Dynamics of ghost free Massive Gravity

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Based on:

arXiv:2302.04876 - with Jan Kozuszek, Claudia De Rham, Andrew Tolley
arXiv:2409.18802 - with Emma Albertini and Jan Kozuszek
arXiv:2410.19491 - with Jan Kozuszek

***5th EPS Conference on Gravitation: Unlocking Gravity Through Computation* (Dec '24)**

Plan for talk

- *Goal*: understand whether ghost free massive gravity is a viable theory of gravity — *a computational problem!*

- Massive gravity — linear and non-linear
- Spherical dynamics; numerical and analytic work shows spherical symmetry is problematic
- Full numerical analysis to go beyond non-generic spherical case

Massive gravity

- Massive gravity in asymptotically flat 4d has a long history
- Initially considered by *Fierz* + *Pauli* in linear theory — there is a unique ghost free mass term
  - General mass term d.o.f. 2 → 6 with one a ghost
  - For *Fierz-Pauli* mass no ghost (due to extra constraint), giving 2 (spin-2) + 2 (spin-1) + 1 (spin-0) d.o.f.
- The (*Boulware-Deser*) ghost generally returns at non-linear level
- Except for dRGT massive gravity which is ghost free non-linearly

Motivation

- Adding a mass is perhaps the most natural IR modification
- Cosmology provides an interesting motivation — perhaps a cosmological mass can `explain' dark energy
- Clearly a mass would have to be small $\sim 10^{-30} eV$ ~ few MPc, but cosmological masses are still viable
- In particular GW170817 does not provide strong constraint

Linear massive gravity

- Linearizing about flat space we see the van *Dam-Veltman-Zakharov* (vDVZ) discontinuity as $m \to 0$
- … but also see linear theory breaks down, the *Vainshtein* effect

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

$$h = -\frac{2}{3m^2}T$$

$$\chi_\mu = \partial^\rho h_{\rho\mu} - \tfrac{1}{2}\partial_\mu h$$

$$-\frac{1}{2}(\nabla^2 - m^2)h_{\mu\nu} + \partial_{(\mu}\chi_{\nu)} = T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T$$

dRGT massive gravity

- Key observation of dRGT is requirement of reference metric $f_{\mu\nu}$
    - What should this be? Here we will take it as Minkowski.
- Using this introduce *symmetric* vierbein;

$$g_{\mu\nu} = (f^{-1})^{\alpha\beta} K_{\alpha\mu} K_{\beta\nu}$$

dRGT massive gravity

Next-to-Minimal mass term

- Using this we may write the theory as;

$$S = \int d^4x\sqrt{g}\left(\frac{1}{2}R - \frac{1}{2}m_1^2\mathcal{L}^{(1)} - \frac{1}{2}m_2^2\mathcal{L}^{(2)} + \mathcal{L}^{(matter)}[g]\right)$$

Minimal mass term

$$\mathcal{L}_1 = 2K^\mu{}_\mu - 6$$

$$\mathcal{L}_2 = \frac{1}{2}(K^\mu{}_\mu)^2 - \frac{1}{2}K^\mu{}_\nu K^\nu{}_\mu - 3$$

Graviton mass $m^2 = m_1^2 + m_2^2$

- This theory admits Minkowski vacuum solution

- There is one more mass term which we ignore here — it is thought not to be compatible with phenomenology

G. Chkareuli,D. Pirtskhalava ’12; L. Berezhiani, G. Chkareuli, C. de Rham, G. Gabadadze, A. Tolley ’13

dRGT massive gravity

- Yields massive Einstein equations;

$$E_{\mu\nu} \equiv G_{\mu\nu} + m_1^2 M^{(1)}_{\mu\nu} + m_1^2 M^{(2)}_{\mu\nu} - T_{\mu\nu} = 0$$

$$M^{(1)}_{\mu\nu} = -K_{\mu\nu} + K g_{\mu\nu} - 3 g_{\mu\nu}$$

$$M^{(2)}_{\mu\nu} = \frac{1}{2} K_{\mu\alpha} K^{\alpha}{}_{\nu} - \frac{1}{2} K K_{\mu\nu} - \frac{1}{4}\left(K_{\alpha\beta}K^{\alpha\beta} - K^2\right) g_{\mu\nu} - \frac{3}{2} g_{\mu\nu}$$

∇^μ

- The Bianchi identities imply;

$$0 = V_\mu \equiv \nabla^\nu \left(m_1^2 M^{(1)}_{\mu\nu} + m_1^2 M^{(2)}_{\mu\nu} \right)$$

- This vector equation is only one-derivative, thus a constraint that reduces $10 \rightarrow 6$ d.o.f.

