_{k,(\bar{c}c)}(-\partial^2)\, c$$

Clearly, the regulator terms break BRST invariance. However, let us remind that the transverse gauge field is gauge invariant in the Abelian case. Hence, we could try to employ the following BRST-invariant construction:

$$\Delta S_k^{(A^T)} = \frac{1}{2}\int \mathrm{d}^4x\; A^{\mathrm{T}}_\mu\, \mathscr{R}^{\mu\nu}_{k,(A)}(-\partial^2)\, A^{\mathrm{T}}_\nu$$

with

dressed gauge field

$$A^{\mathrm{T}}_\mu = A_\mu - \frac{\partial_\mu}{\partial^2}\partial\cdot A$$

$$\Delta S_k^{(A^T)} = \Delta S_k^{(A)} + \int \mathrm{d}^4x\; \mathscr{F}_k(A)\, \partial\cdot A$$

Collecting the ghost terms:

$$S_{\mathrm{FP}}[0,0,\bar{c},c] + \Delta S_k^{(\bar{c}c)} = \int \mathrm{d}^4x\; \bar{c}\left(-\partial^2 + \mathscr{R}_{k,(\bar{c}c)}(-\partial^2)\right) c$$

Gauge-invariant coarse-graining

We now employ the Landau gauge. As a first step we integrate out the FP ghosts and replace the gauge-field regulator by the (dressed) gauge-invariant regulator

$$Z_k[J] = \int [\mathscr{D}\mu]_{\mathrm{FP}} \left(\det \mathscr{M}_{\mathrm{FP,k}} \right) \mathrm{e}^{-S[A,b,\bar{c},c] - \Delta S_k^{(A^T)} + \int \mathrm{d}^4 x\, \mathscr{F}_k(A)\, \partial\cdot A + S_{\mathrm{sources}}}$$

Ghost sector:

$$\det \mathscr{M}_{\mathrm{FP,k}} = \det\left(\partial^2 + \mathscr{R}_{\mathrm{k},(\bar{\mathrm{c}}\mathrm{c})}(-\partial^2)\right) = \exp\left[\mathrm{Tr}\,\ln\left(\partial^2 + \mathscr{R}_{k,(\bar{c}c)}(-\partial^2)\right)\right]$$

From the gauge-fixing term:

$$\int \mathrm{d}^4x\; b\, \partial_\mu A_\mu \longrightarrow \int \mathrm{d}^4x\; b\, \partial_\mu A_\mu - \int \mathrm{d}^4x\; \mathscr{F}_k(A)\, \partial\cdot A$$

Consequently

$$\int \mathrm{d}^4x\; \tilde{b}\, \partial_\mu A_\mu \longrightarrow \tilde{b} = b - \mathscr{F}_k(A)$$

trivial Jacobian

Gauge-invariant coarse-graining

Integrating out the redefined LN field:

$$Z_k[J] = \int [\mathscr{D}A]\, \delta(\partial \cdot A)\, \mathrm{e}^{-S_{\mathrm{M}}[A] - \Delta S_k^{(A^T)} + S_{\mathrm{sources}}}$$

Ghost sector decouples - field-independent.

In the Landau gauge: $A_\mu^{\mathrm{T}} \to A_\mu$

In this sense, in the Landau gauge condition, the gauge field can be replaced by a gauge-invariant field with no extra cost. Hence, the regulator can be written in terms of gauge-invariant fields.

This is quite similar to the gauge-invariant flow equation proposed by C. Wetterich recently. It is related to the standard flow equation by an appropriate field-redefinition in the Landau gauge.

What about matter fields?

