A lapse Wick rotation for the FRG

Max Niedermaier

PITT PACC
Department of Physics and Astronomy
University of Pittsburgh

in collaboration with R. Banerjee

Quantum Spacetime and the Renormalization Group
Sardinia, Oct. 3, 2023

Outline

Towards Lorentzian FRG

FRG computations have mostly used Euclidean signature. The very definition of a Lorentzian FRG is nontrivial.

Tension between maintaining ‘covariance’ and ‘finiteness of RHS’.

Approaches include:

– Fehre et al (2021) [Symanzik cutoff, spectral functions]

– R.B.-M.N. (2022) [spatial cutoff]

– D’Angelo et al (2022) [Symanzik cutoff, T-product]

Here: Wick rotation approach from Euclidean to near Lorentzian regime.
Wick rotate in lapse not in time on real manifold.

- Maintains in qualified sense ‘covariance’ and ‘finiteness of RHS’.
- Allows ‘apples-to-apples’ comparison with Euclidean, heat kernel rooted results.

Notions of Wick rotation in (non-static) spacetimes

- Locally approximate smooth metrics with complex analytic metrics $g_{\mu\nu}(z)$ [Moretti 2000, Strohmaier-Verch-Wollenberg 2002, Gerard-Wrochna 2010]. Coordinate Wick rotation.
- Use Vielbein frame $g_{\mu\nu} = \eta_{\rm IJ} E^{\rm I}_\mu E^{\rm J}_\nu$, and complexify $\eta_{\rm IJ}$. Leads to "admissible complex metrics" [Louko-Sorkin 1997, Samuel 2016, Kontsevich-Segal 2021, Visser 2022].
- Real rank-1 deformation of Lorentzian metric: $g^{\varepsilon}_{\mu\nu} = g_{\mu\nu} + 2\varepsilon n_\mu n_\nu$, n_μ unit time-like and $\frac{1}{2} \neq \varepsilon \in [0,1]$; $\varepsilon = \frac{1}{2}$ singular. Corresponds to rescaling of the lapse, $N^2 \mapsto (1-2\varepsilon)N^2$ [Baldazzi et al. 2018] and implements a Wick-flip, $N^2 \mapsto -N^2$. [Dasgupta-Loll 2001].

Here: Wick rotation in the lapse $N \mapsto e^{-i\theta}N$, $\theta \in (0,\pi)$ on a real manifold. Amounts to $g^{\theta}_{\mu\nu} = g_{\mu\nu} + (1 - e^{-2i\theta})n_\mu n_\nu$, a complex, metric dependent rank-1 deformation [Candelas-Raine, 1979, use external V_μ],

Lapse Wick rotation

Take foliated $1+d$ manifolds M as basic. Come with a temporal function $T(y)$ whose level surfaces $T = t$ are d dim. hypersurfaces Σ_t. ADM decomposition of metric yields triples $(N, N^i, \mathrm{g}_{ij})_{\epsilon_g}$ with

$$g^{\epsilon_g}_{\mu\nu}(y)dy^\mu dy^\nu = \epsilon_g N^2 dt^2 + \mathrm{g}_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,,$$

in coords. $y^\mu = (t, x^i)$, $i = 1, \ldots, d$.
The foliation frame $(Ndt, e^1, \ldots e^d)$, $e^i = dx^i + N^i dt$, comprises $1+d$ 1-forms that are coordinate independent (inv. under passive diffeos) for fixed T. With $Ndt = n_\mu dy^\mu$ the complexified metric

$$g^\theta_{\mu\nu} = g^{\epsilon_g}_{\mu\nu} - (\epsilon_g + e^{-2i\theta})n_\mu n_\nu\,, \quad \theta \in [0, \pi]\,,$$

is a $\binom{0}{2}$ tensor under passive diffeos for fixed T.