Constraints

$$0 = V_\mu \equiv \nabla^\nu \left(m_1^2 M^{(1)}_{\mu\nu} + m_1^2 M^{(2)}_{\mu\nu} \right)$$

- Let us rewrite out vector constraint as;

$$\xi_\alpha = K_{\alpha\beta}\eta^{\beta\mu}V_\mu = 0$$

- Then it takes an elegant form; $0 = \xi^\mu = V^{\mu\alpha\beta\sigma}\partial_{[\alpha}K_{\beta]\sigma}$

- Then we may construct the scalar constraint as;

$$\Pi = \frac{1}{2}\left(m_1^2 g^{\mu\nu} + m_2^2 K^{\mu\nu} \right) E_{\mu\nu} + \nabla \cdot \xi$$

C. Deffayet, J. Mourad, and G. Zahariade ‘12

dRGT massive gravity

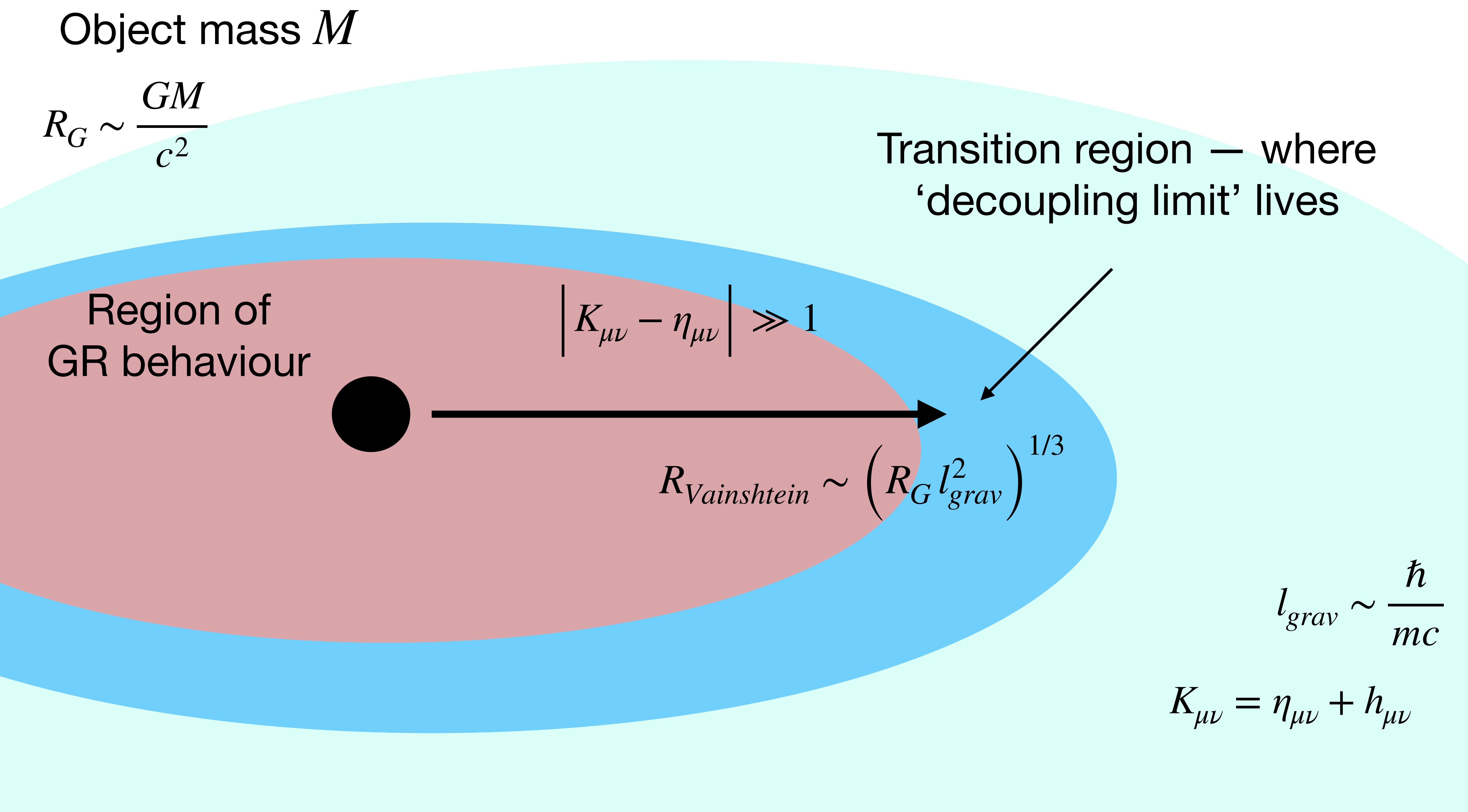
- Phenomenology complicated — understanding how GR behaviour is recovered requires detailed non-linear study

- Taking $m \to 0$ we may imagine the solution being a GR solution in some particular coordinates such that $V_\mu = 0$

- Called ‘Vainshtein mechanism’

- However must be highly ‘non-linear’

dRGT massive gravity



dRGT massive gravity

- Minimal theory does allow Vainshtein mechanism in spherical symmetry [Renaux-Petel ’14]
- Previous “state of the art” was that phenomenological constraints imply that we need the non-minimal term and;

$$m_1^2, m_2^2 > 0$$

- Then weak field analysis in static spherical symmetry argued that Vainshtein mechanism works [Koyama, Niz, Tasinato ’11; Berezhiani, Chkareuli, Gabadadze ’13]

