Invariants: a perspective for mathematical and numerical relativity

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5th EPS Conference on Gravitation
10 Dec 2024

Focus in Numerical and Mathematical Relativity

- Numerical Relativity
  - modeling of physically interesting matter/geometry configurations
  - focus on Black Hole dynamics
  - Einstein equations written and solved in specific gauge (time slicings, coordinates, frames) adapted to specific configuration
- Mathematical Relativity
  - general properties of (near) exact matter/geometry solutions
  - focus on stability of Black Holes and Cosmologies
  - Einstein equations written and solved in specific gauge (time/sphere slicings, coordinates, frames) adapted to exact background solutions

Invariants

- **Curvature Invariants**
  - A spacetime geometry (M, g) defines the **Riemann curvature** tensor R_{abcd} of the metric g_{ab}.
  - The curvature tensor gives rise to higher differential and polynomial order **curvature invariants**:

$$R_{abcd}R^{cd}{}_{ef}, \quad \nabla_e R_{abcd}, \quad (R_a{}^{bcd}\nabla_b\nabla_c R_{defh})R^h{}_{ijk}, \quad \dots$$

  - Reveal local geometric properties, independent of the choice of coordinates.
    - The blow up of the *Kretschmann scalar* $R_{abcd}R^{abcd}$ reveals the singularity inside a Schwarzschild Black Hole.
  - Have uses in classifying spacetime geometries, constructing theories of modified gravities, ...
- **Cartan Invariants** are a further construction that uses a frame $e_a^{(i)}$ in addition to g_{ab} and R_{abcd}, which may be more sensitive in some situations. [Stephani *et al* (CUP, 2003); Ch.9] (Not discussed here further.)

Some Interesting Questions

(A) When are two spacetimes (M, g) and (M_0, g_0) (presented in different coordinates, say) physically the same (diffeomorphic)?
- Comes up when classifying exact solutions.

(B) How to measure the closeness of a perturbed dynamical spacetime to the reference background solution?
- Comes up while studying dynamical Black Hole ring-down to Schwarzschild or Kerr in Numerical Relativity.

(C) How close is a dynamical spacetime to an "expected limit" in an asymptotic regime?
- Comes up while studying the stability of asymptotically flat black hole solutions or asymptotically de Sitter cosmologies in Mathematical Relativity.

- Curvature invariants naturally arise in solutions to **(A)**.
- **(B)** and **(C)** are difficult to answer directly if the gauge adapted to the dynamical geometry is not automatically adapted to the reference geometry in desired regime!
  - A gauge adapted to the asymptotic past, may not be adapted after evolution to the asymptotic future.
- Could the curvature invariants from (A) help? **Maybe!**

Example: Initial Data Closeness to Schwarzschild BH

Consider an arbitrary time slicing of a spacetime $\mathbb{R} \times \Sigma = M$. The metric g_{ab} induces initial data (g_{ij}, K_{ij}) on every time slice Σ. Decomposing the Einstein equations $R_{ab} = 0$ w.r.t. the slicing gives *evolution* $\mathcal{E}_{ij} = 0$ and *constraint* $\mathcal{H} = 0, \mathcal{M}_i = 0$ equations, the latter are intrinsic to the (g_{ij}, K_{ij}) data on each 3-slice Σ.

The Weyl curvature tensor W_{abcd} equivalently decomposes w.r.t. the slicing into the *electric* E_{ij} and *magnetic* tensors B_{ij}. Modulo $\mathcal{E}_{ij} = 0$, the E_{ij} and B_{ij} are also intrinsic to the (g_{ij}, K_{ij}) data on each slice Σ.

Definition (García-Parrado, Valiente Kroon '07)

Introduce the following "mysterious" slice-adapted curvature invariants:

$$\rho := \left(\tfrac{1}{2} B_i{}^j B^{il} E_{jl} - \tfrac{1}{6} E_{ij} E_l{}^j E^{il}\right)^{1/3},$$

$$P := -\tfrac{1}{2} E^{ij} K_{ij} - \rho K - \tfrac{1}{6\rho} \epsilon^{jk}{}_i \left(E^{il} D_k B_{lj} + B^{il} D_k E_{lj}\right),$$

$$P_i := D_i \rho, \ \tilde{E}_i := E_{ij} P^j, \ \tilde{B}_i := B_{ij} P^j,$$

$$\gamma^2 := P_i P^i, \ \Omega := \tilde{E}_i P^i, \ \alpha := \frac{\gamma^2 - P^2}{9\rho^2} + 2\rho, \ Y^2 := \frac{\gamma^2 \rho - \Omega}{27 \alpha \rho^3},$$

$$Y_i := \frac{\epsilon_i{}^{jk} P_j \tilde{B}_k - P \tilde{E}_i + \rho P P_i}{27 \alpha \rho^3 Y},$$

Example: Initial Data Closeness to Schwarzschild BH

Theorem (García-Parrado, Valiente Kroon '07)