Changing the foliation

Changing T to T' (i.e. a new scalar function on M) amounts to changing the foliation. Under these active diffeos the 1-forms $(Ndt, e^1, \ldots, e^d)$ are not invariant. E.g.

$$N' dt' = \frac{N}{\sqrt{\left(\frac{\partial t'}{\partial t} - \frac{\partial t'}{\partial x^i} N^i\right)^2 + \epsilon_g N^2 \frac{\partial t'}{\partial x^j} \frac{\partial t'}{\partial x^k} \mathrm{g}^{jk}}} \left[\left(\frac{\partial t'}{\partial t} - \frac{\partial t'}{\partial x^i} N^i\right) dt + \frac{\partial t'}{\partial x^i} e^i\right].$$

The same applies to $(e^1, \ldots, e^d)$ and the 1-forms $E^{\mathrm{I}} = E^{\mathrm{I}}_{\mu} dy^{\mu}$ in a Vielbein formulation, $g^{\epsilon_g}_{\mu\nu} = \eta^{\epsilon_g}_{\mathrm{IJ}} E^{\mathrm{I}}_{\mu} E^{\mathrm{J}}_{\nu}$. On a foliated real manifold complexification of the metric always refers to a fiducial foliation $\{\Sigma\}$. Write, $\sqrt{\epsilon_g} = +1, +i$ for $\epsilon_g = +1, -1$, and define the lapse Wick rotation wrt $\{\Sigma\}$ by

$$\mathfrak{w}_{\Sigma} : \ (N, N^i, \mathrm{g}_{ij})_{\epsilon_g} \ \mapsto \ (i\epsilon_g^{-1/2} e^{-i\theta} N, N^i, \mathrm{g}_{ij})_{\epsilon_g} , \quad \theta \in [0, \pi] .$$

Result: The Wick flip relating $\theta = 0$ to $\theta = \pi/2$ is independent of the fiducial foliation.

Induced complexified metric

Set $N_\theta = e^{-i\theta} N$ such that $\mathfrak{w}_\Sigma(\epsilon_g N^2 dt^2 + \dots) = -N_\theta^2 dt^2 + \dots$. Then:

$$N'_\theta = \frac{N_\theta}{\sqrt{\left(\frac{\partial t'}{\partial t} - \frac{\partial t'}{\partial x^i} N^i\right)^2 - N_\theta^2 \frac{\partial t'}{\partial x^j}\frac{\partial t'}{\partial x^k} \mathrm{g}^{jk}}},$$

$$N_\theta^{i\,\prime} = -\frac{\left(\frac{\partial x'^i}{\partial t} - \frac{\partial x'^i}{\partial x^j} N^j\right)\left(\frac{\partial t'}{\partial t} - \frac{\partial t'}{\partial x^k} N^k\right) - N_\theta^2 \frac{\partial x'^i}{\partial x^j}\frac{\partial t'}{\partial x^k} \mathrm{g}^{jk}}{\left(\frac{\partial t'}{\partial t} - \frac{\partial t'}{\partial x^j} N^j\right)^2 - N_\theta^2 \frac{\partial t'}{\partial x^j}\frac{\partial t'}{\partial x^k} \mathrm{g}^{jk}},$$

$$\mathrm{g}_{ij}^{\theta\,\prime} = \left(\frac{\partial x^k}{\partial x'^i} + \frac{\partial t}{\partial x'^i} N^k\right)\left(\frac{\partial x^l}{\partial x'^j} + \frac{\partial t}{\partial x'^j} N^l\right)\mathrm{g}_{kl} - N_\theta^2 \frac{\partial t}{\partial x'^i}\frac{\partial t}{\partial x'^j}$$

define a complexified metric g^θ

$$\begin{aligned} g^\theta &= -N_\theta'^2 dt'^2 + \mathrm{g}'^\theta_{ij}(dx'^i + N_\theta'^i dt')(dx'^j + N_\theta'^j dt') \\ &= -N_\theta^2 dt^2 + \mathrm{g}_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,. \end{aligned}$$

Full, real Diffeomorphism group is realized nonlinearly.

Complexified scalar field action

Consider $1+d$ form of scalar field action:

$$S_{\epsilon_g}[\chi, g] = \int dt d^d x \sqrt{\mathrm{g}} \Big\{ \frac{1}{2N} e_0(\chi)^2 + \frac{\epsilon_g}{2} N \mathrm{g}^{ij} \partial_i \chi \partial_j \chi + \epsilon_g N\, U(\chi) \Big\} ,$$

where $S_+ > 0$ for $U(\chi) \geq 0$, and $e_0 = \partial_t - \mathcal{L}_{\vec{N}}$. In fiducial foliation define complexified action

$$S^\theta[\chi, g] := S_+[\chi, g]|_{N \mapsto i e^{-i\theta} N} = -i S_-[\chi, g]|_{N \mapsto e^{i\theta} N}$$
$$= \sin\theta\, S_+[\chi, g] - i \cos\theta S_-[\chi, g] .$$

Exponential is damping $e^{-S^\theta} = e^{-\sin\theta S_+} e^{i\cos\theta S_-}$ for $\theta \in (0, \pi)$. The underlying complex metric is admissible in fiducial foliation.

