GENERAL RELATIVITY MEETS GEODESY

Eva Hackmann

for the TerraQ collaboration

The 5th EPS Conference on Gravitation
Unlocking Gravity through Computations
Prague, 09-11 December 2024

University of Bremen

Irrigation

Droughts

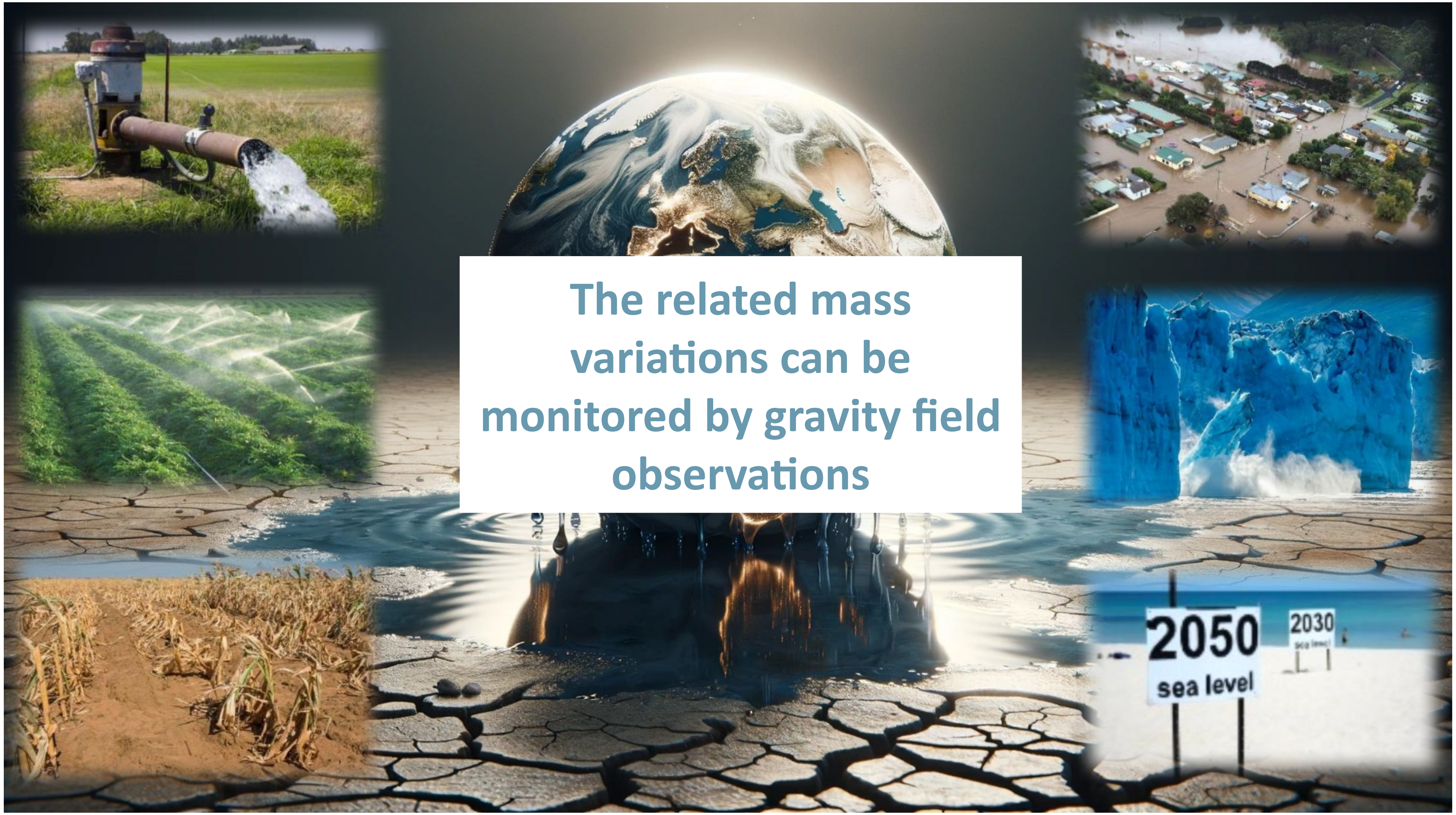
Flooding

Ice melting

Sea level change

Irrigation

Droughts



The related mass variations can be monitored by gravity field observations