Physical (dressed) matter fields:

Let us write down the action for QED in the Landau gauge:

$$S_{\text{QED}}[\Phi] = S_{\text{M}}[A] + S_{\text{FP}}[A, b, \bar{c}, c] + S_{\text{D}}[\bar{\psi}, \psi, A]$$

$$S_{\text{D}}[\bar{\psi}, \psi, A] = \int \mathrm{d}^4x \left(\bar{\psi}\, \gamma_\mu D_\mu \psi - m\, \bar{\psi} \psi \right)$$

$$D_\mu = \partial_\mu - igA_\mu$$

We can define a gauge-invariant (dressed) field as follows:

[Dirac, Lavelle-McMullan,...]

$$\psi^h = \exp\left(-ig \frac{\partial \cdot A}{\partial^2} \right) \psi$$

$$\bar{\psi}^h = \bar{\psi} \exp\left(ig \frac{\partial \cdot A}{\partial^2} \right)$$

Gauge-invariant dressed fermions

Physical (dressed) matter fields:

Gauge-invariant regulator

$$\Delta S_k^{(\bar\psi^h \psi^h)} = \int \mathrm{d}^4x \;\bar\psi^h \,\mathscr{R}_{k,(\bar\psi\psi)}(-\partial^2)\,\psi^h \quad \longrightarrow \quad \Delta S_k^{(\bar\psi\psi)} = \int \mathrm{d}^4x \;\bar\psi \,\mathscr{R}_{k,(\bar\psi\psi)}(-\partial^2)\,\psi$$

In the Landau gauge

Once again, in the Landau gauge, the gauge-invariant (non-quadratic) regulator collapses into a quadratic expression: "physical gauge"

This is some sort of miraculous property of the Landau gauge!

In the Landau gauge, we can write the regulated path integral with dressed fields which engender a gauge-invariant meaning to it or, conversely, a physical meaning to the Landau gauge.

One can map the standard regularized path integral to the gauge-invariant regularized path integral by a change of variables in the Landau gauge.

Non-Abelian gauge theories

Let us consider now

$$Z[J] = \int \mathcal{D}A \, \mathrm{e}^{-S_{\mathrm{YM}}[A] + \int \mathrm{d}^d x \, J^a_\mu(x) A^a_\mu(x)}$$

generating functional of correlation functions

$$S_{\mathrm{YM}}[A] = \frac{1}{4} \int \mathrm{d}^4 x \; F^a_{\mu\nu} F^a_{\mu\nu}$$

Yang-Mills Action

Gauge transformation

$$A'_\mu = U^\dagger A_\mu U + \frac{i}{g} U^\dagger \partial_\mu U$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$U \in SU(N) \qquad A_\mu = A^a_\mu T^a$$

$$[T^a, T^b] = i f^{abc} T^c$$

Gauge-fixed path integral (Landau gauge)

$$Z[J] = \int \mathcal{D}A \, \delta(\partial_\mu A^a_\mu) \, \det \mathcal{M}_{\mathrm{FP}} \, \mathrm{e}^{-S_{\mathrm{YM}}[A] + \int \mathrm{d}^d x \, J^a_\mu(x) A^a_\mu(x)}$$

$$\mathcal{M}^{ab}_{FP} = -\, \partial_\mu D^{ab}_\mu$$

$$D^{ab}_\mu = \delta^{ab} \partial_\mu - g f^{abc} A^c_\mu$$

FP operator is field dependent

Introducing FP ghosts and LN field

$$Z[J] = \int [\mathscr{D}\mu]_{\text{FP}} \, \mathrm{e}^{-S[A,b,\bar{c},c]+S_{\text{sources}}}$$

$$S[A,b,\bar{c},c] = S_{\text{YM}} + \int \mathrm{d}^4x \left(b^a \partial_\mu A_\mu^a + \bar{c}^a \partial_\mu D_\mu^{ab} c^b \right)$$

BRST Transformations

$$sA_\mu^a = -D_\mu^{ab} c^b$$

$$sc^a = \frac{g}{2} f^{abc} c^b c^c$$

$$s\bar{c}^a = b^a$$

$$sb^a = 0$$

$$s^2 = 0$$

Couple external sources to the non-linear BRST transformations:

$$S_{\text{ext}} = \int \mathrm{d}^4x \left[\Omega_\mu^a (sA_\mu^a) + L^a (sc^a) \right]$$

Slavnov-Taylor Identity

$$\mathcal{S}(\Gamma) = \int \mathrm{d}^4x \left(\frac{\delta\Gamma}{\delta A_\mu^a} \frac{\delta\Gamma}{\Omega_\mu^a} + \frac{\delta\Gamma}{\delta c^a} \frac{\delta\Gamma}{\delta L^a} + b^a \frac{\delta\Gamma}{\delta \bar{c}^a} \right) = 0$$

Can we introduce a gauge-invariant regulator for the gauge field?