Result: S^θ is invariant under the nonlinear $(N_\theta, N^i, \mathrm{g}_{ij})_- \mapsto (N'_\theta, {N'_\theta}^i, {\mathrm{g}^{\theta\prime}}_{ij})_-$ transfs. Its real part $\mathrm{Re} S^\theta$ remains positive.
Hence g^θ is admissible (in all foliations).

Complexified Hessian

Prepare $(\delta^2 S^\theta/\delta\chi\delta\chi)(\varphi) =: i\mathcal{D}_\theta(\varphi)\mathbb{1}$, $\mathbb{1} = (N\sqrt{\mathrm{g}})^{-1}\delta(t-t')\delta^d(x-x')$. Explicitly

$$i\mathcal{D}_\theta = ie^{i\theta}\nabla_t^2 + ie^{-i\theta}[-\nabla_s^2 + U''(\varphi)] = \sin\theta\,\mathcal{D}_+ + i\cos\theta\,\mathcal{D}_- \,,$$

$$\nabla_t^2 := \sqrt{\mathrm{g}}^{-1}N^{-1}e_0\big(\sqrt{\mathrm{g}}N^{-1}e_0\big)\,, \quad \nabla_s^2 := N^{-1}\sqrt{\mathrm{g}}^{-1}\partial_i\big(N\sqrt{\mathrm{g}}\mathrm{g}^{ij}\partial_j\big)\,.$$

Here, $\mathcal{D}_+, \mathcal{D}_-$ are the Euclidean, Lorentzian signature Hessians. Note:

$\mathcal{D}_+ > 0$ for $U''(\varphi) \geq 0$ is elliptic,
$\mathcal{D}_-$ is hyperbolic wave operator.

$[\mathcal{D}_+, \mathcal{D}_-] \neq 0\,!$

Math challenge later on: make sense out of

$$e^{-s\,i\mathcal{D}_\theta},\ s > 0\,,$$

$$[i\mathcal{D}_\theta + ie^{-i\theta}z]^{-1}\,,\ z > 0\,.$$

Lapse Wick rotated FRG

Want to compare Euclidean with Lorentzian signature results on 'apples-to-apples' basis. From now on take Euclidean signature as basic, $\epsilon_g = +1$ in $\mathfrak{w}_\Sigma$, i.e. $N \mapsto ie^{-i\theta} N$.

- Aim at interpolating versions of generating functionals W_k^θ, Γ_k^θ, $\theta \in (0, \pi)$.
- Assume Euclidean functional integral for $W_{+,k} = W_k^{\pi/2}$ to be well defined, leading to FRGs for $W_{+,k} = W_k^{\pi/2}$ and $\Gamma_{+,k} = \Gamma_k^{\pi/2}$.
- Then: by the same standards of rigor, the functional integral for W_k^θ and the FRGs for W_k^θ, Γ_k^θ are well-defined, for all $\theta \in (0, \pi)$.

Consider wlog scalar fields in background field formalism, $W_k^\theta = W_k^\theta[J, \varphi]$, $\Gamma_k^\theta[\phi, \varphi]$.

Where to place the phases

Action: Admissibility in functional integral ensured by
$S^\theta[\chi, g] := S_+[\chi, g]|_{N \mapsto ie^{-i\theta}N} = \sin\theta\, S_+[\chi, g] - i\cos\theta\, S_-[\chi, g]$.

Source term: $J \cdot \chi = \int dt d^d x (N\sqrt{\mathrm{g}}) J\chi)(t, x)$ gives:
$J \cdot \chi|_{N \mapsto ie^{-i\theta}N} = ie^{-i\theta} J \cdot \chi$.

Modulator term: Set
$R_{+,k}(t, x; t', x')|_{N \mapsto ie^{-i\theta}N} =: -ie^{i\theta} R_k^\theta(t, x; t', x')$, as limiting
$\mathbb{1} = (N\sqrt{\mathrm{g}})^{-1}\delta(t - t')\delta^d(x - x')$ is not phase modified.

