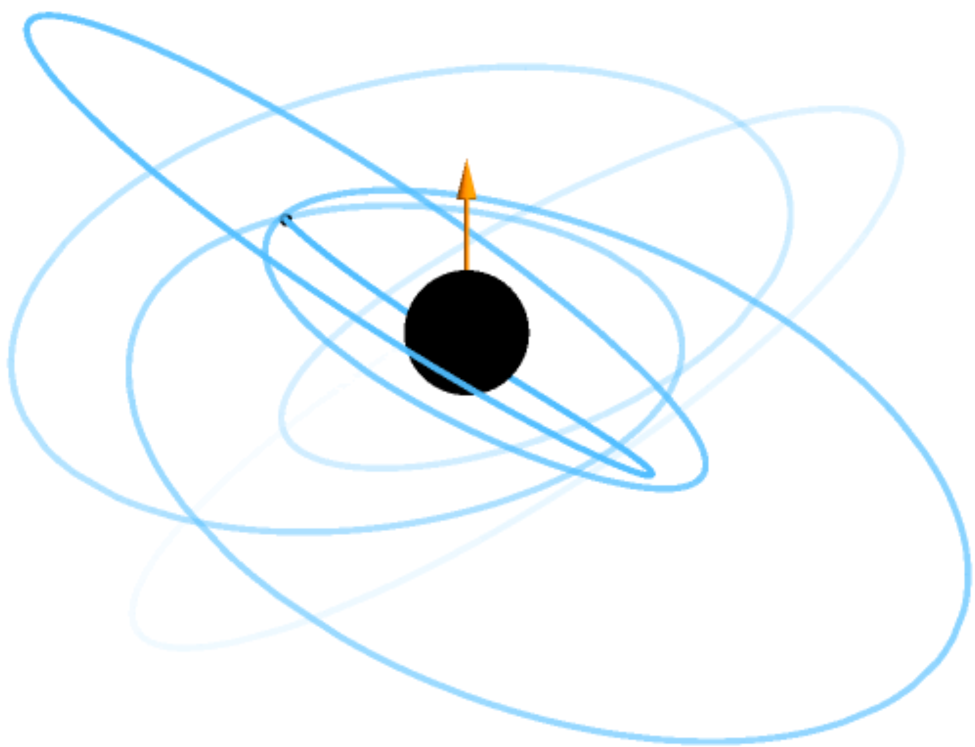
Gravitational waveform models for extreme mass ratio inspirals via the self-force approach



Niels Warburton
University College Dublin

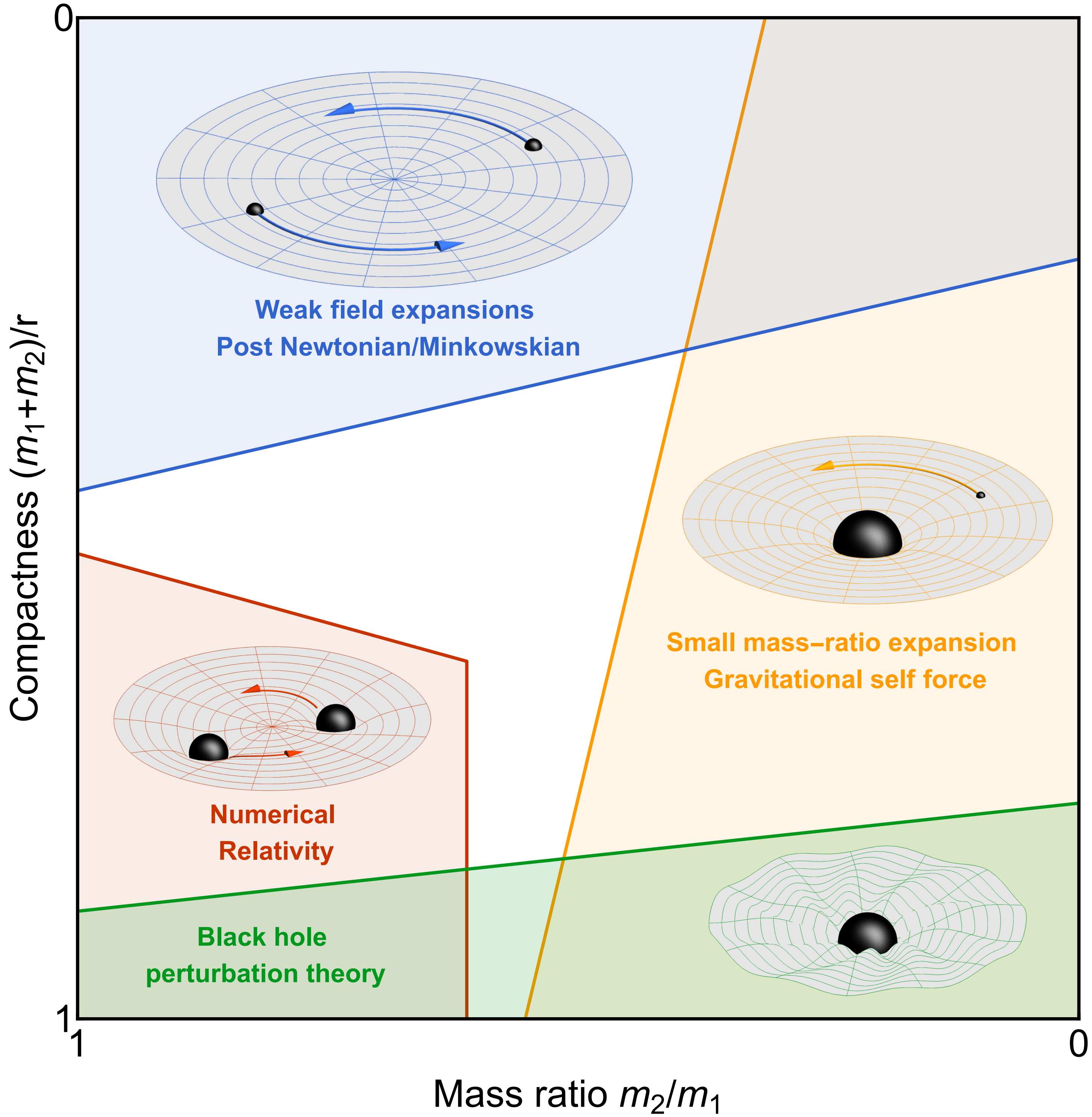
5th European Physics Society Conference on Gravitation
Unlocking Gravity Through Computation
11th December 2024

THE ROYAL SOCIETY

Science Foundation Ireland sfi For what's next

UCD DUBLIN

Compact binary parameter space



[Image credit: LISA Consortium Waveform Working Group]

Modelling goals

Our EMRI waveform models must be
accurate, efficient and extensive

Modelling goals

We need the phase error in the waveform to be 'small' $\implies$ we must include adiabatic and post-adiabatic corrections

Our EMRI waveform models must be accurate, efficient and extensive

Modelling goals

We need the phase error in the waveform to be 'small' $\Longrightarrow$ we must include adiabatic and post-adiabatic corrections

Our EMRI waveform models must be accurate, efficient and extensive

Search and parameter estimation requires millions of templates $\Longrightarrow$ each waveform must be computed in less than 1 second

Modelling goals

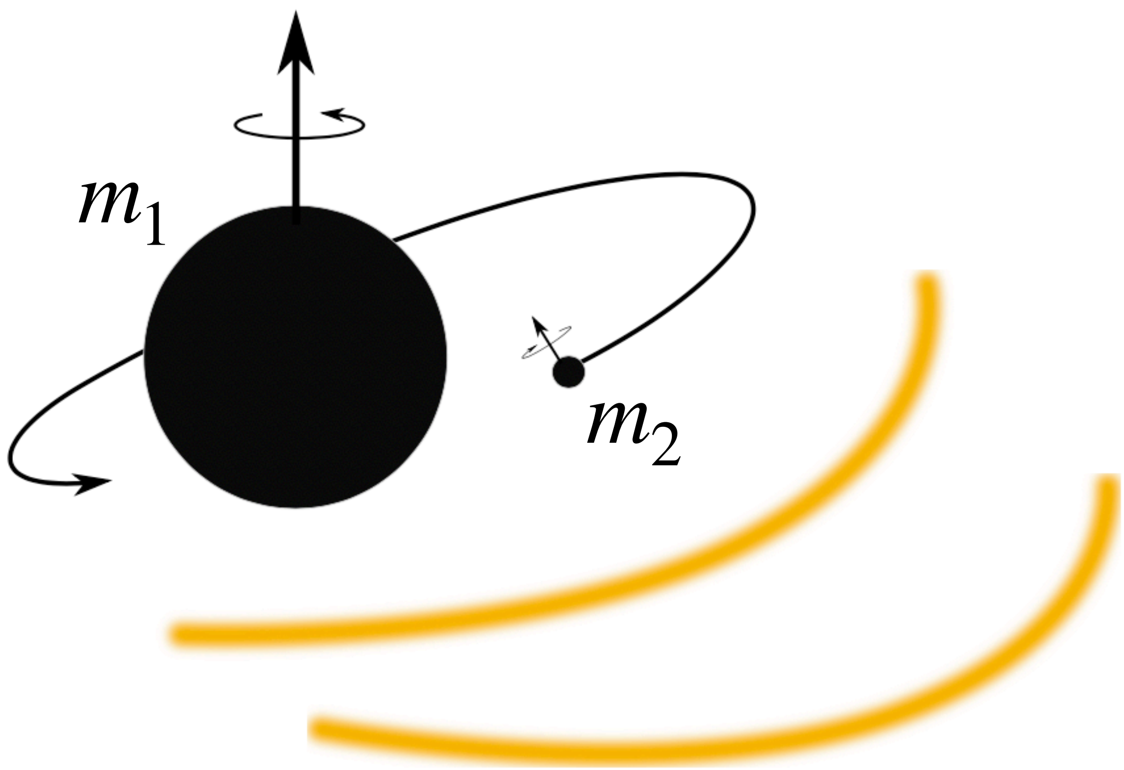
We need the phase error in the waveform to be 'small' $\implies$ we must include adiabatic and post-adiabatic corrections

EMRIs will be very generic $\implies$ our models must span the full parameter space of eccentric, precessing systems

Our EMRI waveform models must be accurate, efficient and extensive

Search and parameter estimation requires millions of templates $\implies$ each waveform must be computed in less than 1 second

Gravitational self-force approach



[Image credit: Adam Pound]