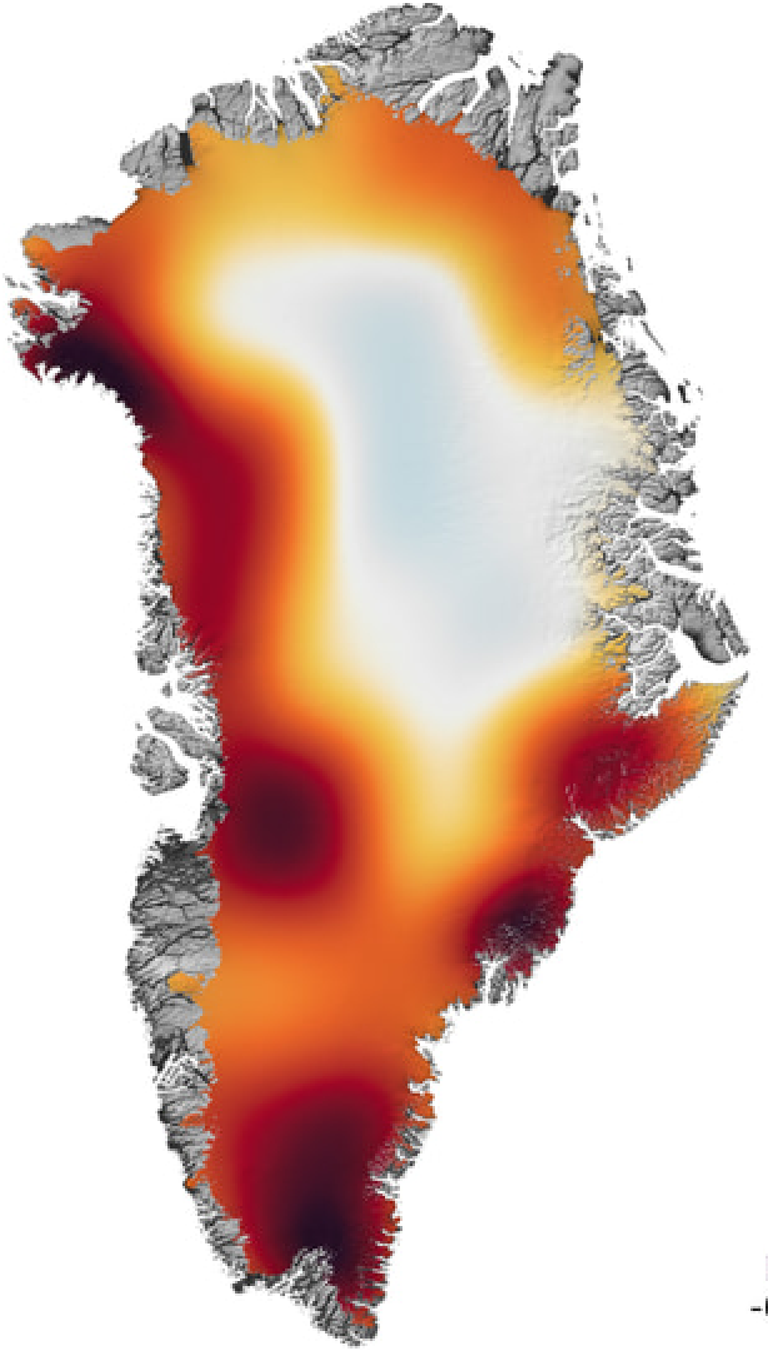
Flooding

Ice melting

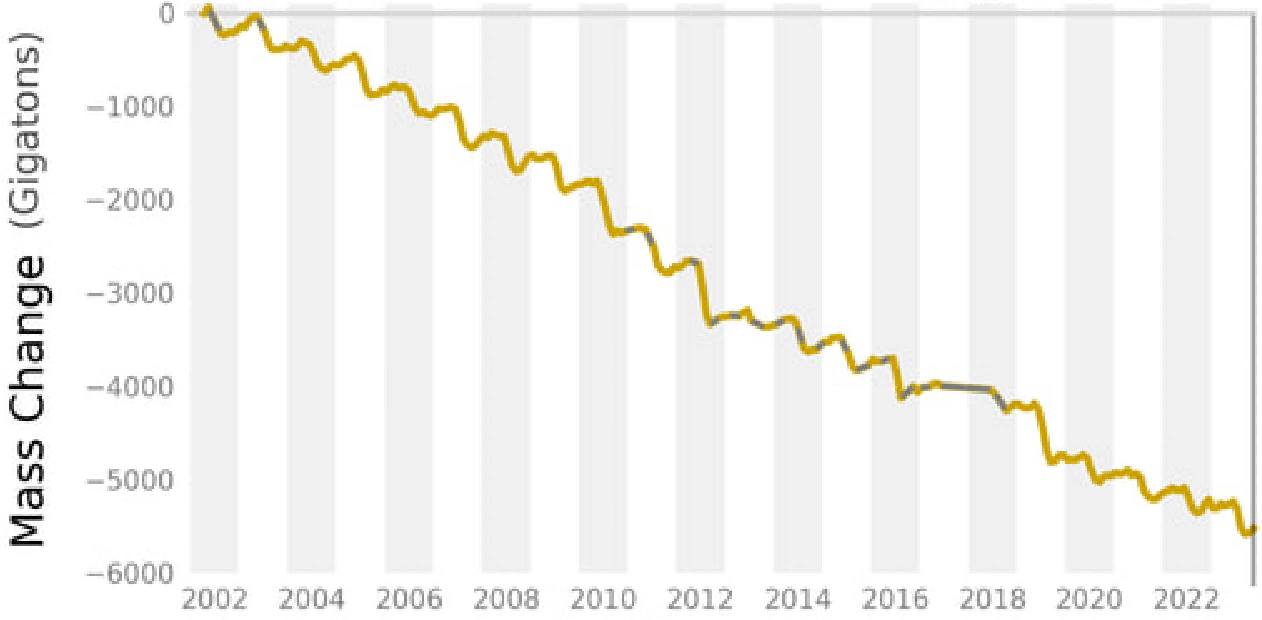
Sea level change

2023-11

GRACE AND GRACE-FO Observations OF Greenland Land Ice Mass Changes

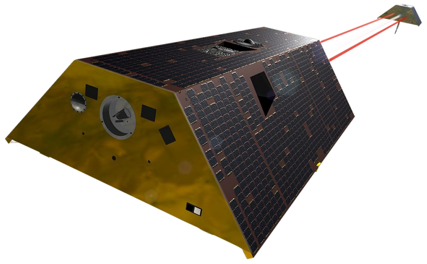


Average Mass Loss:
269 Gigatons/year



2023-11

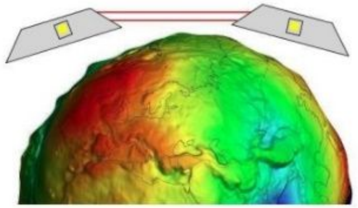