Legendre transform: Want $\Gamma_k^\theta[\phi, \varphi] = S^\theta[\varphi + \phi] + O(\hbar)$. Need:
$\Gamma_k^\theta[\phi, \varphi] = ie^{-i\theta}\Big\{ J_k^\theta[\phi, \varphi] - W_k^\theta[J_k^\theta[\phi, \varphi], \varphi] - \frac{1}{2}\phi \cdot R_k^\theta \cdot \phi \Big\}$,
$\frac{\delta W_k^\theta}{\delta J}\Big|_{J = J_k^\theta} = \phi$, $\frac{\delta \Gamma_k^\theta}{\delta \phi} = ie^{-i\theta}\{ J_k^\theta - R_k^\theta \cdot \phi \}$.

Lapse Wick rotated Wetterich Equation

Starting from

$$e^{\frac{ie^{-i\theta}}{\hbar} W_k^\theta[J,\varphi]} = \int d\mu(\chi)\, e^{-\frac{1}{\hbar} S_k^\theta[\chi,\varphi] + \frac{ie^{-i\theta}}{\hbar} J\cdot\chi}\,, \quad \theta \in (0,\pi)\,,$$

obtain

$$\partial_k \Gamma_k^\theta[\phi,\varphi] = \frac{\hbar}{2} ie^{-i\theta} \mathrm{Tr}\{\partial_k R_k^\theta(\varphi) \cdot G_k^\theta[\phi,\varphi]\}$$

$$\left[\frac{\delta^2 \Gamma_k^\theta}{\delta\phi\delta\phi} + ie^{-i\theta} R_k^\theta\right] \cdot G_k^\theta = \mathbb{1}\,, \quad \theta \in (0,\pi)\,.$$

Normalizations are such that the Euclidean, Lorentzian limits are: $\Gamma_k^{\pi/2} = \Gamma_{+,k}$, $\lim_{\theta\to 0^+} \Gamma_k^\theta = -i\Gamma_{-,k}$ (whenever defined).

Perturbative solution

- Set $\varphi = 0$ for simplicity. Inserting Ansätze

$$\Gamma_k^\theta[\phi] = S^\theta[\phi] + \sum_{n\geq 1} \hbar^n \Gamma_{k,n}^\theta[\phi]\,, \qquad G_k^\theta[\phi] = G_{k,0}^\theta[\phi] + \sum_{n\geq 1} \hbar^n G_{k,n}^\theta[\phi]\,,$$

gives recursive system of flow equations

$$[(S^\theta)^{(2)} + ie^{-i\theta} R_k^\theta] \cdot G_{k,0}^\theta = \mathbb{1}\,,$$

$$[(S^\theta)^{(2)} + ie^{-i\theta} R_k^\theta] \cdot G_{k,n}^\theta[\varphi] = -\sum_{l=1}^{n} (\Gamma_{k,l}^\theta)^{(2)} \cdot G_{k,n-l}^\theta\,, \quad n \geq 1\,,$$

$$k\partial_k \Gamma_{k,n}^\theta = \frac{1}{2} ie^{-i\theta} \mathrm{Tr}\{k\partial_k R_k^\theta \cdot G_{k,n-1}^\theta\}\,, \quad n \geq 1\,.$$

- In principle this iteratively defines
$G_{k,0}^\theta \;\rightarrow\; \Gamma_{k,1}^\theta \;\rightarrow\; G_{k,1}^\theta \;\rightarrow\; \Gamma_{k,2}^\theta \;\rightarrow\; \ldots$

Solution at one-loop by "heat kernel" resolution

Integrating the $\partial_k \Gamma^\theta_{k,1}$ equation between $k = \mu, \Lambda$ get formally

$$\Gamma^\theta_{\mu,1} = \Gamma^\theta_{\Lambda,1} - \frac{\hbar}{2} i e^{-i\theta} \int_\mu^\Lambda dk \, \mathrm{Tr}\{\partial_k R^\theta_k \cdot [(S^\theta)^{(2)} + i e^{-i\theta} R^\theta_k]\}$$
$$= \Gamma^\theta_{\Lambda,1} + \frac{\hbar}{2} \mathrm{Tr} \ln \left[(S^\theta)^{(2)} + i e^{-i\theta} R^\theta_\mu\right] - \frac{\hbar}{2} \mathrm{Tr} \ln \left[(S^\theta)^{(2)} + i e^{-i\theta} R^\theta_\Lambda\right].$$

