Tutorial Sessions:
ASQG meets computer tensor algebra

DAY 1: General coding advice and conceptual introduction

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Before you start coding

- know and understand 100% what you want to compute – not 90%, not 99%, there is no room for insecurity, coding is rigorous, and you will get nonsense if you don't know what you are doing
- start easy, build up slowly – there is no point in wanting to start out with creating the most general code
- check the documentation of Mathematica – if you feel that there should be a function which does what you want to do, there probably is, don't reinvent the wheel
- test every single function of your code, test more than just trivial examples (edge cases are important), reproduce known results before computing new ones
- perform sanity checks – should this take this long/short? is the result reasonable? have an expectation of what should come out!
- assume your code to be wrong until thoroughly tested – the questions is not whether there are bugs, but how many
- ask experienced colleagues for help if you are running against a wall, check out notebooks that have been published along articles, use stackexchange/forums/...
- document your code! you maybe remember now what you did, but in 2 months? 5 years?
- optimisation: as much as needed, but don't overdo it, diminishing returns; usually tradeoff between performance and generality

Conceptual introduction into the flow equation

Goal of today: making sense out of the flow equation

$$\dot{\Gamma} = \frac{1}{2}\text{Tr}\left[\left(\Gamma^{(2)} + \mathfrak{R}\right)^{-1} \dot{\mathfrak{R}}\right] . \tag{1}$$

Things that we will discuss today:

- approximation schemes and simplifications = choosing an ansatz for Γ
- metric perturbations = computing $\Gamma^{(2)}$
- gauge-fixing and regulator = making the inversion well-defined

- inversion = performing the inversion
- heat kernel = performing the trace

Biased selection of useful references:

- recent book chapters on Asymptotic Safety [1–8]
- books [9, 10]
- recent FRG review [11]
- heat kernel [12–14]
- advanced technical topics [15–17]

Approximation schemes

- cannot solve (1) exactly, so we have to make approximations
- background field method: split metric into arbitrary background plus fluctuations, gauge fixing and regularisation break diffeomorphism symmetry
- first choice that you have to make: background field approximation or fluctuation computation; here only the former
- within this approximation, have to make further approximations:
  - derivative expansion:

$$\Gamma = \frac{1}{16\pi G_N} \int \sqrt{g} \left[-2\Lambda + R + aR^2 + bR^{\mu\nu}R_{\mu\nu} + c\mathfrak{E} + \mathcal{O}(\partial^6) \right] . \tag{2}$$

  - curvature expansion:

$$\Gamma = \frac{1}{16\pi G_N} \int \sqrt{g} \left[-2\Lambda + R + Rf_{RR}(\Delta)R + R^{\mu\nu}f_{RicRic}(\Delta)R_{\mu\nu} + c\mathfrak{E} + \mathcal{O}(\mathcal{R}^3) \right] . \tag{3}$$

  - choice of useful background:

$$\Gamma = \int \sqrt{g} \left[f(R) + \mathcal{O}(DR, S, C) \right] . \tag{4}$$

- finding a basis is non-trivial and dimension-dependent [18]
- approximation scheme dictates which simplifications you can use
  - derivative expansion up to $\mathcal{O}(\partial^2)$: $C_{\mu\nu\rho\sigma} \simeq 0$, $S_{\mu\nu} \simeq 0$, $D_\mu R \simeq 0$
  - derivative expansion up to $\mathcal{O}(\partial^4)$: $D_\alpha R_{\mu\nu\rho\sigma} \simeq 0$, $RC_{\mu\nu\rho\sigma} \simeq 0$, ... [17]