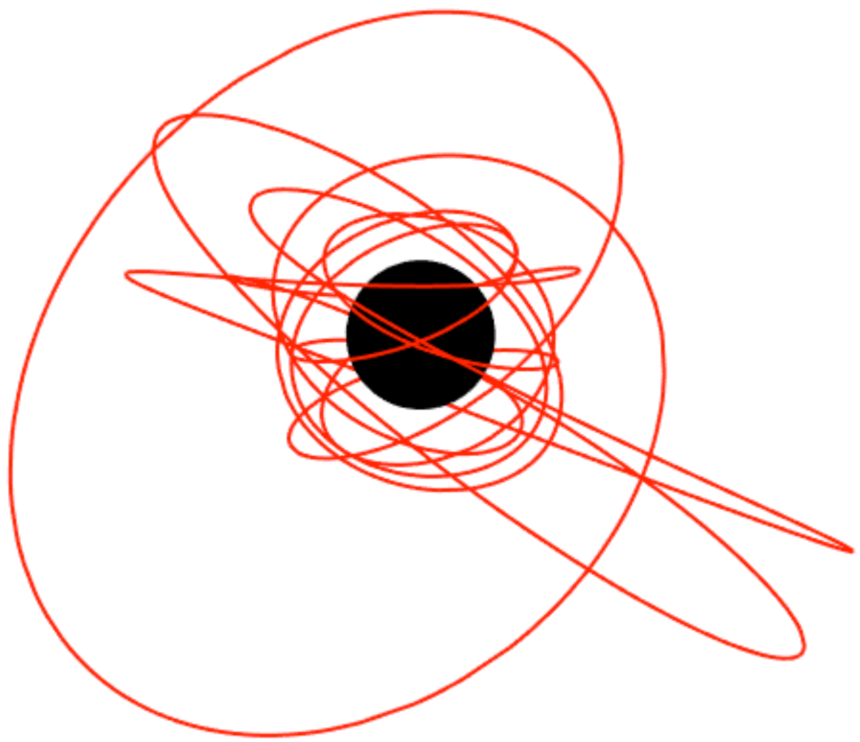
- $\epsilon = 1/q = m_2/m_1 \ll 1$
- Small body perturbs spacetime:

$$g_{\alpha\beta} = g^{\text{Kerr}}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \ldots$$

- Perturbation affects m_2's motion:

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon f^\mu_{(1)} + \epsilon^2 f^\mu_{(2)} + \ldots$$

Zeroth order: geodesics in Kerr



[Image created using the BHPToolkit KerrGeodesics package]

- constants $J_A = (m_1, \chi_1, E, L_z, Q)$
  - Energy E
  - angular momentum L_z
  - Carter constant Q
- phases $\varphi_A = (\varphi_r, \varphi_\theta, \varphi_\phi)$ with frequencies $\Omega_A(J_B)$
- Simple ODEs:

$$\frac{d\varphi_A}{dt} = \Omega_A(J_B)$$

$$\frac{dJ_A}{dt} = 0$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

- evolution due to the self-force:

$$\frac{d\tilde{\varphi}_A}{dt} = \Omega_A(\tilde{J}_B)$$

$$\frac{d\tilde{J}_A}{dt} = \epsilon\left[F_A^{(0)}(\tilde{J}_B) + \epsilon F_A^{(1)}(\tilde{J}_B) + \mathcal{O}(\epsilon^2)\right]$$

- waveform:

$$h_{\ell m} = \sum_{k^i}\left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3)\right] e^{-i(m\tilde{\varphi}_\phi + k^i\tilde{\varphi}_i)}$$

- Secondary spin: (i) new slow parameters
  (ii) new precession phase

Accuracy and post-adiabatic counting

phases: $$\tilde{\varphi}_A = \epsilon^{-1}\varphi_A^{(0)}(\Omega_B) + \epsilon^{0}\varphi_A^{(1)}(\Omega_B) + \mathcal{O}(\epsilon)$$

Accuracy and post-adiabatic counting

phases: $\tilde{\varphi}_A = \epsilon^{-1} \varphi_A^{(0)}(\Omega_B) + \epsilon^0 \varphi_A^{(1)}(\Omega_B) + \mathcal{O}(\epsilon)$

Adiabatic (0PA)

From the orbit averaged piece of first-order self-force $\langle f^\alpha_{(1)} \rangle$

$\langle f^\alpha_{(1)} \rangle$ can be related to the fluxes, thus avoiding a local calculation of the self-force

0PA is sufficient for detection and rough parameter estimation for astrophysics of EMRIs of bright sources

Accuracy and post-adiabatic counting

phases: $\tilde{\varphi}_A = \epsilon^{-1} \varphi_A^{(0)}(\Omega_B) + \epsilon^{0} \varphi_A^{(1)}(\Omega_B) + \mathcal{O}(\epsilon)$

Adiabatic (0PA)

From the orbit averaged piece of first-order self-force $\langle f^{\alpha}_{(1)} \rangle$

$\langle f^{\alpha}_{(1)} \rangle$ can be related to the fluxes, thus avoiding a local calculation of the self-force

0PA is sufficient for detection and rough parameter estimation for astrophysics of EMRIs of bright sources

Post-adiabatic order (1PA)

Two contributions:

- oscillatory pieces of the first order self-force $\breve{f}^{\alpha}_{(1)}$
- second-order orbit averaged self-force $\langle f^{\alpha}_{(2)} \rangle$

Needed to extract all sources

Needed for precision tests of GR

Potential application to IMRIs

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

- Write metric as product of slowly evolving amplitudes and a rapidly evolving phase:

$$h^{(n)}_{\alpha\beta} = \sum_{k^A} h^{(n,k^A)}_{\alpha\beta}(\tilde{J}_A; x^i) e^{-ik^A\tilde{\varphi}_A} \qquad x^i = \{r, \theta, \phi\}$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

- Write metric as product of slowly evolving amplitudes and a rapidly evolving phase:

$$h^{(n)}_{\alpha\beta} = \sum_{k^A} h^{(n,k^A)}_{\alpha\beta}(\tilde{J}_A; x^i) e^{-ik^A \tilde{\varphi}_A} \qquad x^i = \{r, \theta, \phi\}$$

- Substitute this into the Einstein field equations. By treating t as a function of $(\tilde{J}_A, \tilde{\varphi}_B)$ time derivatives can be computed via:

$$\partial_t = \sum_A \dot{\tilde{\varphi}}_A \partial_{\tilde{\varphi}_A} + \dot{\tilde{J}}_A \partial_{J_A} = \sum_A \Omega_A \partial_{\tilde{\varphi}_A} + \epsilon F^{(0)}_A \partial_{J_A} + \mathcal{O}(\epsilon)^2$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

- Write metric as product of slowly evolving amplitudes and a rapidly evolving phase:

$$h^{(n)}_{\alpha\beta} = \sum_{k^A} h^{(n,k^A)}_{\alpha\beta}(\tilde{J}_A; x^i) e^{-ik^A\tilde{\varphi}_A} \qquad x^i = \{r, \theta, \phi\}$$

- Substitute this into the Einstein field equations. By treating t as a function of $(\tilde{J}_A, \tilde{\varphi}_B)$ time derivatives can be computed via:

$$\partial_t = \sum_A \dot{\tilde{\varphi}}_A \partial_{\tilde{\varphi}_A} + \dot{\tilde{J}}_A \partial_{J_A} = \sum_A \Omega_A \partial_{\tilde{\varphi}_A} + \epsilon F^{(0)}_A \partial_{J_A} + \mathcal{O}(\epsilon)^2$$

- Lots of extra detail… (gauge choice, regularisation via matched expansions, numerical methods, etc). The first calculation of $h^{(2)}_{\alpha\beta}$ took ~10 years to work through all the details.

Near-identity averaging transformations

Lynch, van de Meent, NW, Witzany

- What are those $\tilde{J}_A$ variables?

$$\frac{d\tilde{J}_A}{dt} = \epsilon \left[F_A^{(0)}(\tilde{J}_B) + \epsilon F_A^{(1)}(\tilde{J}_B) + \mathcal{O}(\epsilon^2) \right]$$

Near-identity averaging transformations

Lynch, van de Meent, NW, Witzany

- What are those $\tilde{J}_A$ variables? $$\frac{d\tilde{J}_A}{dt} = \epsilon\left[F_A^{(0)}(\tilde{J}_B) + \epsilon F_A^{(1)}(\tilde{J}_B) + \mathcal{O}(\epsilon^2)\right]$$

- Evolution of J_A depends on phases φ_C:

$$\frac{dJ_A}{dt} = \epsilon\left[F_A^{(0)}(J_B, \varphi_C) + \epsilon F_A^{(1)}(J_B, \varphi_C) + \mathcal{O}(\epsilon^2)\right]$$

Near-identity averaging transformations

Lynch, van de Meent, NW, Witzany

- What are those $\tilde{J}_A$ variables?

$$\frac{d\tilde{J}_A}{dt} = \epsilon \left[F_A^{(0)}(\tilde{J}_B) + \epsilon F_A^{(1)}(\tilde{J}_B) + \mathcal{O}(\epsilon^2) \right]$$

- Evolution of J_A depends on phases φ_C:

$$\frac{dJ_A}{dt} = \epsilon \left[F_A^{(0)}(J_B, \varphi_C) + \epsilon F_A^{(1)}(J_B, \varphi_C) + \mathcal{O}(\epsilon^2) \right]$$

- Introduce a near-identity (averaging) transformation (NIT):

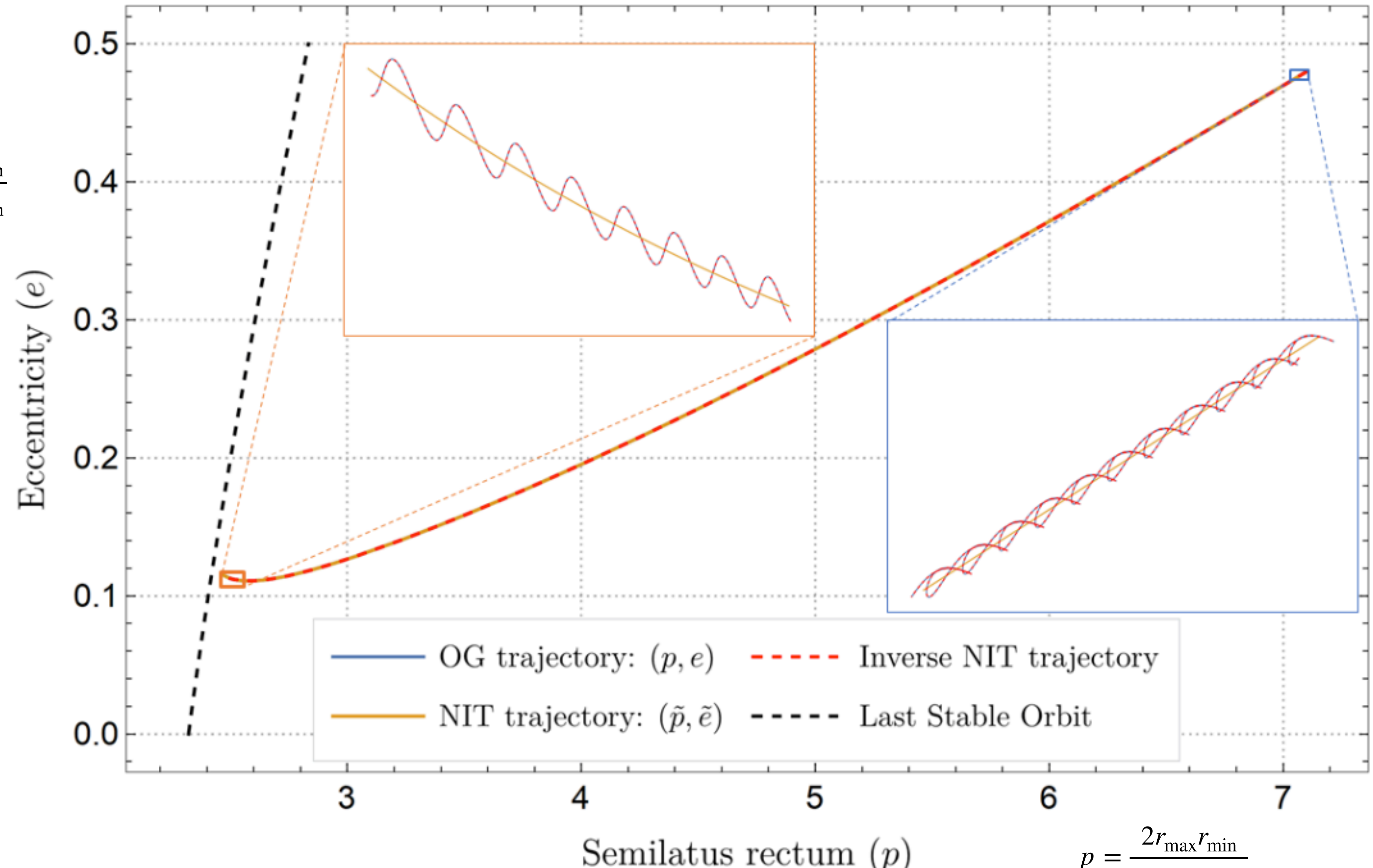
$$\tilde{J}_A = J_A + \epsilon Y_A^{(1)}(J_B, \varphi_C) + \epsilon^2 Y_A^{(2)}(J_B, \varphi_C) + \mathcal{O}(\epsilon^3)$$