The un-regularized trace-log of the R_k regularized Hessian occurs.
In FRG practice on replaces this by
the regularized trace-log of the R_k un-regularized Hessian.
Here, set

$$F^\theta_{\mu,\Lambda}(\Delta) := \ln\left(\frac{\Delta + i e^{-\theta} R_\mu(\Delta)}{\Delta + i e^{-\theta} R_\Lambda(\Delta)}\right) \overset{!}{=} \int_0^\infty ds \, \widetilde{F}^\theta_{\mu,\Lambda}(s) e^{-s\Delta}, \quad \mathrm{Re}\Delta > 0\,.$$

Digression: regulator triples

- Consider regulators that are differences:
  $F^{\theta}_{\mu,\Lambda}(\Delta) = f^{\theta}_{\Lambda}(\Delta) - f^{\theta}_{\mu}(\Delta)$, $0 \leq \mu \leq \Lambda$, $\mathrm{Re}\Delta > 0$,
  μ RG scale, Λ UV cutoff. Write $\partial_\mu F^{\theta}_{\mu,\,\cdot}(\Delta) = -\partial_\mu f^{\theta}_{\mu}(\Delta)$.
- Via $f^{\theta}_{k}(\Delta) = -\ln[1 + ie^{-i\theta}k^2 r(\Delta/k^2)/\Delta]$ get correspondence to FRG type $r(\cdot)$
- Need also inverse Laplace transform $\tilde{f}^{\theta}_{k}(s)$, s heat kernel time, and its inverse Laplace transform $\tilde{\tilde{f}}^{\theta}_{k}(z)$, z resolvent variable.
- Want for $k \to \infty$ the behavior: $f^{\theta}_{k}(\Delta) \sim \ln(\Delta/k^2)$, $\tilde{f}^{\theta}_{k}(s) \sim -1/s$, $\tilde{\tilde{f}}^{\theta}_{k}(z) \sim -1$. Simple closed forms for triples: $f^{\theta}_{k}(\Delta)$, $\tilde{f}^{\theta}_{k}(s)$, $\tilde{\tilde{f}}^{\theta}_{k}(z)$.

Example:

$$\tilde{\tilde{f}}^{\theta}_{k}(z) = -\frac{k^2}{k^2+z}e^{-z/k^2} \quad \text{gives } Ei(\cdot) \text{ expressions for } \tilde{f}^{\theta}_{s}(s) \text{ and } f^{\theta}_{k}(\Delta)\,.$$

Regularized trace-log and pre-EPA

Write $(S^\theta)^{(2)} = i\mathcal{D}_\theta \mathbb{1}$ and interpret Δ as $i\mathcal{D}_\theta$, the R_k un-regularized Hessian. For suitable $\underbrace{\mathrm{Spec}(i\mathcal{D}_\theta)}$ adopt an operator version to write

?! in right half plane ?!

$$\Gamma^\theta_{\mu,1} = \Gamma^\theta_{\Lambda,1} + \frac{\hbar}{2}\mathrm{Tr}F^\theta_{\mu,\Lambda}(i\mathcal{D}_\theta) = \Gamma^\theta_{\mu,1} + \frac{\hbar}{2}\int_0^\infty ds\, \widetilde{F}^\theta_{\mu,\Lambda}(s)\, \mathrm{Tr}[\underbrace{e^{-is\mathcal{D}_\theta}}]$$

?! semi-group ?!