Metric perturbations

- need to compute Hessian $\Gamma^{(2)}$
- here: linear parameterisation

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{5}$$

- have to compute $\Gamma[\bar{g} + h]$ up to quadratic order in h (higher order if fluctuation computation)
- metric determinant:

$$\begin{aligned}
\sqrt{\det g_{\mu\nu}} &= \sqrt{\det\left[\bar{g}_{\mu\nu} + h_{\mu\nu}\right]} \\
&= \sqrt{\det\left[\bar{g}_{\mu\rho}\left(\delta^{\rho}{}_{\nu} + h^{\rho}{}_{\nu}\right)\right]} \\
&= \sqrt{\det \bar{g}}\sqrt{\det\left(\mathbb{1} + \mathbb{h}\right)} \\
&= \sqrt{\det \bar{g}}\sqrt{\exp \operatorname{tr} \ln\left(\mathbb{1} + \mathbb{h}\right)} \\
&= \sqrt{\det \bar{g}} \exp\left[-\frac{1}{2}\sum_{n\geq 1}\frac{1}{n}\operatorname{tr}\left[(-\mathbb{h})^{n}\right]\right]
\end{aligned} \tag{6}$$

- metric inverse:

$$\begin{aligned}
g^{-1} &= (\bar{g} + h)^{-1} \\
&= \left(\bar{g}\left[\mathbb{1} + \mathbb{h}\right]\right)^{-1} \\
&= \left[\mathbb{1} + \mathbb{h}\right]^{-1}\bar{g}^{-1} \\
&= \sum_{n\geq 0}(-\mathbb{h})^{n}\,\bar{g}^{-1} \\
\Rightarrow g^{\mu\nu} &= \bar{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu}{}_{\alpha}h^{\alpha\nu} + \ldots
\end{aligned} \tag{7}$$

- Christoffel symbol:

$$\Gamma^{\alpha}{}_{\mu\nu} = \bar{\Gamma}^{\alpha}{}_{\mu\nu} + \frac{1}{2}g^{\alpha\beta}\left[\bar{D}_{\mu}h_{\beta\nu} + \bar{D}_{\nu}h_{\beta\mu} - \bar{D}_{\beta}h_{\mu\nu}\right] \tag{8}$$

- from these, one can compute curvature tensors and their covariant derivatives
- to compute Hessian:
  - insert expansions of geometric quantities up to order of interest
  - expand everything out
  - perform partial integrations and sort covariant derivatives to bring the quadratic part into the form

$$\Gamma[\bar{g} + h]|_{h^2} = \frac{1}{2}\int \sqrt{\bar{g}}\, h_{\mu\nu}\Gamma^{(2)\mu\nu\rho\sigma}h_{\rho\sigma} \tag{9}$$

Gauge-fixing and regulator

- to make inverse well-defined, have to add gauge-fixing and regularisation
- standard gauge-fixing condition:

$$\mathcal{F}_\alpha \equiv \mathfrak{F}_\alpha{}^{\mu\nu} h_{\mu\nu} = \frac{1}{\sqrt{16\pi G_N\, \alpha}} \left(\delta_\alpha{}^{(\mu} \bar{D}^{\nu)} - \frac{1+\beta}{d} \bar{g}^{\mu\nu} \bar{D}_\alpha \right) h_{\mu\nu} \tag{10}$$

  α, β: gauge-fixing parameters

- implementation via extra contribution to Γ:

$$\Gamma_{\text{gf}} = \frac{1}{2} \int \sqrt{\bar{g}}\, \mathcal{F}_\alpha \bar{g}^{\alpha\beta} \mathcal{F}_\beta \tag{11}$$

  quadratic in the fluctuation h – breaks full diffeomorphism symmetry (as it must)

- the gauge-fixing procedure also requires Fadeev-Popov ghost - action is related to gauge-fixing operator acting on a diffeomorphism

$$\Gamma_{\text{gh}} = \int \sqrt{\bar{g}}\, \bar{c}^\mu\, \mathfrak{F}_\mu{}^{\alpha\beta}\, D_{(\alpha} c_{\beta)}\,, \tag{12}$$