A globally hyperbolic spacetime $(\mathbb{R}\times\Sigma, g_{ab})$ *is exactly a Schwarzschild solution* ***iff*** *the* (g_{ij}, K_{ij}) *data on each* Σ *(a) solves the* $\mathcal{E}_{ij}=0$ *evolution equations, (b) solves the* $\mathcal{H}=0$, $\mathcal{M}_i=0$ *constraint equations, (c) each of the following slice invariants vanishes*

$$B_{ij}+\frac{1}{\rho}(B_i{}^k E_{kj}+B_j{}^k E_{ki}),\ E_{ij}-\frac{1}{\rho}(B_i{}^k B_{kj}-E_i{}^k E_{kj})-2\rho h_{ij},$$
$$\tilde{B}_j P^j,\ \tilde{E}_k\epsilon^k{}_{li}P^l-P\tilde{B}_i,\ (P^2+\gamma^2)B_{ij}+2PE_{k(i}\epsilon^k{}_{j)l}P^l-2P_{(i}\tilde{B}_{j)},\qquad(*)$$
$$2B_{l(i}\epsilon_{j)}{}^{lk}P_kP-2P_{(i}\tilde{E}_{j)}+E_{ji}(P^2+\gamma^2)-\rho P_jP_i+h_{ji}((-P^2+\gamma^2)\rho+\Omega)+27\alpha\rho^3Y_jY_i,$$
$$D_{(i}Y_{j)}-YK_{ij},\quad D_iD_jY-\mathcal{L}_YK_{ij}-Y(r_{ij}+K^l_lK_{ij}-2K_{il}K^l_j)$$

while $\rho\neq 0$, $\frac{1}{9\rho^2}(\gamma^2-P^2)+2\rho>0$ *(*ϵ_{ijk} *— Levi-Civita tensor of* (Σ, g_{ij})*).*

This is a gauge and time slicing independent way of measuring the closeness of initial data to Schwarzschild geometry.

A **quantitative measure of closeness** to Schwarzschild can be obtained from a sum of squares of the tensors in $(*)$.

Intermission: IDEAL Characterizations

The following notion is a key stepping stone to the results of García-Parrado & Valiente Kroon.

- **Q:** Given a model geometry (M_0, g_0), is it possible to verify when (M, g) belongs to its local isometry class by checking a list of equations

$$T_{(\alpha)}[g] = 0 \quad (\alpha = 1, 2, \cdots, A),$$

  where each $T_{(\alpha)}[g]$ is a **tensor covariantly constructed** from g, the curvature, and its derivatives?
- If Yes, we call this an ***IDEAL*** (**I**ntrinsic, **D**eductive, **E**xplicit, **AL**gorithmic) *characterization* of the local isometry class of (M_0, g_0). We may call the $T_{(\alpha)}[g]$ the *IDEAL tensors*.
- Generalizes to (M, g, Φ), **including matter** (tensor) fields, if we use covariant tensor equations of the form $T_{(\alpha)}[g, \Phi] = 0$.

Initial Data via IDEAL Characterization

Theorem (Ferrando, Sáez 1998)

Given a metric g_{ab}, defining the spacetime curvature invariants

$$\rho = -(\tfrac{1}{12}\,\mathrm{tr}\, W^3)^{1/3}, \quad S_{abcd} = W_{abcd} - \tfrac{1}{6}(g_{ac}g_{bd} - g_{bc}g_{ad}),$$
$$\alpha = \tfrac{1}{9}(\nabla \ln \rho)^2 - 2\rho, \qquad P_{ab} = ({}^*W)_a{}^c{}_b{}^d \nabla_c \rho \nabla_d \rho,$$

the following is an IDEAL characterization of Schwarzschild of mass M*:*

$$\begin{array}{ll} R_{ab} = 0, & S_{abcd}S^{cd}{}_{ef} + 3\rho S_{abef} = 0, \\ P_{ab} = 0, & \rho/\alpha^{3/2} - \mathsf{M} = 0, \end{array} \quad \textit{while} \quad \begin{array}{l} \rho \neq 0, \\ \alpha > 0. \end{array} \qquad (**)$$

- García-Parrado & Valiente Kroon have found a "propagation identity" for the above IDEAL tensors compatible with the Einstein evolution equations: $(**)$ at $t = 0 \implies (**)$ for all t.
- Their list of initial data invariants is just the 3+1 decomposition of the IDEAL tensors from the lhs of $(**)$!

Potential Further Developments

- Do such notions of initial data closeness exist for other geometries?
- Have they found applications in Numerical or Mathematical GR?

geometry	IDEAL characterization	Initial Data characterization	NumGR application	MathGR application
Schwarzschild	Ferrando, Sáez CQG **15** 1323 (1998)	García-Parrado, Valiente Kroon PRD **75** 024027 (2007)	✗	✗
Kerr	Ferrando, Sáez CQG **26** 07501(2009)	García-Parrado PRD **92** 124053 (2015)	Bhagwat *et al* PRD **97** 104065 (2018)	✗
FLRW	Canepa, Dappiaggi, IK CQG **35** 035013 (2018)	✗	✗	✗
...	...	✗ (wip)	✗	✗

- When (M, g) is perturbatively close to (M_0, g_0), what measure of the size of $h_{ab} = g_{ab} - (g_0)_{ab}$ is independent of (linear) gauge?
- For every IDEAL tensor $T_{(\alpha)}[g]$ characterising (M_0, g_0), its linearization $\dot{T}_{(\alpha)}[g_0; h] = 0$ on pure gauge $h_{ab} = \nabla_{(a}\xi_{b)}$ modes (is a linear gauge invariant).
  [Stewart, Walker ProcRSLA **341** 49 (1974)]
- The equivalence $h_{ab} = \nabla_{(a}\xi_{b)} \iff \forall\alpha : \dot{T}_{(\alpha)}[g_0; h] = 0$, can be rigorously verified case by case, using the methods from [IK CQG **36** 185012 (2019)].

Discussion

- The focus on highly adapted coordinates/frames/time slicings in Numerical or Mathematical Relativity may obscure the closeness of a solution to a reference geometry, in a regime where the adaptation fails.
- Curvature invariants may provide a gauge independent way of measuring closeness in such regimes (e.g., IDEAL characterizations of spacetimes and initial data).
- While some applications already exist, there is room to use them more in Numerical and Mathematical Relativity.

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Thank you for your attention!