By $\partial/\partial\mu$ and $\mu \mapsto k$, $U''(\varphi) \mapsto U''_k(\varphi)$ get a precursor of flow in Effective Potential Approximation

?! does $e^{-is\mathcal{D}_\theta}$ have a kernel ?!

$$\begin{aligned} &ie^{i\theta}\int dtd^dx\, N\sqrt{g}\,\partial_k U_k \\ &= \frac{\hbar}{2}\int_0^\infty ds\,\partial_k \widetilde{F}^\theta_{k,\cdot}(s)\int dtd^dx\, N\sqrt{g}\,\overbrace{K^\theta_s(t,x;t,x)}\Big|_{U''\mapsto U''_k}, \end{aligned}$$

The lapse Wick rotated heat kernel

Theorem (R.B, M.N, in preparation, 70 pp)

Let $i\mathcal{D}_\theta = \sin\theta\,\mathcal{D}_+ + i\cos\theta\,\mathcal{D}_-$ *be the Hessian of* S^θ *with* $\theta \in (0, \pi/2]$. *Then:*

(a) $i\mathcal{D}_\theta$ *is an unbounded operator on a Sobolev domain dense in* L^2. *The same holds for its adjoint, and* $[i\mathcal{D}_\theta]^* = i\mathcal{D}_{\pi-\theta}$, *including domains.*

(b) *The spectrum of* $i\mathcal{D}_\theta$ *is contained in a wedge of the right half plane,* $|\arg\lambda| \leq \pi/2 - \theta$.

(c) *The resolvent* $[i\mathcal{D}_\theta + ie^{-i\theta}z]^{-1}$ *exists for* $|\arg(ie^{-i\theta}z)| < \pi/2+\theta$ *(*$z > 0$ *in particular) and obeys norm bounds that qualify* $i\mathcal{D}_\theta$ *as the generator of an unique analytic semigroup,* $\zeta \mapsto e^{-\zeta i\mathcal{D}_\theta}$, $|\arg\zeta| < \theta$.

Theorem cont.

(d) $e^{-si\mathcal{D}_\theta}$, $s \geq 0$, *is a strongly continuous semigroup on* L^2, *which is unique and contractive,* $\|e^{-si\mathcal{D}_\theta}\| \leq 1$, $s \geq 0$.

(e) $e^{-si\mathcal{D}_\theta}$, $s > 0$, *acts as an integral operator on* L^2 *functions with a kernel* $K_s^\theta(t,x;t',x')$ *that is jointly smooth in* (s,t,x,t',x'), *and obeys* $K_s^\theta(t,x;t',x')^* = K_s^{\pi-\theta}(t',x';t,x)$.

(f) *The kernel admits an asymptotic expansion of the form*

$$K_s^\theta(t,x;t',x') \asymp \frac{(-ie^{i\theta})^{\frac{d-1}{2}}}{(4\pi s)^{\frac{d+1}{2}}} e^{-\frac{1}{2s}\sigma_\theta(t,x;t',x')} \sum_{n\geq 0} A_n^\theta(t,x;t',x')(ie^{-i\theta}s)^n,$$

where σ_θ *is the Synge function of* g^θ *and* $A_n^\theta = A_n|_{N\mapsto ie^{-i\theta}N}$ *are the standard heat kernel coeffs evaluated on* g^θ. *On the RHS the termwise limit* $\theta \to 0^+$ *is trivial.*

Heat kernel induced Green's function

Given K_s^θ, $\theta \in (0, \pi/2]$ one can define a unique Green's function via

$$G_z^\theta(t,x;t',x') = \int_0^\infty ds\, e^{-sie^{-i\theta}z} K_s^\theta(t,x;t',x')\,, \quad z > 0\,.$$

It solves $[i\mathcal{D}_\theta + ie^{-i\theta}z]G_z^\theta = \mathbb{1}$, and property (f) of K_s^θ implies

$$G_z^\theta \asymp \frac{2}{(4\pi)^{\frac{d+1}{2}}} \sum_{n\geq 0} A_n^\theta \Big(\frac{2z}{ie^{-i\theta}\sigma_\theta}\Big)^{\frac{d-1}{2}-n} K_{\frac{d-1}{2}-n}\Big([2z\, ie^{-i\theta}\sigma_\theta]^{1/2}\Big)$$

$\asymp$ Hadamard parametrix as $\sigma_\theta \to 0$

However, not every solution of $[i\mathcal{D}_\theta + ie^{-i\theta}z]G_z^\theta = \mathbb{1}$, having the Hadamard property in σ_θ derives from a heat kernel.

