Synthetic curvature for GR and beyond

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Curvature beyond smooth spacetimes

Why at all?

- physically relevant *models* (matched spacetimes, impulsive waves, ...)
- *PDE* point-of-view
- *singularities* vs *curvature blow-up* — *CCH* of Penrose
- approaches to *Quantum Gravity* (no metric, e.g. causal sets)

Why does it matter?

Basic geometric properties change even if $g \in C^{1,\alpha}$.

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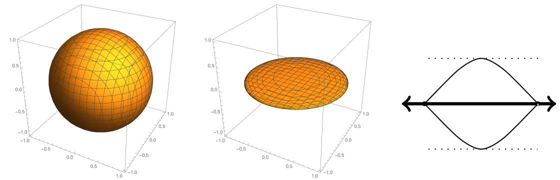

Squeezing a sphere: Equator still geodesic but it's always shorter to deviate into hemispheres (Hartman-Wintner '52)

How to detect curvature: A glimpse on Riemannian world

Sectional curvature $\mathrm{Sec}(X,Y) = \dfrac{\langle R(X,Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2}$

Theorem (Toponogov) $\mathrm{Sec} \geq K \iff$

For all (small) geodesic triangles $\triangle abc$ in (M,h) consider a comparison triangle $\triangle \bar{a}\bar{b}\bar{c}$ in the 2D space of const. curvature K. Then for all for all p, q on its sides and corresponding comparison points $\bar{p}$, $\bar{q}$

$$d_h(p,q) \geq \bar{d}(\bar{p},\bar{q}).$$

Triangle condition

- needs no manifold structure
- only distances between pts.
- works on metric spaces

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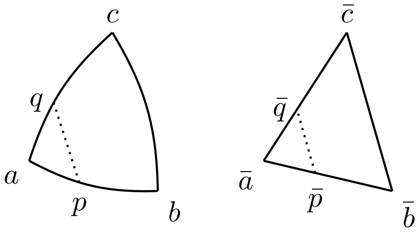

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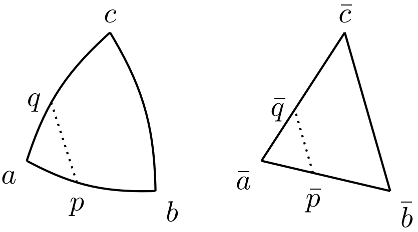

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Sectional curvature bounds for metric spaces

Definition (Length space) (X, d) with d intrinsic, i.e.

$$d(x,y) = \inf\{L(\gamma) \mid \gamma \text{ from } x \text{ to } y \text{ continuous}\}$$

geodesics $\gamma : [0,1] \to X$ with $L(\gamma) = d(\gamma(0), \gamma(1))$

Definition (Synthetic curvature bounds)

A length space has curvature bounded below by K if for all (small) triangles $\triangle abc$ and their comparison triangles $\triangle \bar{a}\bar{b}\bar{c}$ in a space of constant curvature K and all points p, q on its sides and corresponding $\bar{p}$, $\bar{q}$

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curvature bounded below / above: *Alexandrov spaces* / *CAT(K)-spaces*
rich theory since the 1980-ies: GH-convergence, Gromov compactness thm.

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Kulkarni (1979): If Sec is bounded below (above), then it is constant.

Definition ("Correct" curvature bounds, Andersson-Howard 1998)

A smooth Lorentzian manifold has $\mathrm{Sec} \geq K$ if *spacelike* sectional curvatures $\geq K$ and *timelike* sectional curvatures $\leq K$.

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How to go beyond Lorentzian manifolds?

Riemannian manifolds	$\subsetneq$	metric (length) spaces
Lorentzian mfs. / spacetimes	$\subsetneq$	?

What is the analogue of metric (length) spaces in the *Lorentzian setting*?