- Can choose $Y_A^{(n)}$ to remove dependency on the phase in the equations for $\tilde{J}_A$

Near-identity averaging transformations

Lynch, van de Meent,
NW, Witzany

$$e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$$



$$p = \frac{2 r_{max} r_{min}}{r_{max} + r_{min}}$$

Phase space trajectory computation goes from taking hours to to taking milliseconds

Native rapid waveform generation

Offline step

- solve field equations for amplitudes $h^{(n)}_{lmk^i}$ and forcing functions $F^{(n-1)}_A$ on a grid of $\tilde{J}_A$ values

Online step

- solve ODEs for $\tilde{\varphi}_A$ and $\tilde{J}_A$
- Add up the mode amplitudes $h^{(n)}_{\ell mk^i}$ at each sample time
- `FastEMRIWaveforms` (**FEW**) software package can compute a 2-year long waveform in ~ 10 - 100ms

Offline calculations: field equations

$$G_{\alpha\beta}[g^{\rm Kerr}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

- Expand and use multiscale expansion to get:

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)}] = T^{(1)}_{\alpha\beta}$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = S_{\rm eff}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

Offline calculations: field equations

$$G_{\alpha\beta}[g^{\text{Kerr}}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

- Expand and use multiscale expansion to get:

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)}] = T^{(1)}_{\alpha\beta}$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = S_{\text{eff}}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- The effective source, S_{eff}, is non-compact due to contributions from $h^{(1)}$ and its parametric derivative $\partial_{J_A} h^{(1)}$

Offline calculations: field equations

$$G_{\alpha\beta}[g^{\text{Kerr}}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

- Expand and use multiscale expansion to get:

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)}] = T^{(1)}_{\alpha\beta}$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = S_{\text{eff}}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- The effective source, S_{eff}, is non-compact due to contributions from $h^{(1)}$ and its parametric derivative $\partial_{J_A} h^{(1)}$
- S_{eff} also contains terms that regularise the metric perturbation (derived through a rigorous matched asymptotic expansion procedure)

Offline calculations: field equations

$$G_{\alpha\beta}[g^{\mathrm{Kerr}}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

- Expand and use multiscale expansion to get:

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)}] = T^{(1)}_{\alpha\beta}$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = S_{\mathrm{eff}}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- The effective source, S_{eff}, is non-compact due to contributions from $h^{(1)}$ and its parametric derivative $\partial_{J_A} h^{(1)}$

- S_{eff} also contains terms that regularise the metric perturbation (derived through a rigorous matched asymptotic expansion procedure)

- Solve for: $h^{(n)}_{\alpha\beta} = \sum_{k^A} h^{(n,k^A)}_{\alpha\beta}(\tilde{J}_A; x^i) e^{-ik^A \tilde{\varphi}_A}$ $\qquad x^i = \{r, \theta, \phi\}$

Offline calculations: computational burden

$$\delta G_{\alpha\beta}[h^{(1)}] = T^{(1)}$$

$$\delta G_{\alpha\beta}[h^{(2)}] = S_{\text{eff}}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

Offline calculations: computational burden

$$\delta G_{\alpha\beta}[h^{(1)}] = T^{(1)}$$

$$\delta G_{\alpha\beta}[h^{(2)}] = S_{\rm eff}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- Vast majority of the computation time is spent computing the effective source, $S_{\rm eff}$

Offline calculations: computational burden

$$\delta G_{\alpha\beta}[h^{(1)}] = T^{(1)}$$

$$\delta G_{\alpha\beta}[h^{(2)}] = S_{\rm eff}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- Vast majority of the computation time is spent computing the effective source, $S_{\rm eff}$
- For a single circular orbit around a Schwarzschild black hole the computation time is ~500 CPU hours

Offline calculations: computational burden

$$\delta G_{\alpha\beta}[h^{(1)}] = T^{(1)}$$

$$\delta G_{\alpha\beta}[h^{(2)}] = S_{\text{eff}}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- Vast majority of the computation time is spent computing the effective source, S_{eff}
- For a single circular orbit around a Schwarzschild black hole the computation time is ~500 CPU hours
- To compute an inspiral we need ~20 orbits computed in the offline step so we can accurately interpolate the forcing functions and asymptotic metric amplitudes. This gives ~10,000 CPU hours

Offline calculations: computational burden

$$\delta G_{\alpha\beta}[h^{(1)}] = T^{(1)}$$

$$\delta G_{\alpha\beta}[h^{(2)}] = S_{\text{eff}}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- Vast majority of the computation time is spent computing the effective source, S_{eff}
- For a single circular orbit around a Schwarzschild black hole the computation time is ~500 CPU hours
- To compute an inspiral we need ~20 orbits computed in the offline step so we can accurately interpolate the forcing functions and asymptotic metric amplitudes. This gives ~10,000 CPU hours
- This is of the order of magnitude of a numerical relativity simulation (but we only need to carry out the calculation once for all mass ratios)

Offline calculations: future computational burden

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)}] = T^{(1)}_{\alpha\beta}$$
$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = S_{\text{eff}}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- For generic (eccentric, precessing) orbits we need to compute hundreds more modes per orbit

$$h_{\ell m} = \sum_{k^i} \left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3) \right] e^{-i(m\tilde{\varphi}_\phi + k^i \tilde{\varphi}_i)}$$

Offline calculations: future computational burden

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)}] = T^{(1)}_{\alpha\beta}$$
$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = S_{\rm eff}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- For generic (eccentric, precessing) orbits we need to compute hundreds more modes per orbit

$$h_{\ell m} = \sum_{k^i} \left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3)\right] e^{-i(m\tilde{\varphi}_\phi + k^i \tilde{\varphi}_i)}$$

- For generic orbits we need cover a 4D parameter space (black hole spin, semi-latus rectum, eccentricity, inclination)

Offline calculations: future computational burden

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)}] = T^{(1)}_{\alpha\beta}$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = S_{\rm eff}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- For generic (eccentric, precessing) orbits we need to compute hundreds more modes per orbit

$$h_{\ell m} = \sum_{k^i} \left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3) \right] e^{-i(m\tilde{\varphi}_\phi + k^i \tilde{\varphi}_i)}$$

- For generic orbits we need cover a 4D parameter space (black hole spin, semi-latus rectum, eccentricity, inclination)

- Naive calculation of the required resources: 500 CPU hours $\times 100 \times 20 \times 20 \times 20 \times 20 =$ 8 billion CPU hours

Offline calculations: future computational burden

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)}] = T^{(1)}_{\alpha\beta}$$
$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = S_{\rm eff}(h^{(1)}, \partial_{J_A} h^{(1)}; x^i)$$

- For generic (eccentric, precessing) orbits we need to compute hundreds more modes per orbit

$$h_{\ell m} = \sum_{k^i} \left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3)\right] e^{-i(m\tilde{\varphi}_\phi + k^i \tilde{\varphi}_i)}$$

- For generic orbits we need cover a 4D parameter space (black hole spin, semi-latus rectum, eccentricity, inclination)

- Naive calculation of the required resources: 500 CPU hours $\times 100$ $\times 20 \times 20 \times 20 \times 20 =$ 8 billion CPU hours

- New techniques needed! e.g., better numerics, reduce need for parameter space coverage by using analytic results (PN/PM)

Online step: `FastEMRIWaveforms` (FEW)

Chapman-Bird, Katz, Chua, Speri, Hughes, NW

$$h_{\ell m} = \sum_{k^i} \left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3) \right] e^{-i(m\tilde{\varphi}_\phi + k^i \tilde{\varphi}_i)}$$

- The number of $h^{(n)}_{\ell m k^i}$ that need to be summed at each time step can be in the thousands.

Online step: `FastEMRIWaveforms` (FEW)

Chapman-Bird, Katz, Chua, Speri, Hughes, NW

$$h_{\ell m} = \sum_{k^i} \left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3) \right] e^{-i(m\tilde{\varphi}_\phi + k^i \tilde{\varphi}_i)}$$

- The number of $h^{(n)}_{\ell m k^i}$ that need to be summed at each time step can be in the thousands.

- The waveform amplitudes vary slowly. These amplitudes are sampled on a sparse set of points, summed, and then upsampled

Online step: `FastEMRIWaveforms` (FEW)