Green's function form of trace-log and pre-EPA

Assume that $\widetilde{F}^{\theta}_{\mu,\Lambda}(s)$ admits a realization as

$$\widetilde{F}^{\theta}_{\mu,\Lambda}(s) = \int_0^\infty dz\, e^{-sie^{-i\theta}z}\,\widetilde{\widetilde{F}}{}^{\theta}_{\mu,\Lambda}(z)\,, \quad z > 0\,.$$

Then, the trace-log reads

$$\Gamma^{\theta}_{\mu,1} = \Gamma^{\theta}_{\Lambda,1} + \frac{\hbar}{2}\mathrm{Tr} F^{\theta}_{\mu,\Lambda}(i\mathcal{D}_\theta) = \Gamma^{\theta}_{\mu,1} + \frac{\hbar}{2}\int_0^\infty dz\,\widetilde{\widetilde{F}}{}^{\theta}_{\mu,\Lambda}(z)\,\mathrm{Tr}[G^{\theta}_{z}]\,.$$

The pre-EPA reads

$$\begin{aligned} &ie^{i\theta}\int dtd^d x\, N\sqrt{g}\,\partial_k U_k \\ &= \frac{\hbar}{2}\int dtd^d x\, N\sqrt{g}\int_0^\infty dz\,\partial_k \widetilde{\widetilde{F}}{}^{\theta}_{k,\cdot}(z) G^{\theta}_{z}(t,x;t,x)\Big|_{U''\mapsto U''_k}\,, \end{aligned}$$

Note: dz average of G^{θ}_{z}'s coincidence limit is finite by construction.

Beyond semigroup vacua

Not every (Wick rotated) Green's function having the Hadamard property derives from a (Wick rotated) heat kernel.

Reason: for the underlying homogeneous solutions the Hadamard property only enforces positive frequency in the UV limit. Beyond the UV limit solutions are typically admixtures of positive and negative frequency waves. Upon (any) Wick rotation these become an admixture of exponentially decaying and growing modes. The growing modes spoil L^2 estimates in (t, x) and inverse Laplace transform in z.

Many physically relevant examples on cosmological backgrounds.

Proposal: Solve $[i\mathcal{D}_\theta + ie^{-i\theta}z]G_z^\theta = \mathbb{1}$ directly and aim at 'stand-alone' Green's function computational formalism.

Study here for spatially homogeneous cosmologies (Bianchi I).

Bianchi I basics

Line element: $ds^2 = -N(t)^2 dt^2 + \mathrm{g}_{ij}(t)dx^i dx^j$. No need to diagonalize $\mathrm{g}_{ij}(t)$. Extrinsic curvature $K_{ij}(t) := -(2N)^{-1}\partial_t \mathrm{g}_{ij}$.

Relevance: Coupled to selfinteracting scalar field features prominently in quiescent BKL scenario. Becomes velocity dominated (Kasner-like) towards Big Bang. At late(r) times becomes isotropic (FRW).

Lapse Wick rotation: $N(t) \mapsto e^{-i\theta}N(t)$; write $V(t) := U''(\varphi)(t)$.

Homogeneous wave equation: After spatial Fourier transform
$\{\nabla_t^2 + e^{-2i\theta}[\mathrm{g}^{ij}(t)p_i p_j + V(t) + z]\}T_z^\theta(t,p) = 0$, $\mathrm{Re}(z) > 0$.
For $\theta = 0$ physically relevant solutions approach a (positive frequency) adiabatic vacuum of some order n as $|p|^2 := \delta^{ij}p_i p_j \to \infty$.
$T_z^{\theta=0}(t,p) \asymp [2\Omega_p^{(n)}(t)]^{-1} \exp\{-i\int_{t_0}^t dt'(N\mathrm{g}^{-1/2})(t')\Omega_p^{(n)}(t')\}[1 + O(|p|^{-2n})]$
For finite $|p|$ not positive frequency in general.
Hadamard property requires '$n \to \infty$'.

‘Stand-alone’ Green’s function

Given $T_z^\theta(t,p)$ set $\overline{T}_z^\theta(t,p) := T_z^{\theta=0}(t,p)^*|_{N \mapsto e^{-i\theta}N}$ and

$G_z^\theta(t,t',p) := \theta(t-t')T_z^\theta(t,p)\overline{T}_z^\theta(t',p) + \theta(t'-t)T_z^\theta(t',p)\overline{T}_z^\theta(t,p)$.

This solves $[i\mathcal{D}_\theta + ie^{-i\theta}z]G_z^\theta = \mathbb{1}$ without underlying heat kernel K_s^θ.