Dynamics and 3+1 decomposition

[Kozuszek, de Rham, Tolley, TW ’23]

- In our variables the action looks very nice;

$$S = \int |K| \left(-\frac{1}{2} A_{(1)}^{\alpha\beta\gamma\mu\nu\sigma} \partial_{[\alpha} K_{\beta]\gamma} \partial_{[\mu} K_{\nu]\sigma} - \frac{1}{2} m_1^2 \mathcal{L}_1 - \frac{1}{2} m_2^2 \mathcal{L}_2 + \mathcal{L}_{matter} \right)$$

- We define momenta for K_{it} and K_{ij} ; $P_i = \partial_{[t} K_{i]t}$, $P_{ij} = \partial_{[t} K_{i]j}$

- Due to symmetry there is no $\partial_t K_{tt}$ term — and no momentum for K_{tt}

- Vector constraint - algebraic condition

$$0 = V^{\mu\alpha\beta\sigma} \partial_{[\alpha} K_{\beta]\sigma}$$

Dynamics and 3+1 decomposition

[Kozuszek, de Rham, Tolley, TW '23]

- Now scalar constraint takes the form;

$$\Pi = B^{\alpha\beta\gamma\mu\nu\rho}\partial_{[\alpha}K_{\beta]\gamma}\partial_{[\mu}K_{\nu]\rho} + \text{mass and stress tensor terms}$$

- This is *algebraic* in K_{tt} when writing in terms of momenta P_i, P_{ij}

$$P_i = \partial_{[t}K_{i]t}\ , \quad P_{ij} = \partial_{[t}K_{i]j}$$

- ADM like formulation; traceless part of K_{ij} dynamical

Spherical collapse in minimal model

[Kozuszek, de Rham, Tolley, TW ’23]

- Phenomenology may be bad for minimal model; but it is still a theory of gravity so what happens when matter collapses?

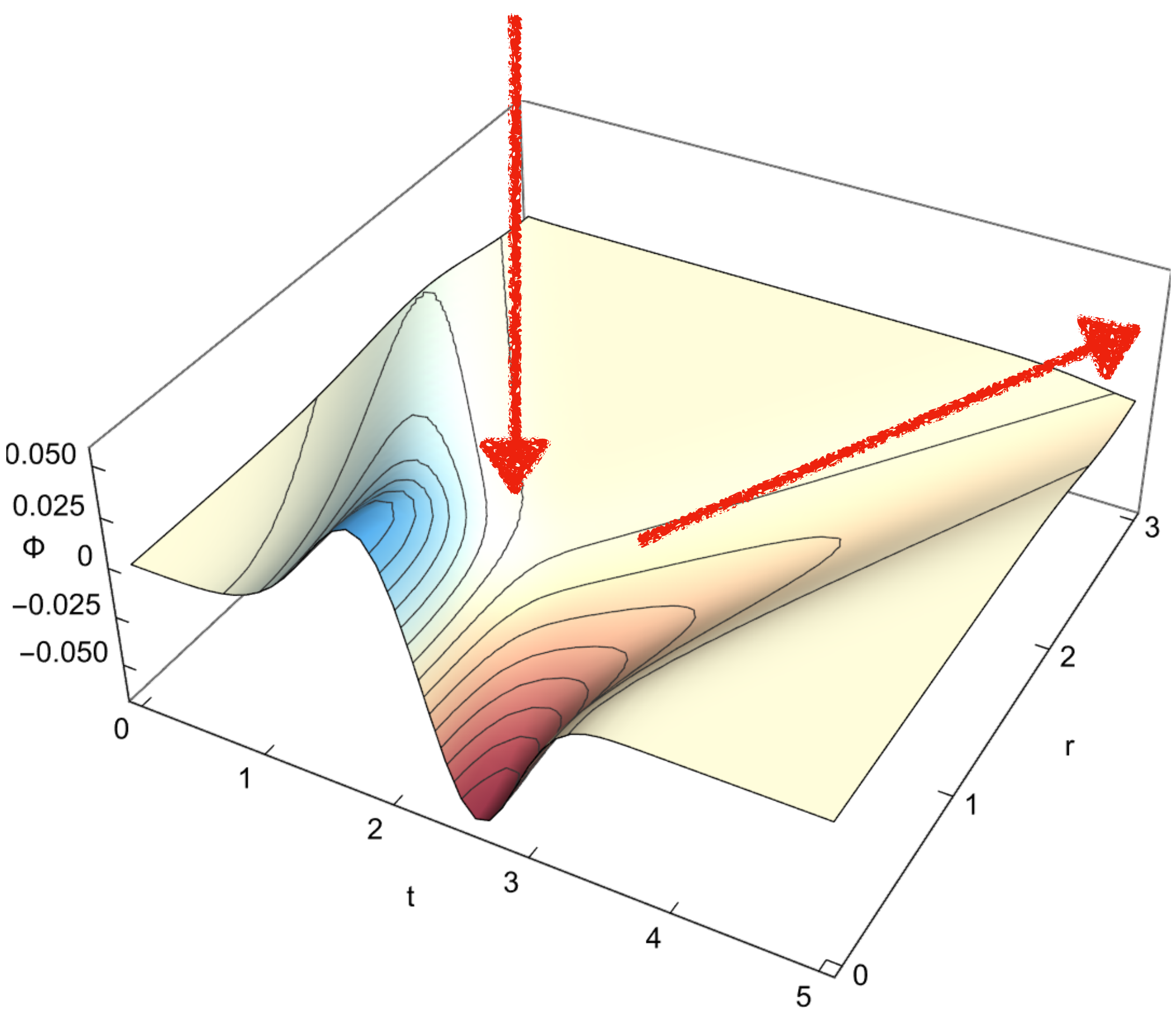
- Take massless(!) scalar field matter Φ and use this dynamical formalism to perform spherical collapse. Choose units so $m = 1$.