Chapman-Bird, Katz, Chua, Speri, Hughes, NW

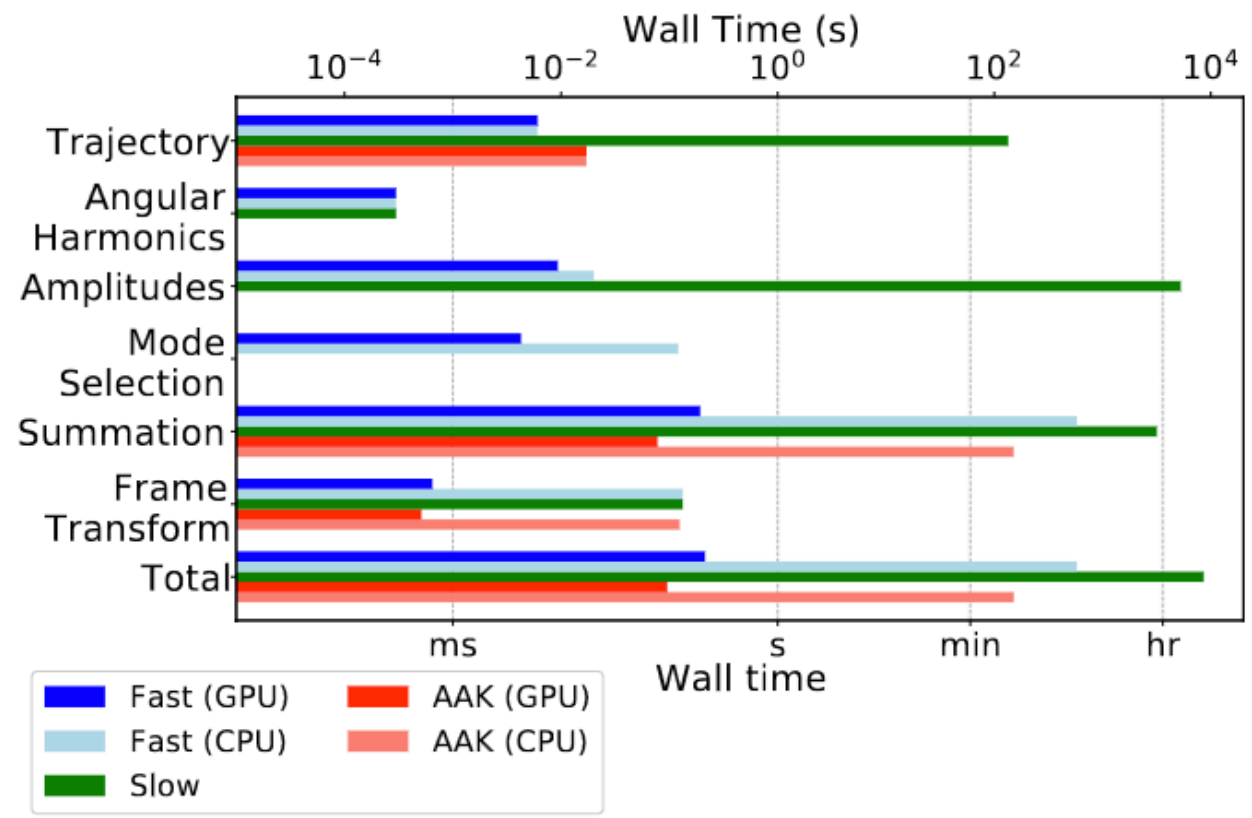
$$h_{\ell m} = \sum_{k^i} \left[\epsilon h^{(1)}_{\ell m k^i}(\tilde{J}_A) + \epsilon^2 h^{(2)}_{\ell m k^i}(\tilde{J}_A) + \mathcal{O}(\epsilon^3) \right] e^{-i(m\tilde{\varphi}_\phi + k^i \tilde{\varphi}_i)}$$

- The number of $h^{(n)}_{\ell m k^i}$ that need to be summed at each time step can be in the thousands.

- The waveform amplitudes vary slowly. These amplitudes are sampled on a sparse set of points, summed, and then upsampled

- GPU acceleration takes generation time down from minutes to milliseconds

- Relativistic adiabatic (0PA) Kerr equatorial model will be publicly available soon

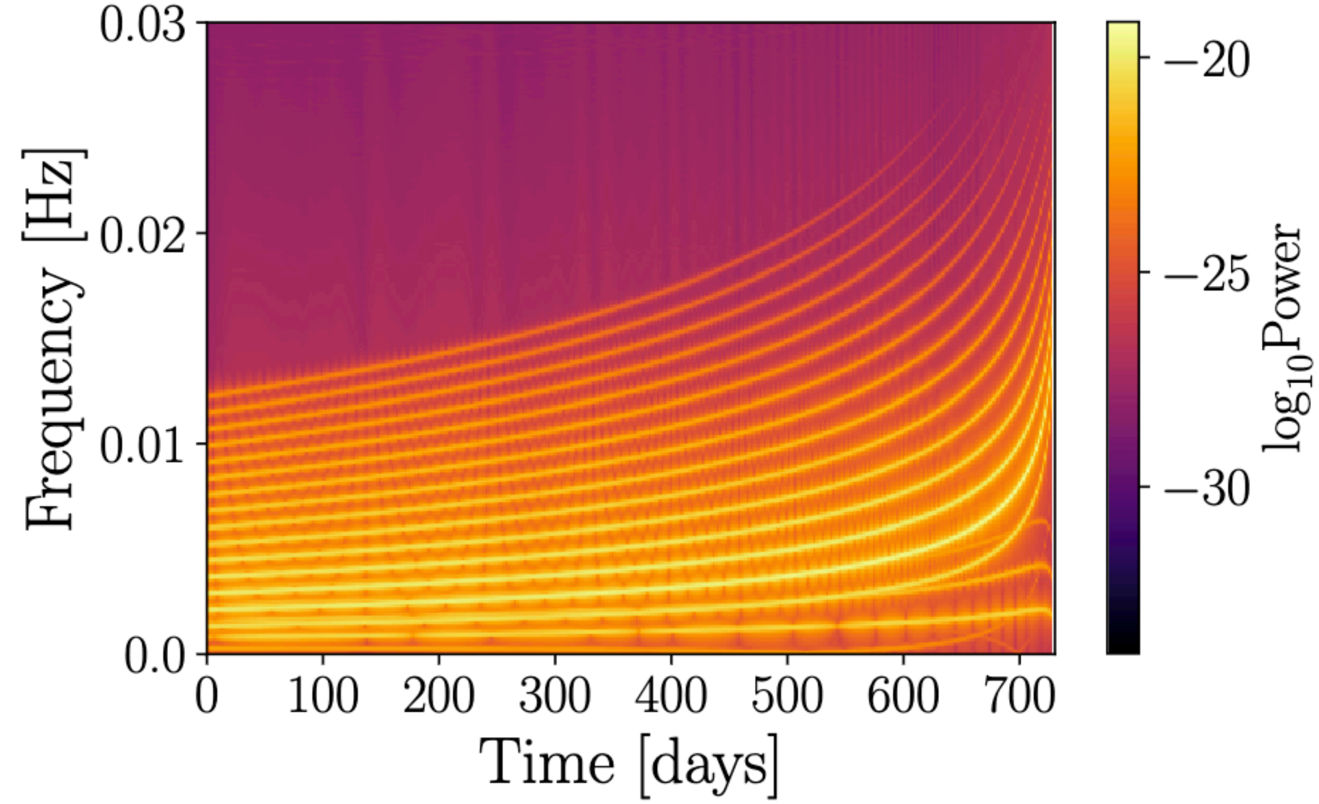


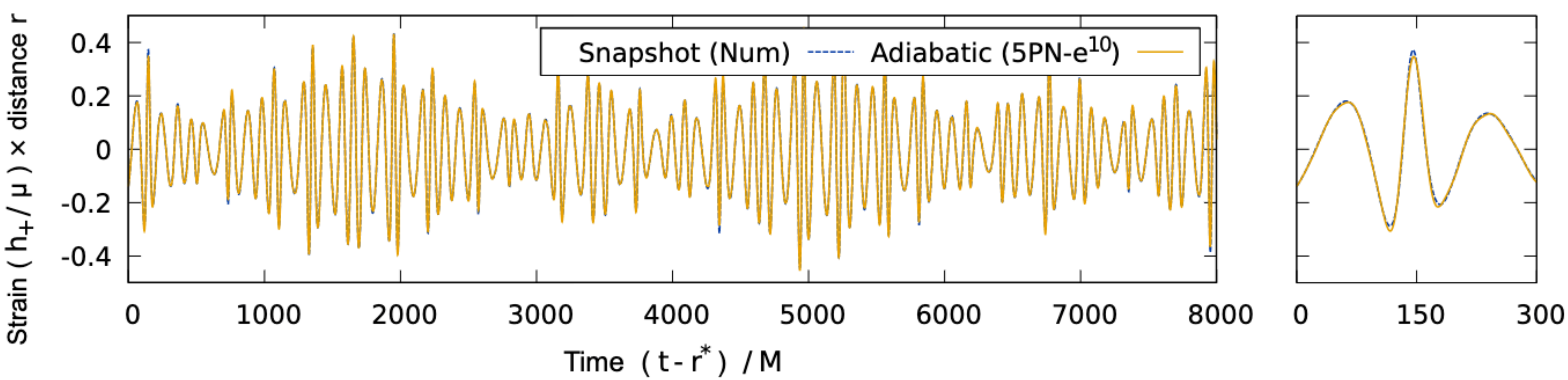
Parameter space coverage at 0PA

Isoyama et al.
FEW developers

In FEW:

- generic orbits in Kerr: 5.5PN-e^{10} approximation
- equatorial orbits in Kerr: full relativistic waveforms in time or frequency domain
- kludge models

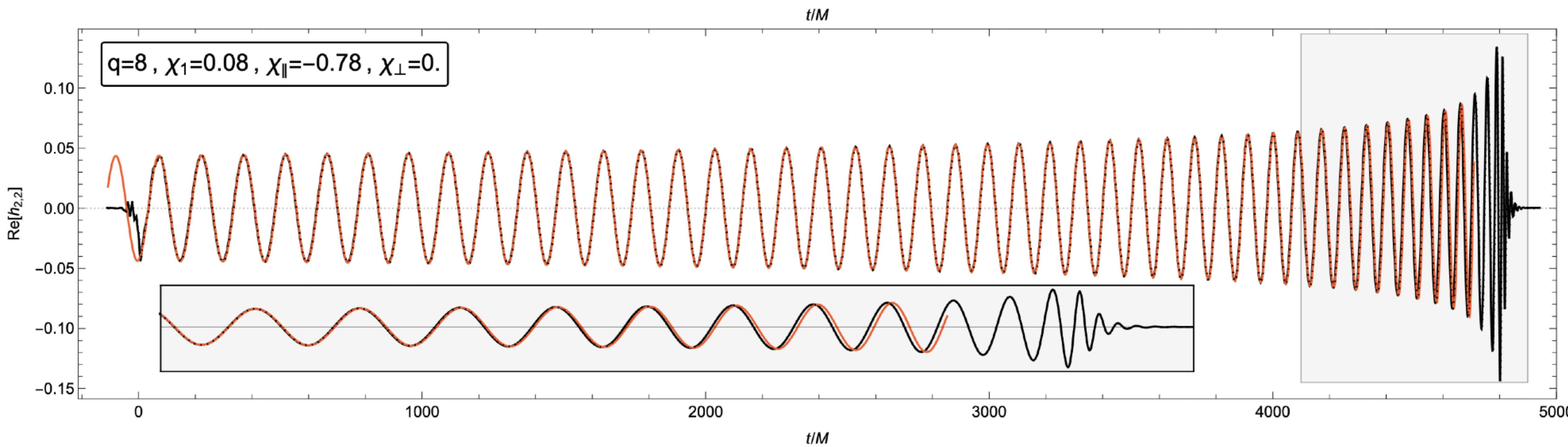




Complete 1PA inspiral waveforms

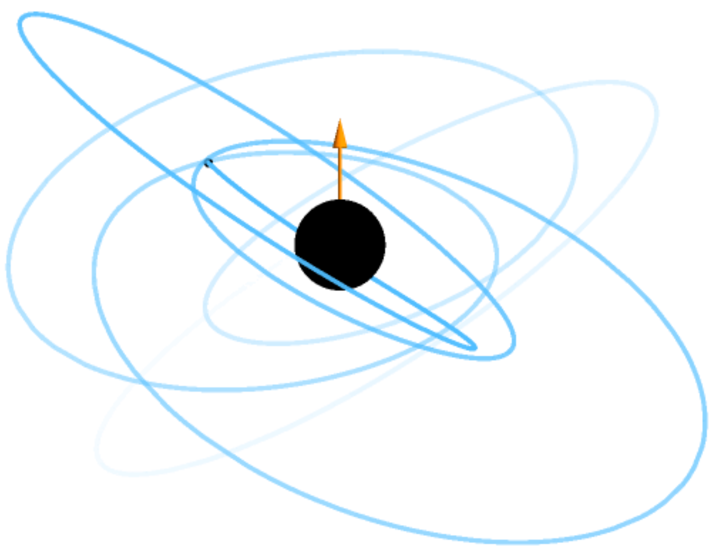
Pound, NW, Wardell,
Durkan, Miller, Mathews