Dressing the gauge field

[Zwanziger, Lavelle-McMullan,...]

For a given gauge field configuration A_μ, we search for U that minimizes the following functional:

$$f_A[U] = \mathrm{Tr}\int \mathrm{d}^4x\ A_\mu^U A_\mu^U$$

Solution (series):

$$A_\mu^h = A_\mu - \frac{\partial_\mu}{\partial^2}\partial\cdot A + ig\left[A_\mu, \frac{1}{\partial^2}\partial\cdot A\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial\cdot A, \partial_\mu\frac{1}{\partial^2}\partial\cdot A\right] + ig\frac{\partial_\mu}{\partial^2}\left[\frac{\partial_\nu}{\partial^2}\partial\cdot A, A_\nu\right] + \frac{ig}{2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial\cdot A}{\partial^2}, \partial\cdot A\right] + \mathcal{O}(A^3)$$

Properties:

- **The dressed field A_μ^h is gauge invariant;**
- **It is transverse, $\partial_\mu A_\mu^h = 0$;**
- **It reduces to the transverse gauge field in the Abelian limit;**
- **Apart from the first term (that is the gauge field itself), all terms contain at least one factor of $\partial\cdot A$.**

By construction:

$$sA_\mu^h = 0$$

Dressing the gauge field

In the Landau gauge: $A_\mu^h \to A_\mu$

We can introduce the following gauge-invariant regulator:

$$\Delta S_k^{(A^h)} = \frac{1}{2}\int \mathrm{d}^4x \; A_\mu^{h,a}\, \mathscr{R}^{ab}_{k,\mu\nu}(-\partial^2)\, A_\nu^{h,b}$$

$$Z_k[J] = \int \mathscr{D}A\, \delta(\partial_\mu A_\mu^a)\, \mathrm{det}\mathscr{M}_{\mathrm{FP}}(A)\, \mathrm{e}^{-S_{\mathrm{YM}}[A]+\int \mathrm{d}^d x\, J_\mu^a(x) A_\mu^a(x) - \Delta S_k^{(A^h)}}$$

The presence of the delta functional allows for the following replacement:

$$\mathrm{det}\mathscr{M}_{\mathrm{FP}}(A) \to \mathrm{det}\mathscr{M}_{\mathrm{FP}}(A^h) \longrightarrow$$

$$\det(-\delta^{ab}\partial^2 + g f^{abc} A_\mu^{h,c}\partial_\mu)$$

$$\downarrow$$

$$\det(\delta^{ab} P_k(-\partial^2) + g f^{abc} A_\mu^{h,c}\partial_\mu)$$

This looks very much with the logic of the gauge-invariant flow equation by C. Wetterich

Recovering the standard flow

In the Landau gauge:

$$\Delta S_k^{(A^h)} = \frac{1}{2}\int \mathrm{d}^4x \; A_\mu^{h,a}\, \mathscr{R}_{k,\mu\nu}^{ab}(-\partial^2)\, A_\nu^{h,b}$$

$$\Delta S_k^{(A^h)} = \Delta S_k^{(A)} + \int \mathrm{d}^4x \; \mathscr{F}^a(A)\,(\partial\cdot A^a)$$

The presence of the delta functional allows for: $\Delta S_k^{(A^h)} \to \Delta S_k^{(A)}$

$$\det \mathscr{M}_{\mathrm{FP,k}}(A^h) \to \det \mathscr{M}_{\mathrm{FP,k}}(A) = \det(-\delta^{ab}\partial^2 + g f^{abc} A_\mu^c \partial_\mu + \delta^{ab}\mathscr{R}_k(-\partial^2))$$