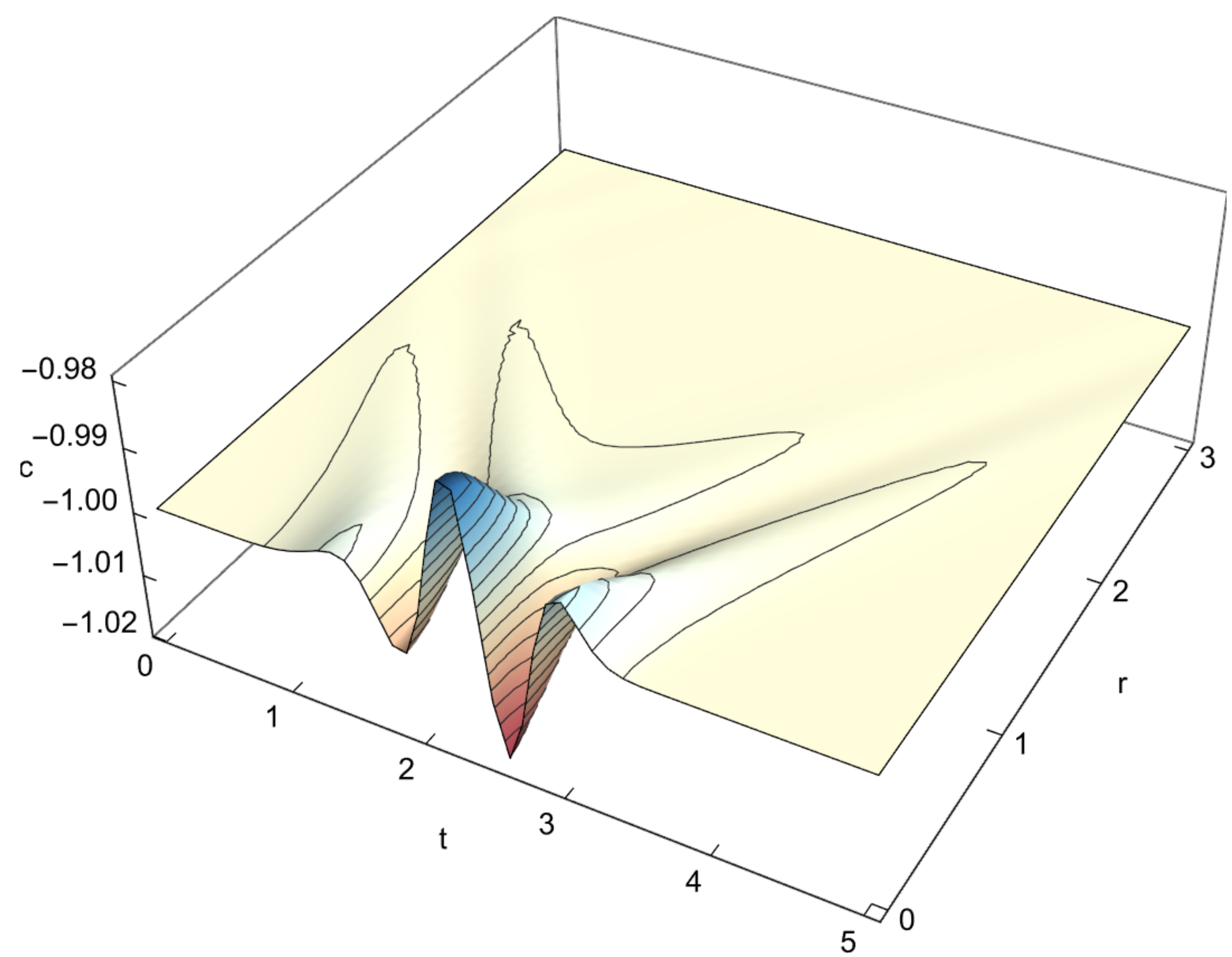
- Note: there is the dynamical spin-0 graviton mode

- We send in a Gaussian shell of scalar field.