Comparison with NR waveform from SXS collaboration



- Complete quasi-circular 1PA inspiral model with generic (precessing) secondary spin, linear-in-ϵ primary spin and evolving m_1 and χ_1
- Implementation in FEW will be public soon

Conclusions



[Movie credit: Philip Lynch]

- Using the post-adiabatic (multiscale) expansion we can compute $h^{(1)}_{\alpha\beta}$ and $h^{(2)}_{\alpha\beta}$
- There is a native, fast waveform generation scheme, which when combined with FEW gives EMRI waveforms in 10s of milliseconds
- Post-adiabatic waveforms agree very remarkably with NR waveforms even for $q \simeq 10$. This suggests we can model IMRIs with 1PA waveforms.
- Once offline step is complete (huge task) the online waveform generation time is roughly the same for all orbital configurations

Extra slides

Adding more physics

- So long as your extra physics acts on a longtime scale (or can be NIT'ed), the equations of motion become:

$$\frac{d\tilde{\varphi}_A}{dt} = \Omega_A(\tilde{J}_B)$$

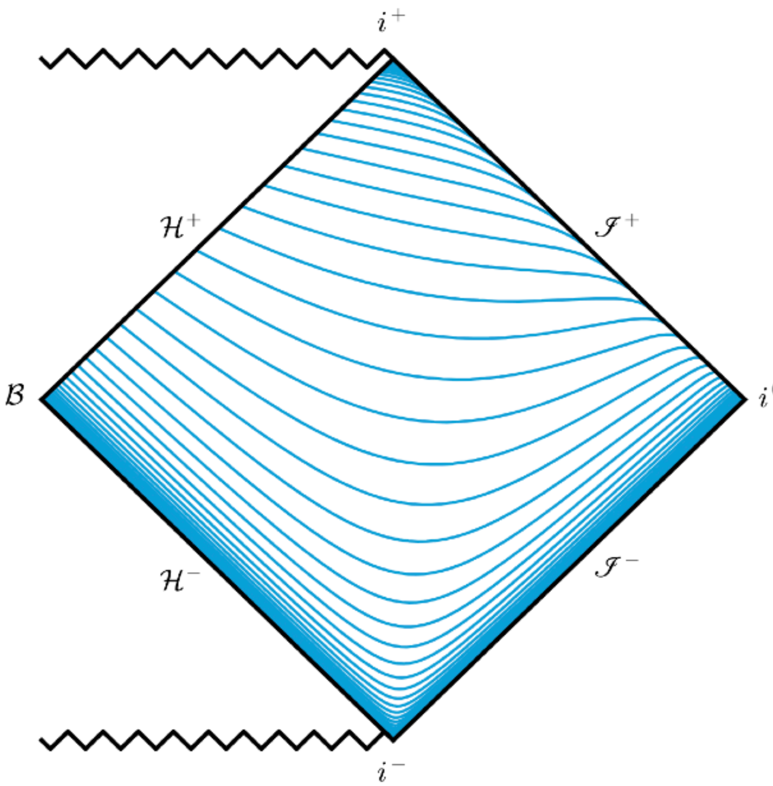
$$\frac{d\tilde{J}_A}{dt} = \epsilon\left[F_A^{(0)}(\tilde{J}_B) + \epsilon F_A^{(1)}(\tilde{J}_B) + \mathcal{O}(\epsilon^2)\right] + \kappa F_A^{(\kappa)}(\tilde{J}_B)$$

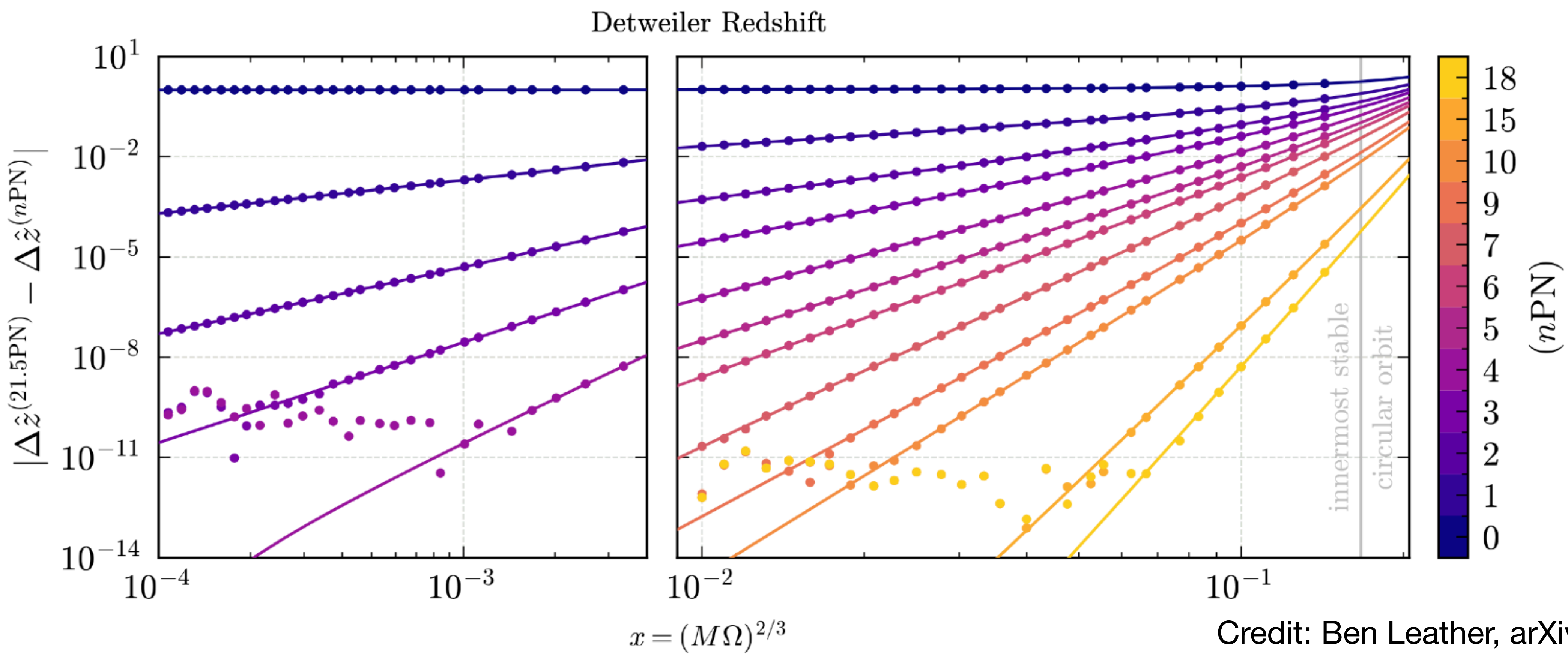
- Examples include:
  - accretion disks
  - third-body perturber (adds new resonances)
  - beyond-GR physics

- Once you have $F_A^{(\kappa)}(\tilde{J}_B)$, the multiscale framework and the modular construction of `FEW` means you can generate waveforms quickly

Offline calculations: solving for $h_{\alpha\beta}^{(n)}$

- To solve for the metric perturbations we are moving to a hyperboloidal, compactified framework
- Solve for the metric perturbation using spectral methods
- Significant improvement over the variations of parameter approach used previously





Credit: Ben Leather, arXiv:2411.14976

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

$$G_{\alpha\beta}[g^{\text{Kerr}}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

$$G_{\alpha\beta}[g^{\text{Kerr}}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

- Expand the Einstein and stress-energy tensors as

$$T_{\alpha\beta} = \epsilon T^{(1)}_{\alpha\beta} + \epsilon^2 T^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3),$$

$$G_{\alpha\beta}[g] = \epsilon \delta G_{\alpha\beta}[h^{(1)}] + \epsilon^2 \left[\delta G_{\alpha\beta}[h^{(2)}] + \delta^2 G_{\alpha\beta}[h^{(1)}, h^{(1)}] \right] + \mathcal{O}(\epsilon^3),$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

$$G_{\alpha\beta}[g^{\text{Kerr}}_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta}] = 8\pi T_{\alpha\beta}$$

- Expand the Einstein and stress-energy tensors as

$$T_{\alpha\beta} = \epsilon T^{(1)}_{\alpha\beta} + \epsilon^2 T^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3),$$

$$G_{\alpha\beta}[g] = \epsilon\delta G_{\alpha\beta}[h^{(1)}] + \epsilon^2\left[\delta G_{\alpha\beta}[h^{(2)}] + \delta^2 G_{\alpha\beta}[h^{(1)}, h^{(1)}]\right] + \mathcal{O}(\epsilon^3),$$

- Write metric as product of slowly evolving amplitudes and a rapidly evolving phase:

$$h^{(n)}_{\alpha\beta} = \sum_m h^{(n,m)}_{\alpha\beta}(\Omega; x^i)e^{-im\varphi_p} \qquad x^i = \{r, \theta, \phi\}$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

$$\epsilon\delta G_{\alpha\beta}[h^{(1)}] + \epsilon^2\left[\delta G_{\alpha\beta}[h^{(2)}] + \delta^2 G_{\alpha\beta}[h^{(1)}, h^{(1)}]\right] = \epsilon T^{(1)}_{\alpha\beta} + \epsilon^2 T^{(2)}_{\alpha\beta}$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

$$\epsilon\delta G_{\alpha\beta}[h^{(1)}] + \epsilon^2 \left[\delta G_{\alpha\beta}[h^{(2)}] + \delta^2 G_{\alpha\beta}[h^{(1)}, h^{(1)}]\right] = \epsilon T^{(1)}_{\alpha\beta} + \epsilon^2 T^{(2)}_{\alpha\beta}$$

- Substitute multiscale expansion into the Einstein field equations. By treating t as a function of (Ω, φ_p) time derivatives can be computed via:

$$\partial_t = \dot{\varphi}_p \partial_{\varphi_p} + \dot{\Omega}\partial_\Omega = \Omega\partial_{\varphi_p} + \epsilon F^{(0)}(\Omega)\partial_\Omega + \mathcal{O}(\epsilon^2)$$

Post-adiabatic (multi-scale) expansion

Hinderer, Flanagan, Miller, Pound, Moxon, Grant

$$\epsilon \delta G_{\alpha\beta}[h^{(1)}] + \epsilon^2 \left[\delta G_{\alpha\beta}[h^{(2)}] + \delta^2 G_{\alpha\beta}[h^{(1)}, h^{(1)}] \right] = \epsilon T^{(1)}_{\alpha\beta} + \epsilon^2 T^{(2)}_{\alpha\beta}$$

- Substitute multiscale expansion into the Einstein field equations. By treating t as a function of (Ω, φ_p) time derivatives can be computed via:

$$\partial_t = \dot{\varphi}_p \partial_{\varphi_p} + \dot{\Omega} \partial_\Omega = \Omega \partial_{\varphi_p} + \epsilon F^{(0)}(\Omega) \partial_\Omega + \mathcal{O}(\epsilon^2)$$