Lifting the regularized FP operator into the Boltzmann weight:

$$Z_k[J] = \int \mathscr{D}A\,\mathscr{D}\bar{c}\,\mathscr{D}c\; \mathrm{e}^{-S[A,\bar{c},c]-\Delta S_k^{(A)}-\Delta S_k^{(\bar{c}c)}+S_{\mathrm{sources}}}$$

back to the "standard" construction

Remark: If a different gauge is employed, the gauge-invariant regulator is not quadratic on the fields.
(special role of the Landau gauge)

Remark: Background Field Method

We can control gauge-parameter and background field dependence by introducing the following extended BRST transformations:

$$A^a_\mu = \bar{A}^a_\mu + a^a_\mu$$

$$\bar{D}^{ab}_\mu a^b_\mu = \alpha b^a$$

$$s a^a_\mu = -D^{ab}_\mu c^b$$

$$s c^a = \frac{g}{2} f^{abc} c^b c^c$$

$$s\bar{c}^a = b^a$$

$$s b^a = 0$$

$$s\alpha = \chi$$

$$s\bar{A}^a_\mu = V^a_\mu$$

$$s V^a_\mu = 0$$

extended Slavnov-Taylor identity

$$\mathcal{S}_{\text{ext}}(\Gamma) = \int \mathrm{d}^4x \left(\frac{\delta\Gamma}{\delta a^a_\mu}\frac{\delta\Gamma}{\Omega^a_\mu} + \frac{\delta\Gamma}{\delta c^a}\frac{\delta\Gamma}{\delta L^a} + b^a \frac{\delta\Gamma}{\delta\bar{c}^a} + V^a_\mu \frac{\delta\Gamma}{\delta\bar{A}^a_\mu} \right) + \chi \frac{\partial\Gamma}{\partial\alpha} = 0$$

Acting with suitable test operators allows for the derivation of identities of the form:

$$\frac{\partial}{\partial\alpha}\langle\Delta_1(x_1)\ldots\Delta_n(x_n)\rangle = \Upsilon(x_1,\ldots,x_n)$$

$$\frac{\delta}{\delta\bar{A}^a_\mu(y)}\langle\Delta_1(x_1)\ldots\Delta_n(x_n)\rangle = \Theta(x_1,\ldots,x_n;y)$$

What about quantum gravity?

Can we define a dressing for the metric fluctuations?

Minimizing functional

$$f_h[\epsilon] = \frac{1}{2}\int \mathrm{d}^4x\sqrt{\bar{g}}\;\bar{g}^{\mu\nu}h^{\epsilon}_{\mu\nu}$$

[Biondo, Eichhorn, Pereira]

Dressed field: $\hat{h}_{\mu\nu}$

It is possible to show that: $\hat{h}_{\mu\nu} \to h_{\mu\nu}$ **for** $\alpha = 0 \qquad \beta = 1$

With:

$$F_\mu[\bar{g}; h] = \bar{\nabla}_\nu h^\nu{}_\mu - \frac{1+\beta}{4}\bar{\nabla}_\mu h$$

One can repeat the same argument as before and construct a gauge-invariant flow equation by employing the gauge-invariant regulator with $\hat{h}$:

$$\Delta S_k^{(\hat{h})} = \frac{1}{2}\int \mathrm{d}^4x\sqrt{\bar{g}}\;\hat{h}_{\mu\nu}\,\mathscr{R}_k^{\mu\nu,\alpha\beta}(-\bar{\nabla}^2)\,\hat{h}_{\alpha\beta}$$

This differs from the gauge-invariant construction by Wetterich; Perhaps there is another dressing that recovers his results!

Conclusions

- Gauge-fixing seems to be unavoidable in order to perform concrete computations;
- Clearly, choosing different gauges should not affect physical quantities;
- However, convenience is always a reasonable criterion for choosing a gauge;
- The Landau gauge in (non-)Abelian gauge theories plays a special role in the sense that it collapses a gauge-invariant (dressed) field to the gauge field;
- The same happens in quantum gravity by choosing the Landau-DeWitt gauge;

- The control of gauge-parameter and background-field dependences can be achieved by an extended Slavnov-Taylor identity (or its modification due to the presence of the regulator);
- This allows for the derivation of the so-called Nielsen identities as well as Landau-Fradkin-Khalatnikov transformations (relating correlation functions computed in different gauges)

Thank You!