Spherical collapse in minimal model

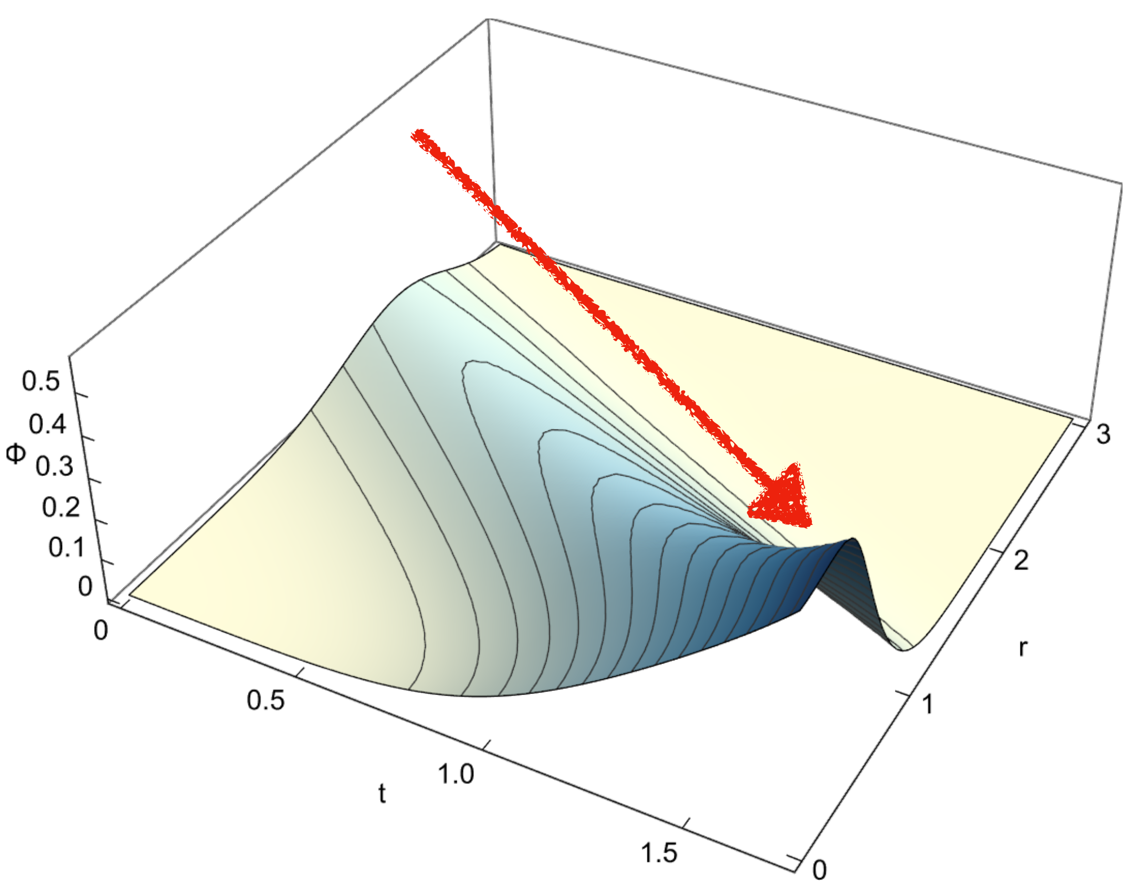
- For a weak pulse it disperses…

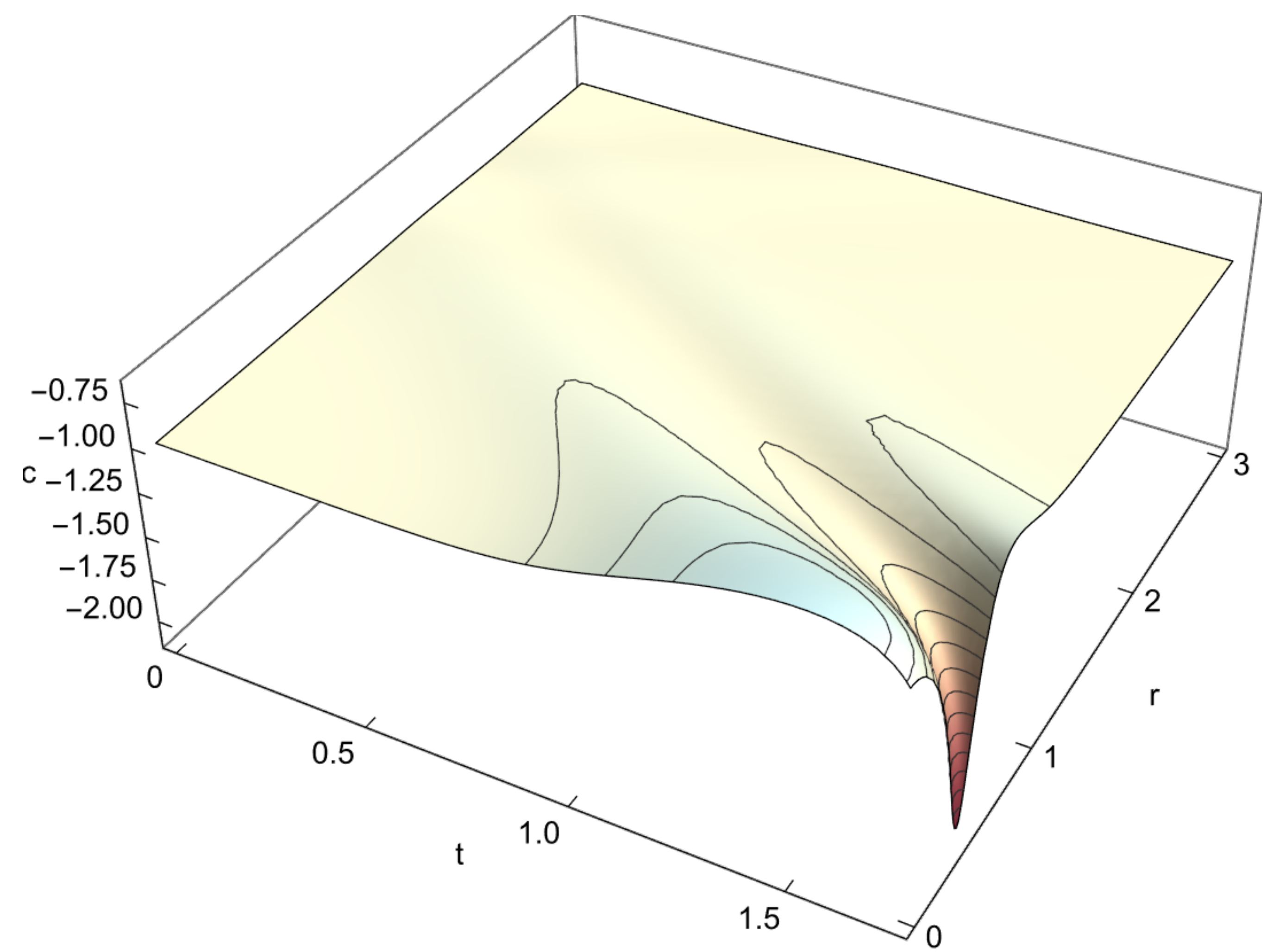