- Expand the linearised and second-order Einstein tensors as

$$\delta G_{\alpha\beta} = \delta G^{[0]}_{\alpha\beta} + \epsilon \delta G^{[1]}_{\alpha\beta} + \mathcal{O}(\epsilon^2)$$

$$\delta^2 G_{\alpha\beta} = \delta^2 G^{[0]}_{\alpha\beta} + \mathcal{O}(\epsilon)$$

Field equations

- Field equations for each m-mode take the form:

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)}] = T^{(1)}_{\alpha\beta}$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)}] = T^{(2)}_{\alpha\beta} - \delta^2 G^{[0]}_{\alpha\beta}[h^{(1)}, h^{(1)}] - \delta G^{[1]}_{\alpha\beta}[h^{(1)}]$$

Field equations

- Field equations for each m-mode take the form:

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)R} + h^{(1)P}] = T^{(1)}_{\alpha\beta}$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R} + h^{(2)P}] = T^{(2)}_{\alpha\beta} - \delta^2 G^{[0]}_{\alpha\beta}[h^{(1)}, h^{(1)}] - \delta G^{[1]}_{\alpha\beta}[h^{(1)}]$$

Field equations

- Field equations for each m-mode take the form:

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)R} + h^{(1)P}] = T^{(1)}_{\alpha\beta}$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R} + h^{(2)P}] = T^{(2)}_{\alpha\beta} - \delta^2 G^{[0]}_{\alpha\beta}[h^{(1)}, h^{(1)}] - \delta G^{[1]}_{\alpha\beta}[h^{(1)}]$$

- Self-force computed from these regular fields

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon f^\mu_{(1)}(h^{(1)R}) + \epsilon^2 f^\mu_{(2)}(h^{(1)R}, h^{(2)R}) + \mathcal{O}(\epsilon^3)$$

Field equations

- Field equations for each m-mode take the form:

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)R}] = T^{(1)}_{\alpha\beta} - G^{[0]}_{\alpha\beta}[h^{(1)P}]$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = T^{(2)}_{\alpha\beta} - \delta^2 G^{[0]}_{\alpha\beta}[h^{(1)}, h^{(1)}] - \delta G^{[1]}_{\alpha\beta}[h^{(1)}] - \delta G^{[0])}_{\alpha\beta}[h^{(2)P}]$$

Field equations

- Field equations for each m-mode take the form:

Mino, Sasaki, Tanaka 1997
Quinn and Wald 1997
MiSaTaQuWa equations

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)R}] = T^{(1)}_{\alpha\beta} - G^{[0]}_{\alpha\beta}[h^{(1)P}]$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = T^{(2)}_{\alpha\beta} - \delta^2 G^{[0]}_{\alpha\beta}[h^{(1)}, h^{(1)}] - \delta G^{[1]}_{\alpha\beta}[h^{(1)}] - \delta G^{[0])}_{\alpha\beta}[h^{(2)P}]$$

Pound 2012
Gralla 2012

Field equations

- Field equations for each m-mode take the form:

Mino, Sasaki, Tanaka 1997
Quinn and Wald 1997
MiSaTaQuWa equations

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)R}] = T^{(1)}_{\alpha\beta} - G^{[0]}_{\alpha\beta}[\underline{h^{(1)P}}]$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = T^{(2)}_{\alpha\beta} - \underline{\delta^2 G^{[0]}_{\alpha\beta}[h^{(1)}, h^{(1)}]} - \delta G^{[1]}_{\alpha\beta}[h^{(1)}] - \delta G^{[0])}_{\alpha\beta}[\underline{h^{(2)P}}]$$

Pound 2012
Gralla 2012

- Non-compact
- Diverges on the worldline

Field equations

- Field equations for each m-mode take the form:

Mino, Sasaki, Tanaka 1997
Quinn and Wald 1997
MiSaTaQuWa equations

$$\delta G^{[0]}_{\alpha\beta}[h^{(1)R}] = T^{(1)}_{\alpha\beta} - G^{[0]}_{\alpha\beta}[h^{(1)P}]$$

$$\delta G^{[0]}_{\alpha\beta}[h^{(2)R}] = T^{(2)}_{\alpha\beta} - \delta^2 G^{[0]}_{\alpha\beta}[h^{(1)}, h^{(1)}] - \delta G^{[1]}_{\alpha\beta}[h^{(1)}] - \delta G^{[0])}_{\alpha\beta}[h^{(2)P}]$$

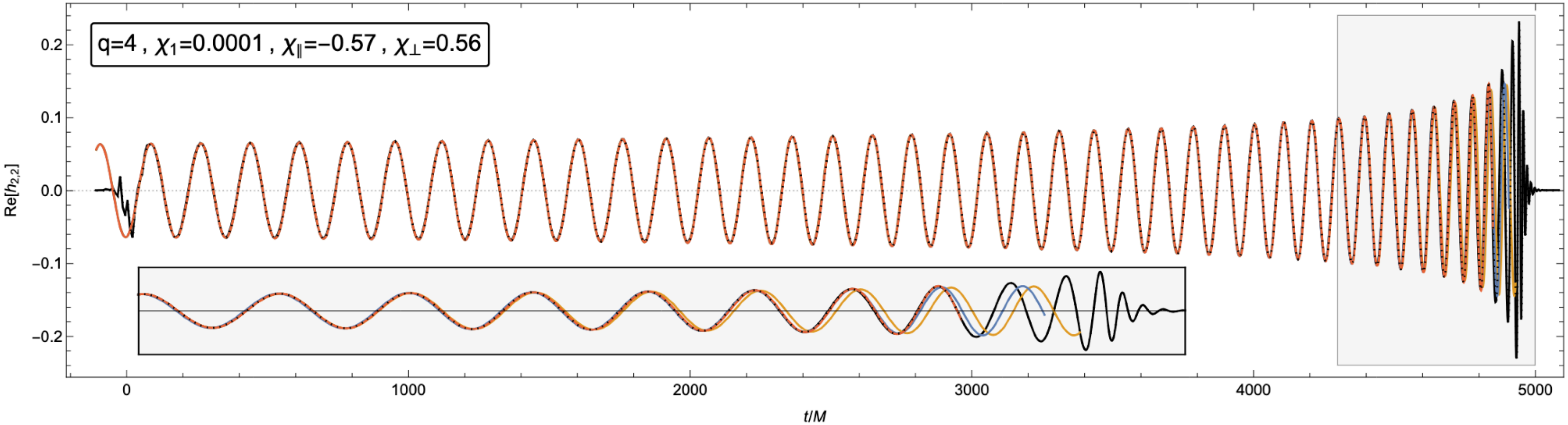
Pound 2012
Gralla 2012

- Non-compact
- Diverges on the worldline

- Non-compact
- $\propto \dot{\Omega}\partial_\Omega h^{(1)}$

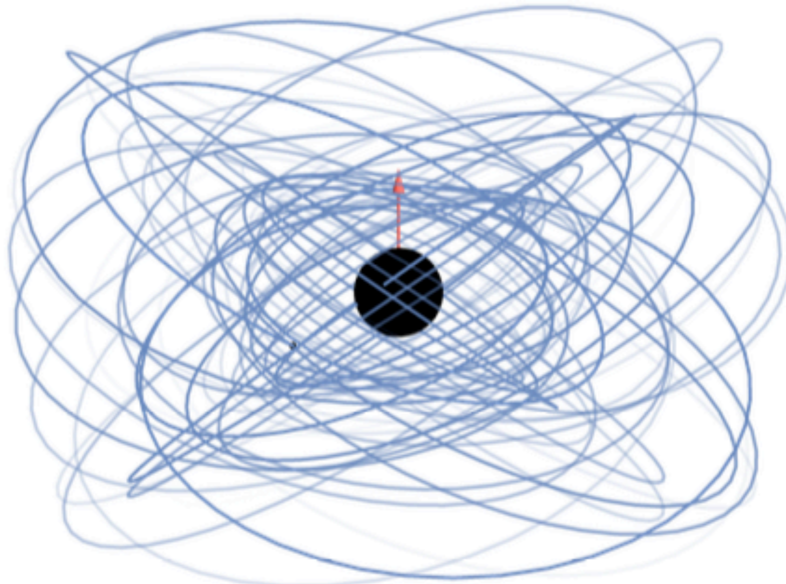
Complete 1PA inspiral waveforms

Comparison with NR waveform from SXS collaboration

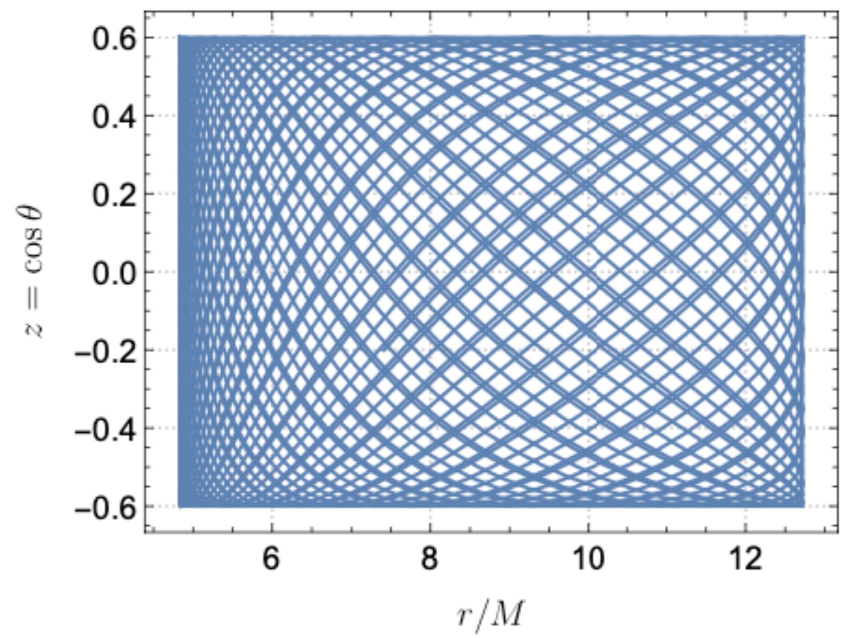


- Complete quasi-circular 1PA inspiral model with generic (precessing) secondary spin, linear primary spin and evolving m_1 and χ_1
- Precession effects only enter the phase at 2PA (amplitudes effected at 1PA)

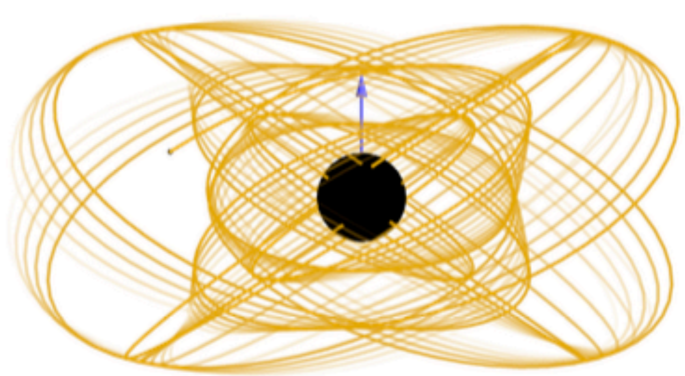
Resonances (0.5PA)