Serious issue:

- natural analogue to distance: time separation function

$$\tau(p,q) = \sup\{L(\gamma) |\ \gamma \text{ future dir. causal from } p \text{ to } q\}$$

- but triangle inequality is *reversed* $\rightsquigarrow$ no metric structure

$\rightsquigarrow$ *Lorentzian (pre-)length spaces* (Kunzinger-Sämann 2018)

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Lorentzian (pre-)length spaces

Causal space: X (metrizable) topological space with *abstract causality:*

$\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$

Abstract time separation: $\tau\colon X \times X \to [0, \infty]$ lower semicontinuous

Definition (Kunzinger-Sämann 2018)

$(X, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if for $p \leq q \leq r$

$$\tau(p,r) \geq \tau(p,q) + \tau(q,r) \quad \text{and} \quad \tau(p,q) \begin{cases} = 0 & \text{if } x \not\leq y \\ > 0 & \Leftrightarrow x \ll y \end{cases}$$

Examples

- *smooth spacetimes* (M, g) with usual time separation function τ
- *Lorentz-Finsler* spacetimes, spacetimes of *low regularity* $(g \in C^0 + \ldots)$
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Timelike curvature via triangle comparison

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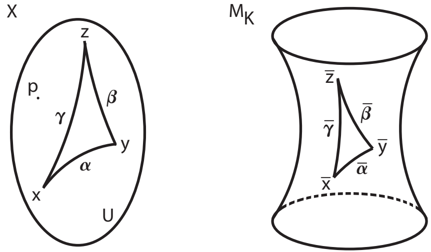

Faithful extension of sectional curvature bounds to "metric" Lorentzian setting

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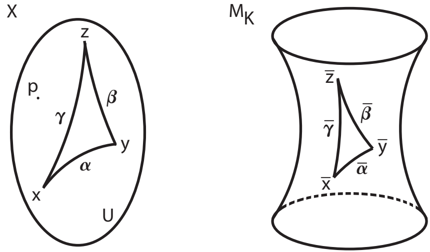

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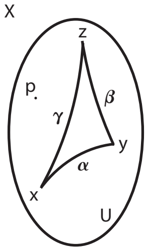

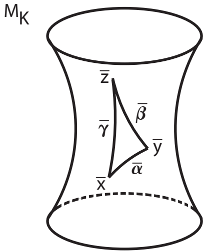

Faithful extension of sectional curvature bounds to "metric" Lorentzian setting

Selected results

Theorem (Kunzinger-Sämann 2018, Beran-Sämann 2022)

In a strongly causal Lorentzian pre-length space with *timelike curvature bounded below* timelike geodesics do *not branch*.

Theorem (Grant-Kunzinger-Sämann 2019)

A timelike geodesically complete spacetime or LLS is *inextendible as a regular LLS*, i.e., any LLS-extension necessarily has unbounded curvature.

Extends (Beem-Ehrlich) and C^0-result (Galloway-Ling-Sbierski 2018).

Splitting theorem (Beran-Ohanyan-Rott-Solis 2023)

Let $(X, \ll, \leq, \tau)$ be a globally hyperbolic LLS with global timelike $K \geq 0$. If X contains a complete timelike line (+ some technical conditions) then it splits into a product $\mathbb{R} \times S$ with S an Alexandrov space with $K \geq 0$.

Generalises smooth Lorentzian as well as synthetic Riemannian results.

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More on Lorentzian (pre-)length spaces

- *causal ladder* (Kunzinger-Sämann 2018, Aké Hau-Cabrera-Solis 2020)
- *generalized cones*, i.e., Lorentzian warped products of length spaces with 1-dim base and singularity theorems (gen. FLRW-spacetimes) (Alexander-Graf-Kunzinger-Sämann 2021)
- *null distance* & Lorentzian length spaces (Kunzinger-S. 2022)
- *gluing* of Lorentzian length spaces (Beran-Rott 2022)
- *time functions* on Lorentzian (pre-)length spaces (Burtscher-García-Heveling 2021)
- Lorentzian Hausdorff *dimension*, *measure* (McCann-Sämann 2021)
- *causal boundaries* (Ake Hau-Burgos-Solis 2023, Burgos-Flores-Herrera 2023)
- . . .
- *machine learning* in spacetimes (Law-Lucas 2023)