Spherical collapse in minimal model

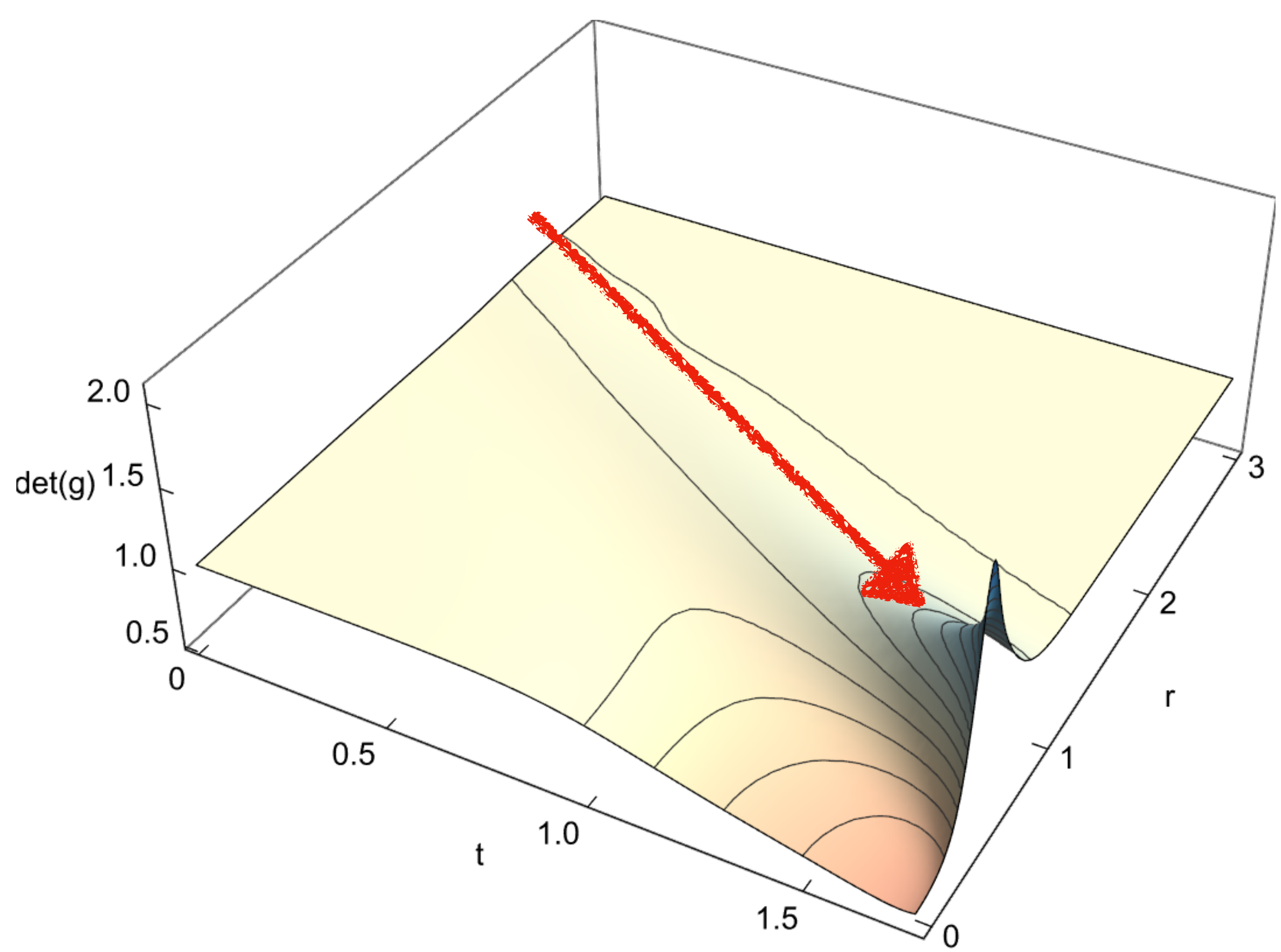
- But for sufficiently strong initial data the evolution breaks down…