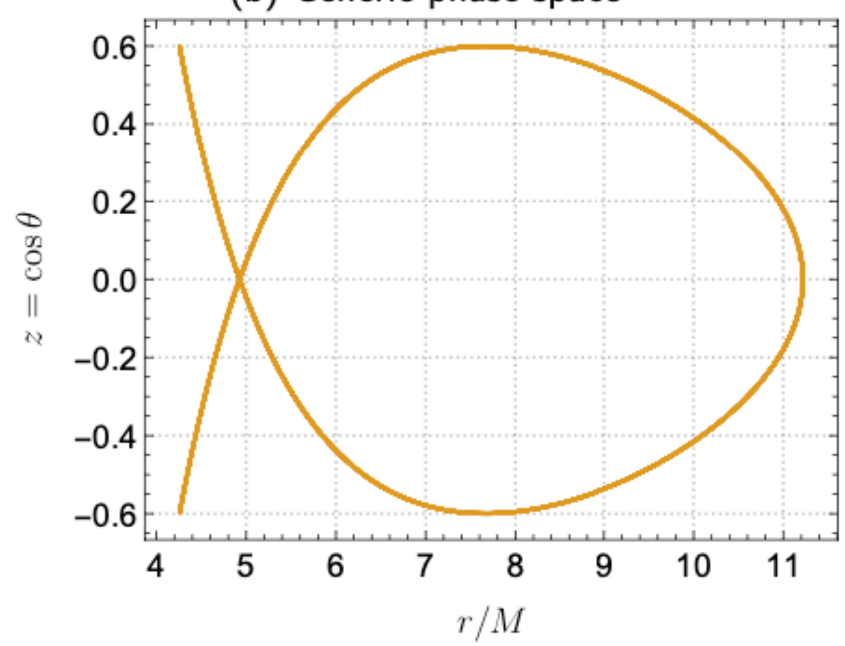
(a) Generic orbit



(b) Generic phase space



(c) Resonant orbit: $q_{r,0} = 0, q_{z,0} = 0$



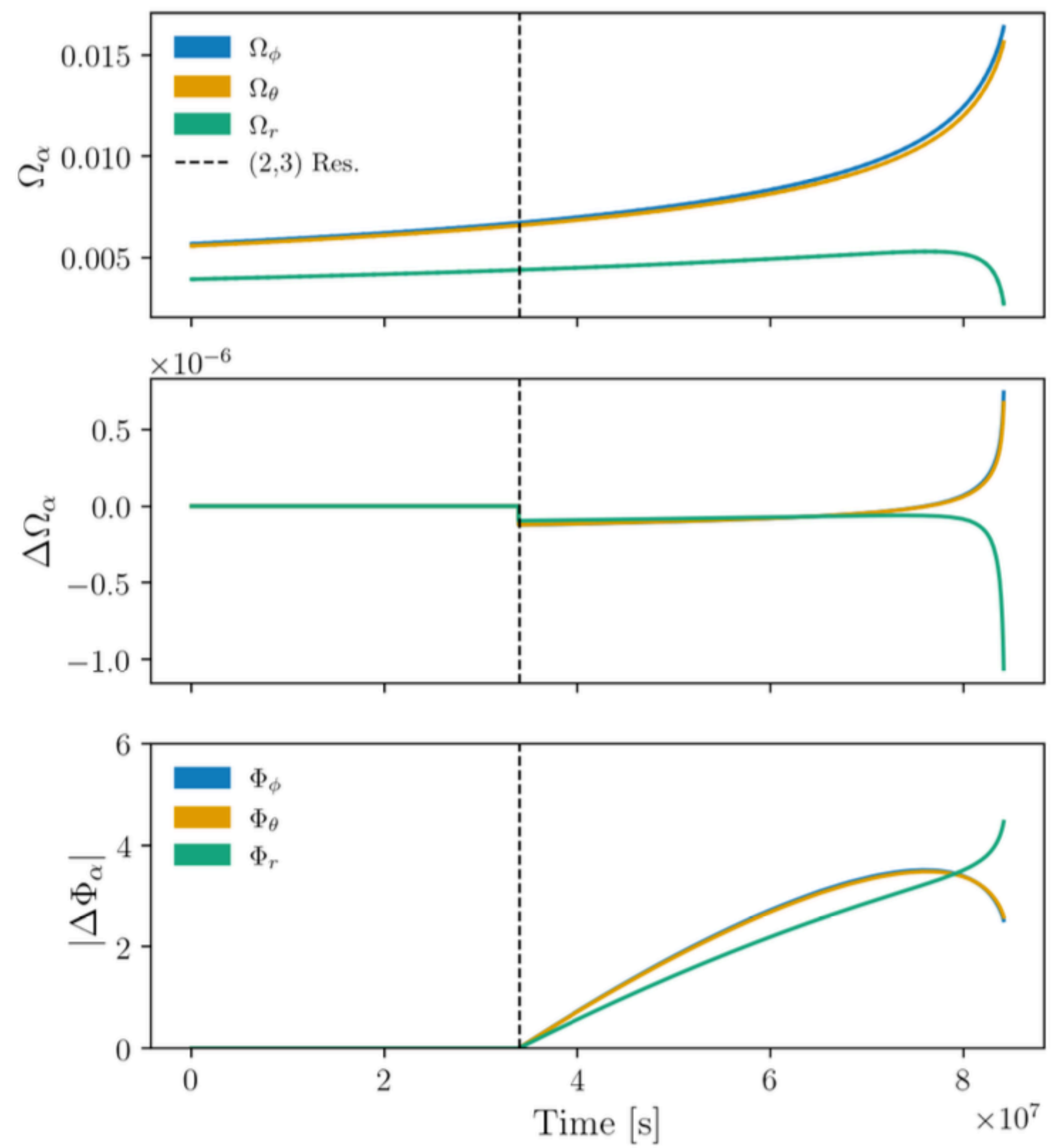
(d) Resonant phase space: $q_{r,0} = 0, q_{z,0} = 0$

[Image credit: Philip Lynch]

Resonances (0.5PA) in FEW

Chapman-Bird, NW

Ω_r/Ω_θ becomes momentarily rational



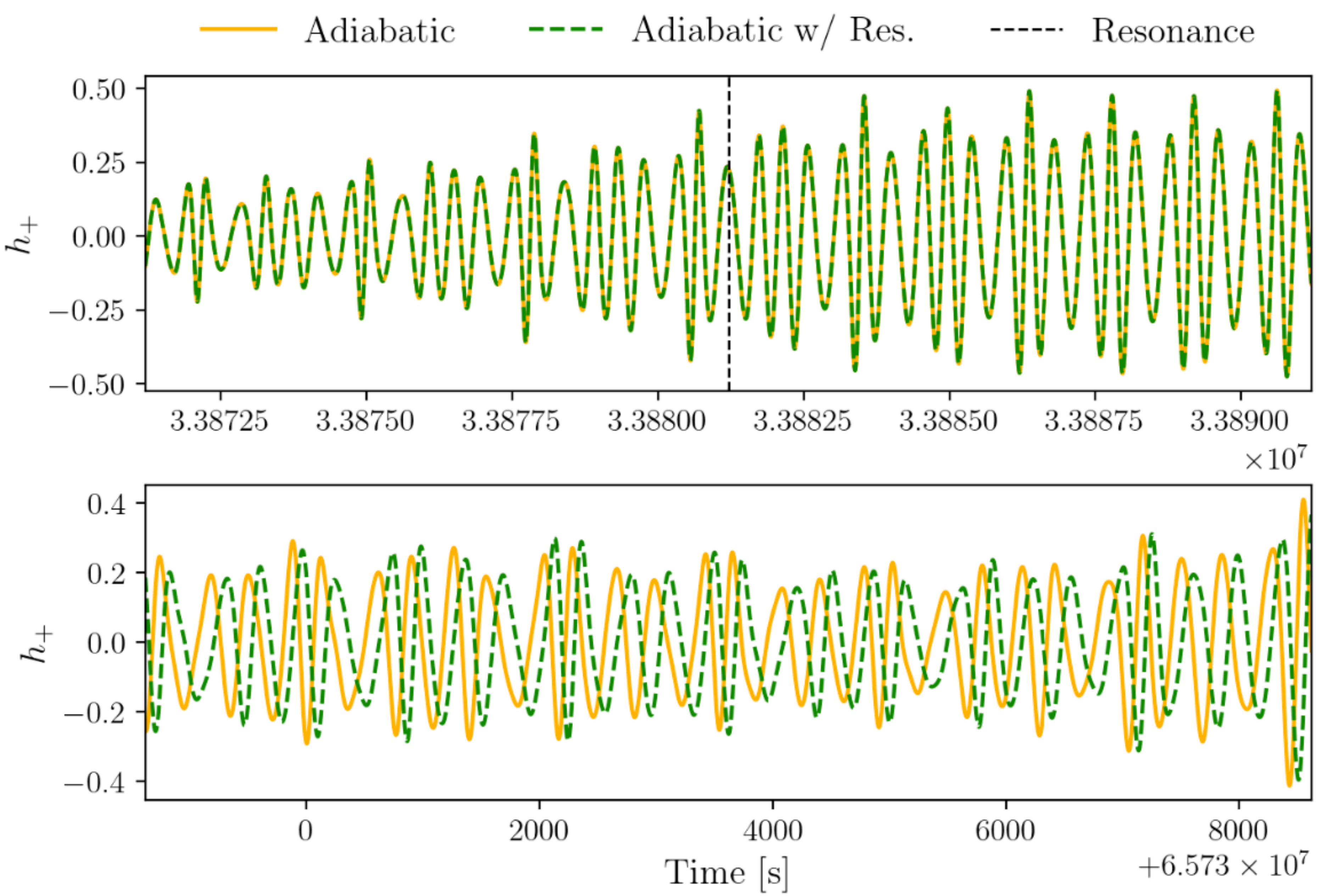
Ω_A "jumps" slightly across the resonance

Leads to a significant phase corrections

$$\tilde{\varphi}_A = \epsilon^{-1}\varphi_A^{(0)}(\Omega_B) + \epsilon^{-1/2}\varphi_A^{(1/2)}(\Omega_B) + \epsilon^{0}\varphi_A^{(1)}(\Omega_B) + \mathcal{O}(\epsilon^{1/2})$$