Further, $\mathcal{G}_z^\theta(t,p) := \mathrm{g}(t)^{-1/2}G_z^\theta(t,t,p)$ solves a Gelfand-Dickey equation

$$2\mathcal{G}_z^\theta(ie^{i\theta}N^{-1}\partial_t)^2\mathcal{G}_z^\theta - (e^{i\theta}N^{-1}\partial_t\mathcal{G}_z^\theta)^2 + 4[w_z + v^\theta](\mathcal{G}_z^\theta)^1 = \mathbb{1}\,,$$

$$w_z := \mathrm{g}(t)^{ij}p_ip_j + z\,, \quad v^\theta(t) := V(t) + e^{2i\theta}\Big[\frac{1}{2}N^{-1}\partial_tK - \frac{1}{4}K^2\Big]\,,$$

wit $K = \mathrm{g}^{ij}K_{ij}$. Regular nonlinear ODE for all $\theta \in [0,\pi]$; defines $\mathcal{G}_z^\theta$ irrespective of positive frequency assumption.

'Stand-alone' asymptotics

Result: $\mathcal{G}^{\theta}_{z}(t,p)$ admits a 'large w_z' asymptotic expansion

$$\mathcal{G}^{\theta}_{z}(t,p) \asymp \frac{1}{2\sqrt{w_z}} \sum_{n\geq 0} (-)^n \mathcal{G}^{\theta}_{z,n}(t,p)\,, \quad w_z := \mathrm{g}(t)^{ij} p_i p_j + z\,,$$

$$\mathcal{G}^{\theta}_{z,n}(t,p) = \frac{1}{w_z^n} \sum_{m=0}^{n} \frac{1}{w_z^m} k_n(t)^{i_1 i_2 \ldots i_m} p_{i_1} p_{i_2} \ldots p_{i_m}\,,$$

where $k_n(t)$ is a $e^{i\theta} N^{-1}\partial_t$ differential polynomial in V, K^{ij}, and

$$\mathcal{G}^{\theta}_{z,1} = \frac{v^{\theta}}{2w_z} + \frac{5}{32} e^{2i\theta} \frac{(N^{-1}\partial_t w_z)^2}{w_z^3} - \frac{1}{8} e^{2i\theta} \frac{(N^{-1}\partial_t)^2 w_z}{w_z^2}\,.$$

Further, a nonrecursive formula for $\mathcal{G}^{\theta}_{z,n}(t,p)$, $n \geq 2$, in terms of $\mathcal{G}^{\theta}_{z,1}$ exists.

EPA for Bianchi I backgrounds

Can convert Green's function version of pre-EPA into EPA proper

$$\partial_k U_k = -ie^{i\theta}\frac{\hbar}{2}\sqrt{g(t)}\int\frac{d^d p}{(2\pi)^d}\int_0^\infty dz\,\partial_k \widetilde{\widetilde{F}}^{\theta}_{k,\cdot}(z)\,\mathcal{G}^{\theta}_z(t,p)\Big|_{U''\mapsto U''_k},$$

where either $\varphi = \varphi(t)$ or $U_k(\cdot) = U_k(t,\cdot)$ for consistency.

- RHS is exactly characterized by Gelfand-Dickey eqn, for all $\theta \in [0,\pi]$, irrespective of positive frequency assumption.
- Asymptotic expansion leads to doable $d^d p$ integrals and expression in terms of $k_n(t)$. Covers the UV regime in a way consistent with very formal use of coeffs A^{θ}_n of non-existent heat kernel.
- Cover IR aspects via small p expansion, etc.

Conclusions

Wick rotation in the lapse provides viable route to define a near Lorentzian FRG that maintains 'covariance' and 'finiteness of RHS', and allows direct comparison with its Euclidean counterpart.

- Asymptotic expansions carry over to Wick rotated case and allow computational access to strictly Lorentzian UV regime.
- Wick rotated semigroup and resolvent and their kernels can rigorously be constructed. The $\theta \to 0^+$ limit is trivial for asymptotic expansions but highly nontrivial for exact semigroup or resolvent.
- Not every (Wick rotated) Green's function having the Hadamard property derives from a (Wick rotated) heat kernel. A stand-alone Green's function formalism is available for spatially homogeneous backgrounds.

Next steps: Green's functions beyond UV asymptotic expansions, ... beyond scalar fields, ...