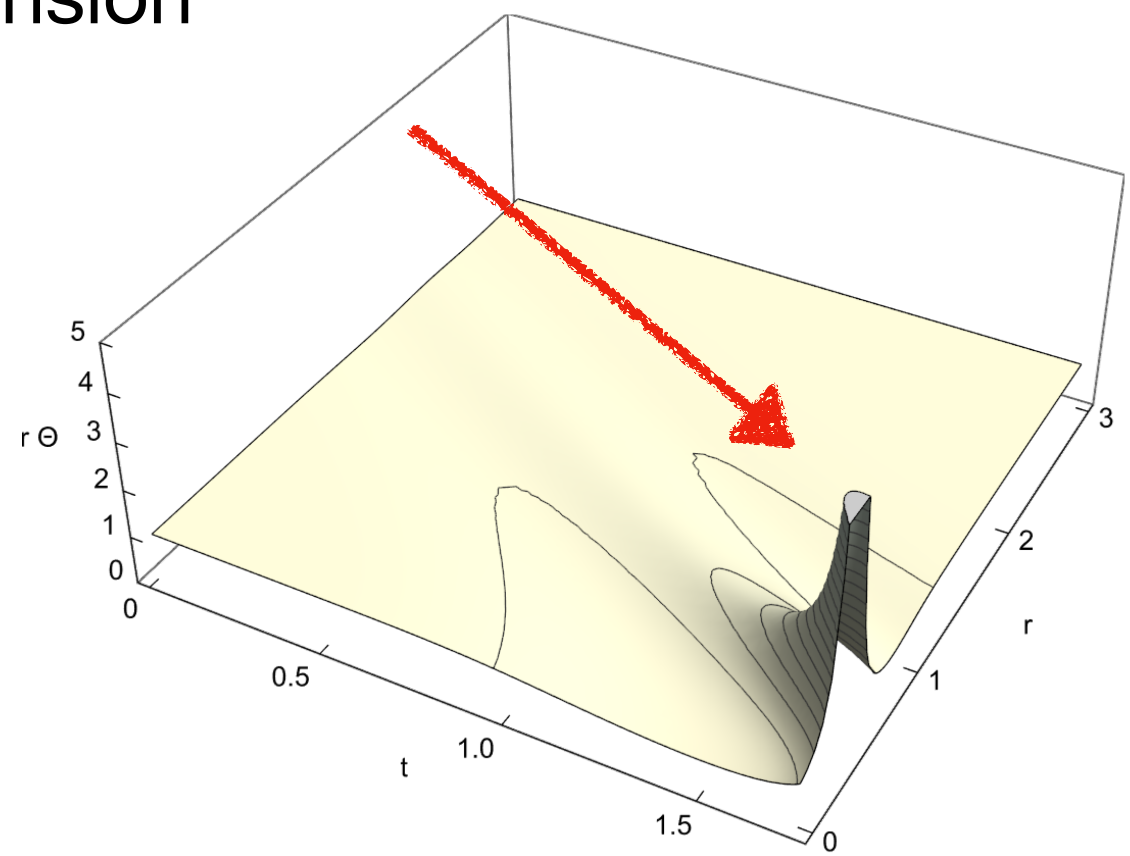
Spherical collapse in minimal model

- It appears that a singularity develops away from the origin; here seen in the metric determinant
- Seems not to be curvature singularity but we are not certain



Spherical collapse in minimal model

- Is this hidden behind a horizon? Apparently not as seen from outer expansion



- If we reduce the graviton mass, strong coupling occurs sooner….

Spherical dynamics for $m \to 0$

[Albertini, Kozuszek, TW '24]

- Assume spherical symmetry (but time dependence)
- Require that $K_{\mu\nu}$ everywhere has the ***same*** signature as that of the Minkowski vacuum solution
- Then writing $m_1^2 = \alpha m^2$, $m_2^2 = \beta m^2$ we find;

$$\frac{8\pi G\rho}{m^2} > \frac{3\alpha^2}{8\beta}\left(\frac{\rho}{P}\right)^3 \qquad \alpha + \beta = 1 \ , \quad \alpha, \beta > 0$$

- Means matter can't be too *non-relativistic* or else one must have a singularity

Beyond spherical symmetry

- Spherical symmetry seems problematic … but it is non-generic. So what about beyond spherical symmetry?