Resonances (0.5PA) in FEW

Chapman-Bird, NW

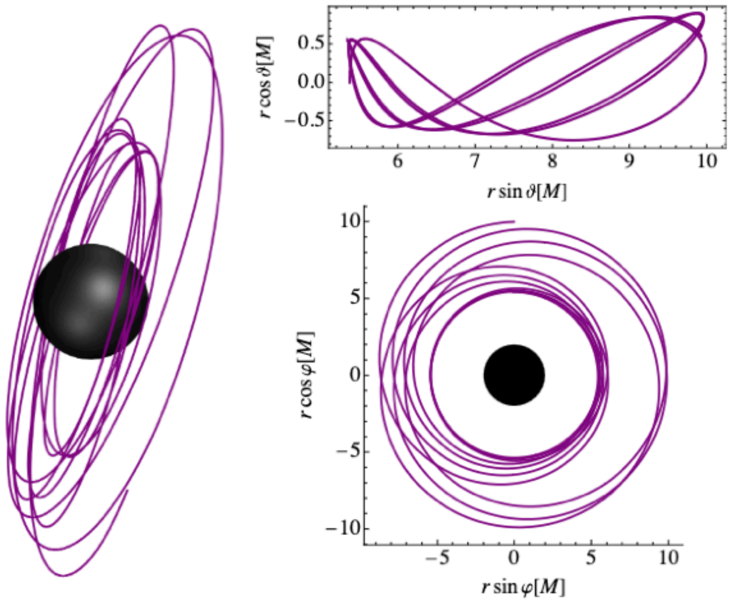


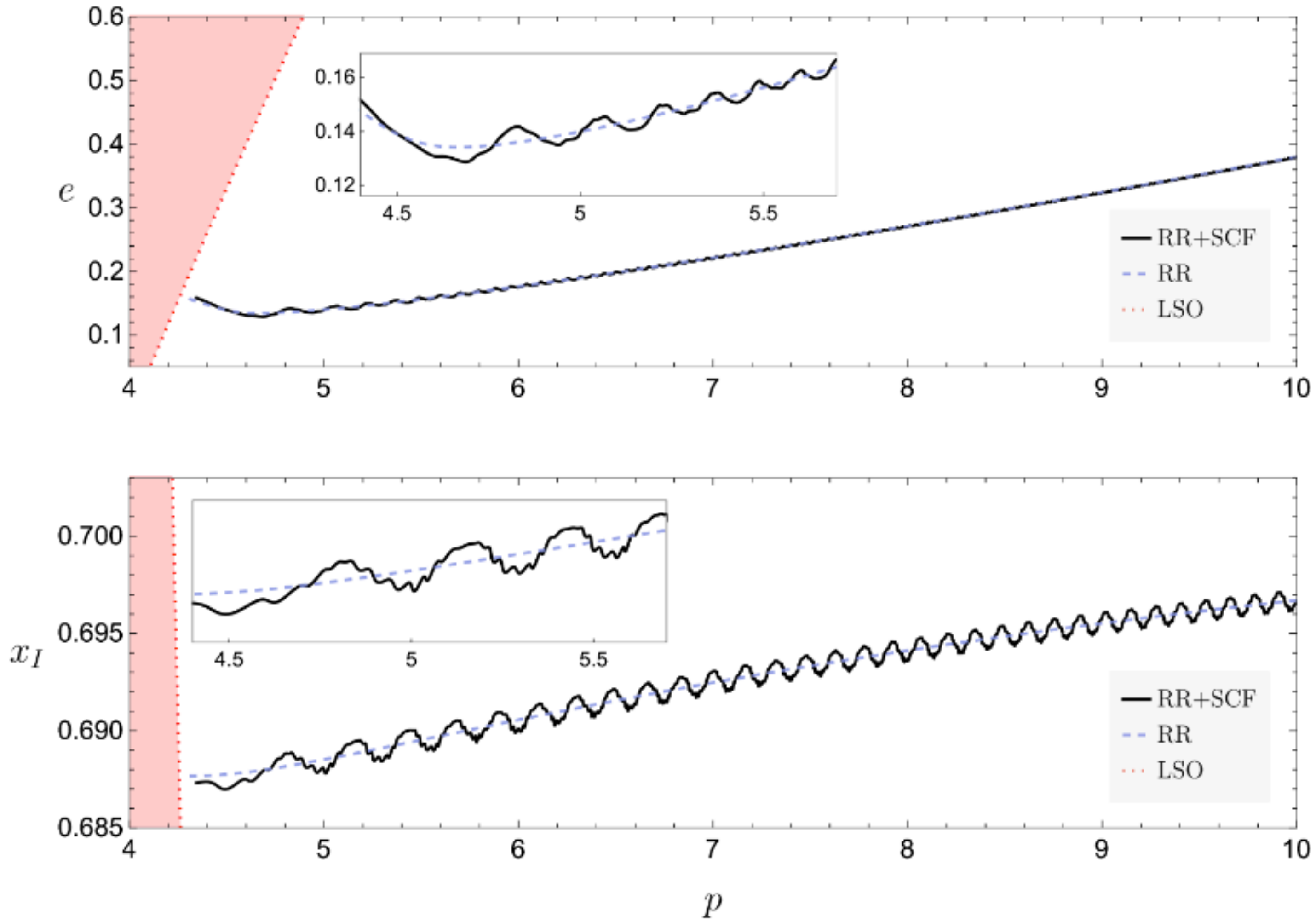
Goal: modular framework in FEW. Given a resonance surface and jump conditions FEW can efficiently model any resonant phenomena.

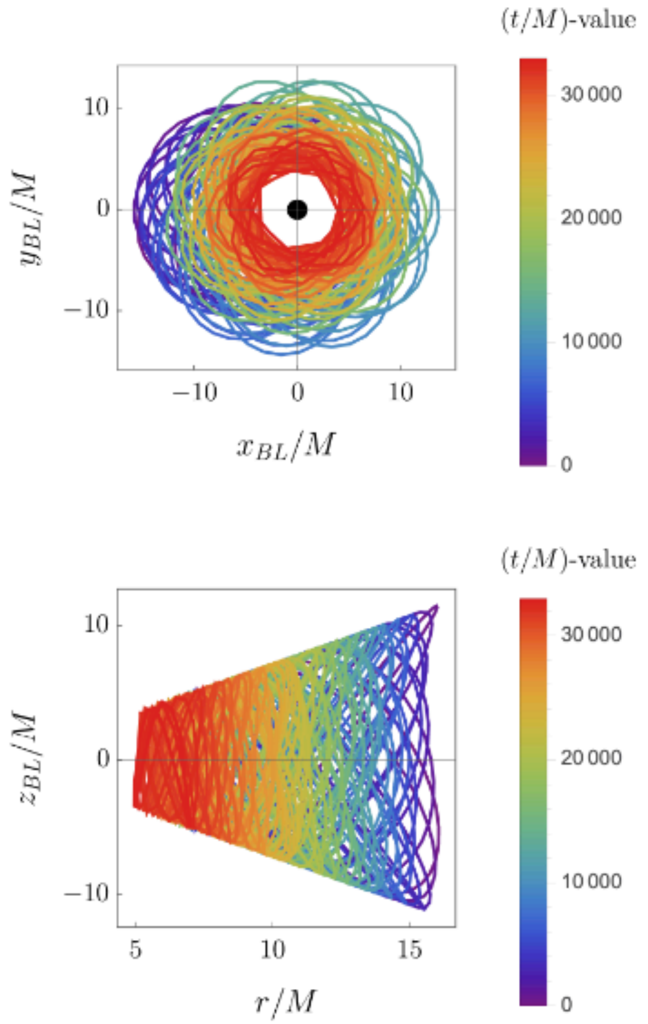
1PA secondary spin effects

Piovano, Witzany
Drummond, Hughes, Lynch et al
Skoupý et al.

- Inspiral trajectory including 1PA conservative effects has been NIT'ed (fast to compute)
- Dissipative correction is computable but still need to tile parameter space and build into FEW

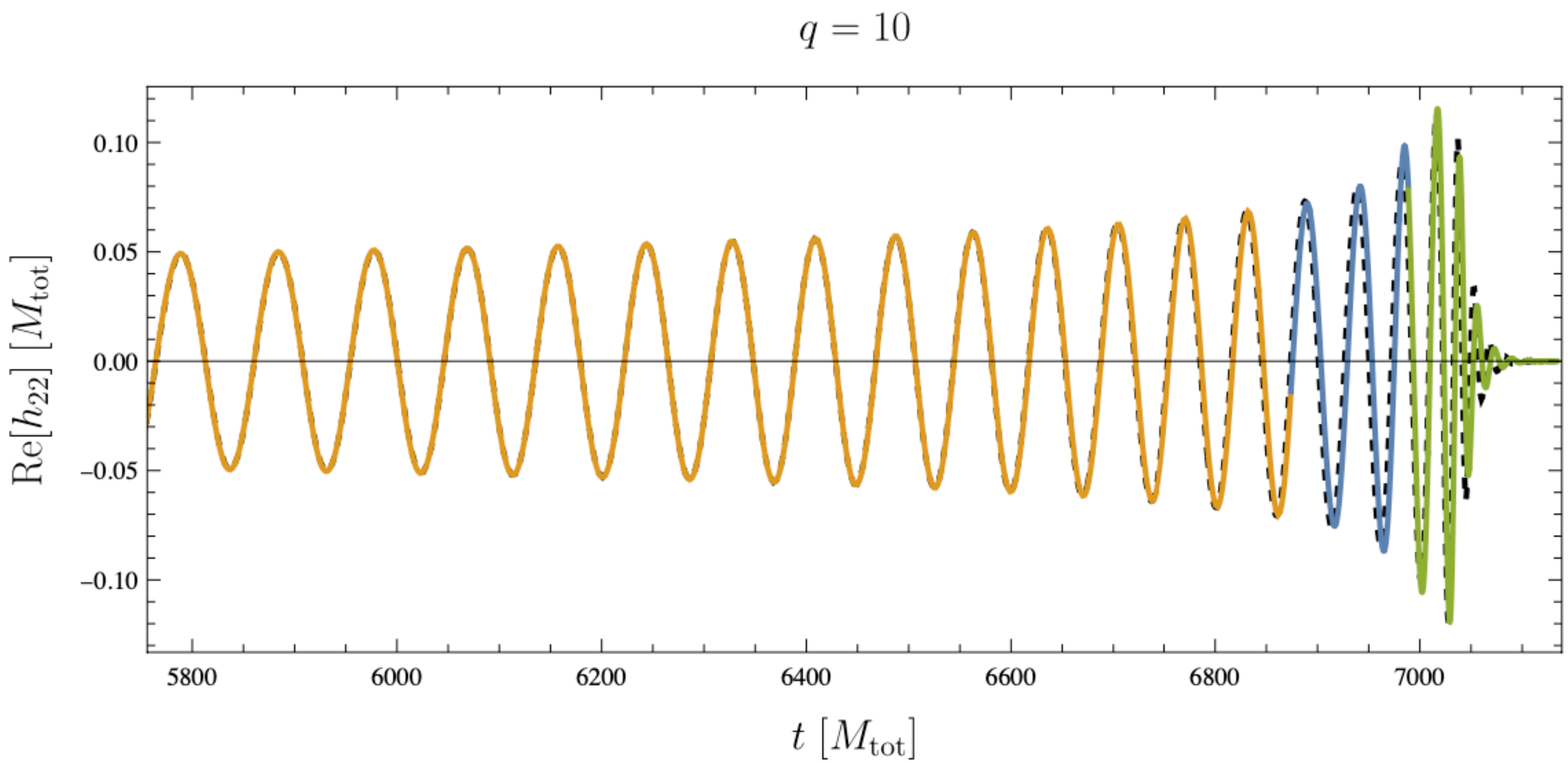






Complete inspiral-merger-ringdown models

Küchler, Compère, Durkan, Pound



- The multiscale expansion used in the inspiral breaks down at the ISCO
- Implement new expansions for the transition-to-plunge and plunge region
- First results appearing. Fast waveform generation speed maintained.