- choosing a well-behaved regulator is extremely difficult in a general gauge
- two straightforward strategies:
  - choose harmonic gauge – only Laplacians in the Hessian, relatively straightforward to regularise: $\bar{\Delta} \mapsto \bar{\Delta} + \mathcal{R}(\bar{\Delta})$
  - use TT decomposition:

$$h_{\mu\nu} = h^{\text{TT}}_{\mu\nu} + \bar{D}_\mu \xi^{\text{T}}_\nu + \bar{D}_\nu \xi^{\text{T}}_\mu + 2\bar{D}_\mu \bar{D}_\nu \sigma - \frac{2}{d} \bar{g}_{\mu\nu} \bar{D}^2 \sigma + \frac{1}{d} \bar{g}_{\mu\nu} h^{\text{Tr}} \tag{13}$$

    with

$$\bar{g}^{\mu\nu} h^{\text{TT}}_{\mu\nu} = 0\,, \qquad \bar{D}^\mu h^{\text{TT}}_{\mu\nu} = 0\,, \qquad \bar{D}^\mu \xi^{\text{T}}_\mu = 0 \tag{14}$$

    removes problematic terms with open derivatives in the Hessian, but has its problems; field redefinitions improve the situation but still not perfect

Inversion

- regularised and gauge-fixed Hessian done, need to invert
- strategy: the "$\mathcal{P}$-$\mathcal{F}$ formula" – split Hessian into part that you know how to invert ($\mathcal{P}$) plus the rest ($\mathcal{F}$):

$$\mathcal{H} = \mathcal{P} + \mathcal{F} \Rightarrow \mathcal{H}^{-1} = (\mathcal{P} + \mathcal{F})^{-1} = \sum_{n \geq 0} \left(-\mathcal{P}^{-1} \mathcal{F} \right)^n \mathcal{P}^{-1} \tag{15}$$

- key point: choose $\mathcal{P}$ and $\mathcal{F}$ in a clever way so that you only need finitely many terms in the sum
- sometimes difficult to compute $\mathcal{P}^{-1}$
- alternatively, relatively generic starting point: flat spacetime propagator

- compute flat propagator in momentum space

$$\mathfrak{G}^{(0)}(p) = G^{\mathrm{TL}}(p^2)\Pi^{\mathrm{TL}} + G^{\mathrm{Tr}}(p^2)\Pi^{\mathrm{Tr}} + \dots \tag{16}$$

- choose convenient ordering of flat propagator and promote to curved spacetime, replacing momenta by covariant derivatives
- compute

$$\mathfrak{G}^{(0)}\mathcal{H} = \mathbb{1} + \mathcal{Y} \tag{17}$$

$\mathcal{Y}$ is automatically at least linear in curvature, and can be *computed*

- compute

$$\mathcal{H}^{-1} = (\mathbb{1} + \mathcal{Y})^{-1}\,\mathfrak{G}^{(0)} \tag{18}$$

using geometric series (truncates at finite order!)

- finally, to compute trace, all functions of Laplacian should be sorted to the right
- need commutators of general functions of Laplacian with covariant derivatives and curvatures – use inverse Laplace transform and Baker-Campbell-Hausdorff:

$$f(\Delta) = \int_0^\infty \mathrm{d}s\, \tilde{f}(s)\, e^{-s\Delta} \tag{19}$$

commutator with either derivative or curvature:

$$\begin{aligned}
[f(\Delta), X] &= \int_0^\infty \mathrm{d}s\, \tilde{f}(s)\, \left[e^{-s\Delta}, X\right] \\
&= \int_0^\infty \mathrm{d}s\, \tilde{f}(s) \sum_{n\geq 1} \frac{(-s)^n}{n!} \left[\Delta, X\right]_n e^{-s\Delta} &(20)\\
&= \sum_{n\geq 1} \frac{1}{n!} \left[\Delta, X\right]_n f^{(n)}(\Delta) &(21)
\end{aligned}$$

with the multicommutator

$$[X,Y]_n = \left[X, [X,Y]_{n-1}\right] \,, \qquad [X,Y]_0 = Y \,, \qquad [X,Y]_1 = [X,Y] = XY - YX \tag{22}$$

Heat kernel

- flow now in the form

$$\dot{\Gamma} = \frac{1}{2}\mathrm{Tr} \sum_i Q_i^{\mu_1\dots\mu_n} D_{\mu_1}\cdots D_{\mu_n} f_i(\Delta) \tag{23}$$

with Q_i multiplication operators – they do not contain covariant derivatives acting to the right
- functional trace = integral over continuous and sum over discrete indices
- flat spacetime: loop momentum integral
- if you know the spectrum of the Laplacian on your background: spectral integral
- generally: heat kernel