- Note: *exact cosmological symmetry is also not allowed!*

Beyond spherical symmetry

- The previous 3+1 dynamical system in principle allows one to go beyond spherical symmetry.

- However it was thought that the theory was probably ill-posed — this is ok as it is an *effective field theory.*

- It isn't obvious that GR is well-posed [Choquet-Bruhat '52]

- Ill-posedness could be cured by higher derivative operators cf. viscous relativistic hydro

- But it would be difficult to simulate.

Well posedness of minimal theory

[Kozuszek, TW '24]

- For minimal theory there is an elegant `harmonic' formulation where vector constraint is removed:
- Recall vector constraint $\xi_\mu = -2(K^{-1})^{\alpha\beta}\partial_{[\mu}K_{\alpha]\beta}$
- Evolve: $\mathcal{E}^H_{\mu\nu} \equiv R_{\mu\nu} - 2\nabla_{(\mu}\xi_{\nu)} + m^2\bar{M}_{\mu\nu} - 8\pi G_N\bar{T}_{\mu\nu} = 0$
- Then: $\nabla^2\xi_\mu + R_\mu{}^\alpha\xi_\alpha = m^2\eta_{\mu\alpha}(K^{-1})^{\alpha\beta}\xi_\beta$
- Ensuring $\xi_\mu \, , \, \dot{\xi}_\mu = 0$ then ensures vector constraint holds

Well posedness of minimal theory

- Then for harmonic formulation of minimal theory we may write the system in first order form using the previous variables as;

$$\underline{\underline{A}}[u] \cdot (\partial_t \underline{u}) + \underline{\underline{P}}^i[u] \cdot (\partial_i \underline{u}) + \underline{C}[\underline{u}] = 0$$

- Define from this $\underline{\underline{M}}(k_i) \equiv -\underline{\underline{A}}^{-1} \underline{\underline{P}}^i k_i$

- If matrix M is diagonalizable with real eigenvalues then well-posed — `strongly hyperbolic'

[see Papallo, Reall '17]

Well posedness of minimal theory

- Analysing this matrix M we found:
    - The linear theory about flat space is well-posed
    - The non-linear theory near flat space is generically well-posed

- Analysis hinges on understanding degenerate eigenvalues
- Interestingly spin-2 graviton always controlled by inverse metric
- Spin 1 modes become birefringent

Summary

- Status of massive gravity is unclear; particularly in spherical symmetry where singularities appear to form generically.

- Can GR behaviour be recovered? It the dynamics well behaved?

- The only way to proceed is numerical, and explore non-symmetric generic dynamics

- We now have a dynamical formulation which appears well-posed for the minimal theory

- Currently working to extend this to non-minimal case; and then the next steps are to implement 3+1 code

The End!

Extra slides

- Status of massive gravity is unclear; particularly in spherical symmetry where singularities appear to form generically.
- Can Vainshtein screening work?
- The only way to proceed is numerical, and explore non-symmetric dynamics
- We now have a dynamical formulation which appears well-posed for the minimal theory
- Currently working to extend this to non-minimal case; and then the next steps are to implement 3+1 code

dRGT massive gravity

- Important for quantum stability of the theory. While the cut off is naturally given by M_{Pl}, massive gravity naively becomes quantum mechanically strongly coupled at the much lower scale;

$$\Lambda_3 = (M_{pl} m^2)^{1/3} \sim \left(1000\, km\right)^{-1}$$

- However, this is computed for fluctuations about flat space. It is believed this scale is much higher expanding about a background where the Vainshtein mechanics is function.

Well posedness of minimal theory

- Then for harmonic formulation of minimal theory we may write the system in first order form using the previous variables as;

$$\underline{\underline{A}}[u] \cdot (\partial_t \underline{u}) + \underline{\underline{P}}^i[u] \cdot (\partial_i \underline{u}) + \underline{C}[\underline{u}] = 0$$

[see Papallo, Reall '17]

- Define from this $\underline{\underline{M}}(k_i) \equiv -\underline{\underline{A}}^{-1} \underline{\underline{P}}^i k_i$

- Then perturbing about a background $\underline{u} = \underline{\bar{u}} + \epsilon\, \delta \underline{u}$

- Formal solution near $x_{(0)}$

$$\delta \underline{u}(t, x^i) = \int d\vec{k} e^{-ik_i (x^i - x^i_{(0)})} \exp(i \underline{\underline{M}}(k_i)(t - t_0)) \cdot \underline{v}(k_i)$$

- If matrix M is diagonalizable with real eigenvalues then;

$$\left\| \exp(i \underline{\underline{M}}(k_i)(t - t_0)) \right\| \leq f(t - t_0) \quad \Longrightarrow \quad \|\delta u\| (t) = f(t - t_0) \|\delta u\| (t_0)$$