Ricci bounds via optimal transport: the basic idea

- *Optimal Transport*: (Monge, Kantorovich) move matter (distribution μ_1) in X in the cheapest / optimal way to (μ_2 in) Y
- *Minimize*

$$\int_{X\times Y} c(x,y)\,\mathrm{d}\pi(x,y)$$

  over couplings $\pi \in \mathcal{P}(X \times Y)$ w. given marginals $(\mathrm{pr}_X)_\sharp\pi = \mu_1$, $(\mathrm{pr}_Y)_\sharp\pi = \mu_2$
- What is *optimal* depends on *cost*,
  distances and *geometry* !
- Turn this on its head:
  define *curvature* by requiring that
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- Riemannian case: cost $c = d$
- Lorentzian case: cost $c = \tau$

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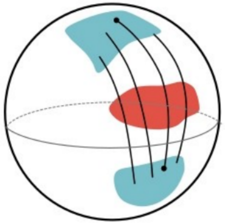

Transporting *clouds* of points on the sphere

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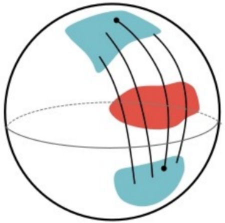

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Ricci Bounds via Optimal Transport: Riemannian case

Thm. (Ric. bds. & displacement convexity, Lott-Villani, Sturm 2006-09)

(M, g) complete Riemannian manifold

$$\mathrm{Ric} \geq 0 \iff (M, d_g, \mathrm{vol}_g) \text{ is an } \mathrm{RCD}(0, n)\text{-space}$$

Definitions. On a metric measure space $(X, d, \mathfrak{m})$ we define

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Ricci Bounds via Optimal Transport: Riemannian case

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Definitions. Measured Lorentzian pre-length space $(X, d, \mathfrak{m} \ll, \leq, \tau)$

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Selected results

Hawking singularity theorem in TCD (Cavaletti-Mondino 2024)

Let $(X, d, \mathfrak{m} \ll, \leq, \tau)$ be a globally hyperbolic measured LLS such that

1. $\mathrm{TCD}(0, N)$ (replaces (SEC)), (some technicalities hold) with
2. a Borel achronal FTC set V w. synthetic mean curvature $\leq H_0 < 0$.

Then $\tau_V \leq D_{H_0,0,N}$ on $I^+(V)$.

Complements low regularity spacetime singularity theorems
(Graf 2020, Kunzinger-Ohanyan-Schinnerl-S. 2022, S. 2023)

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Outlook

(Measured) Lorentzian Length Spaces $(X, d, \mathfrak{m} \ll, \leq, \tau)$

- provide a general mathematical setting for
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- that contains
  - *low regularity spacetimes* but also
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Gives framework for

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  strongly causal $g \in C^{0,1}$, $g \in C^0$ + causally plain
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- ingredients: *causal set* $(X, \leq)$, partial order
  locally finite: $J(x,y) = \{z : x \leq z \leq y\}$ finite
- CS hypothesis: QT of causal sets X; (M,g) approximation of X

$$\mathcal{C}(M, \rho_C) \ \ni \ X \ \longleftrightarrow (M, g)$$

Hauptvermutung of CST

X can be embedded at density ρ_C into two distinct spacetimes iff they are "close".

- $(X, \ll, \leq, \tau)$ is a Lorentzian pre-length space $\qquad (\ll := <)$
  - chain: $C := (x_i)_{i=1}^n$: $x_i < x_{i+1}$
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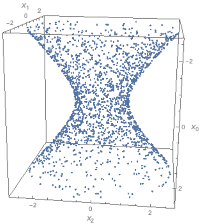

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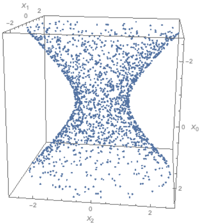

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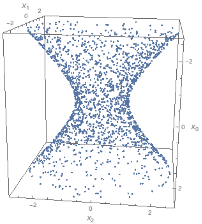

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