- usage:

$$\begin{aligned}\mathrm{Tr}\, Q\, f(\Delta) &= \int_0^\infty \mathrm{d}s\, \tilde{f}(s)\, \mathrm{Tr}\, Q\, e^{-s\Delta} \\ &= \int_0^\infty \mathrm{d}s\, \tilde{f}(s)\, \frac{1}{(4\pi s)^{d/2}} \sum_{n\geq 0} s^n \int \mathrm{d}^d x\, \sqrt{g}\, \mathrm{tr}\, Q\, \overline{A_n} \end{aligned} \tag{24}$$

  and similar for terms with extra derivatives; Tr = functional trace, tr = index trace

- $\overline{A_n}$ known up to sufficiently high order, e.g.:

$$\overline{A_0} = \mathbb{1}\,, \qquad \overline{A_1} = \frac{1}{6} R\, \mathbb{1} \tag{25}$$

- have to undo inverse Laplace transform:

$$\int_0^\infty \mathrm{d}s\, \tilde{f}(s)\, s^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty \mathrm{d}z\, z^{n-1} f(z)\,, \qquad n \geq 1 \tag{26}$$

$$\int_0^\infty \mathrm{d}s\, \tilde{f}(s)\, s^{n} = (-1)^n f^{(n)}(0)\,, \qquad n \geq 0 \tag{27}$$

References

[1] B. Knorr, C. Ripken and F. Saueressig, *Form Factors in Asymptotically Safe Quantum Gravity*, 2210.16072.

[2] A. Eichhorn and M. Schiffer, *Asymptotic safety of gravity with matter*, 2212.07456.

[3] T.R. Morris and D. Stulga, *The Functional f(R) Approximation*, 2210.11356.

[4] R. Martini, G.P. Vacca and O. Zanusso, *Perturbative approaches to non-perturbative quantum gravity*, 2210.13910.

[5] C. Wetterich, *Quantum gravity and scale symmetry in cosmology*, 2211.03596.

[6] A. Platania, *Black Holes in Asymptotically Safe Gravity*, 2302.04272.

[7] F. Saueressig, *The Functional Renormalization Group in Quantum Gravity*, 2302.14152.

[8] J.M. Pawlowski and M. Reichert, *Quantum Gravity from dynamical metric fluctuations*, 2309.10785.

[9] R. Percacci, *An Introduction to Covariant Quantum Gravity and Asymptotic Safety*, vol. 3 of *100 Years of General Relativity*, World Scientific (2017), 10.1142/10369.

[10] M. Reuter and F. Saueressig, *Quantum Gravity and the Functional Renormalization Group: The Road towards Asymptotic Safety*, Cambridge University Press (1, 2019).

[11] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J.M. Pawlowski, M. Tissier et al., *The nonperturbative functional renormalization group and its applications*, *Phys. Rept.* **910** (2021) 1 [2006.04853].

[12] D.V. Vassilevich, *Heat kernel expansion: User's manual*, *Phys. Rept.* **388** (2003) 279 [hep-th/0306138].

[13] K. Groh, F. Saueressig and O. Zanusso, *Off-diagonal heat-kernel expansion and its application to fields with differential constraints*, 1112.4856.

[14] A. Codello and O. Zanusso, *On the non-local heat kernel expansion*, *J. Math. Phys.* **54** (2013) 013513 [1203.2034].

[15] D. Benedetti, K. Groh, P.F. Machado and F. Saueressig, *The Universal RG Machine*, *JHEP* **06** (2011) 079 [1012.3081].

[16] B. Knorr, C. Ripken and F. Saueressig, *Form Factors in Asymptotic Safety: conceptual ideas and computational toolbox*, *Class. Quant. Grav.* **36** (2019) 234001 [1907.02903].

[17] B. Knorr, *The derivative expansion in asymptotically safe quantum gravity: general setup and quartic order*, *SciPost Phys. Core* **4** (2021) 020 [2104.11336].

[18] S.A. Fulling, R.C. King, B.G. Wybourne and C.J. Cummins, *Normal forms for tensor polynomials. 1: The Riemann tensor*, *Class. Quant. Grav.* **9** (1992) 1151.