Ice Mass Change
(meters water equivalent relative to 2002)

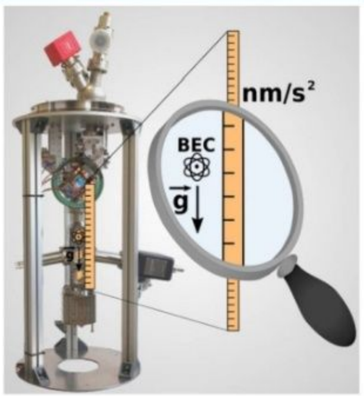


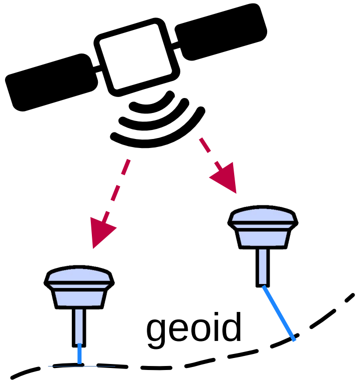
How to observe the gravity field?

Available techniques

- Satellite gravimetry
- Terrestial gravimetry
- GNSS station displacements







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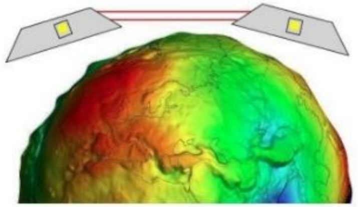
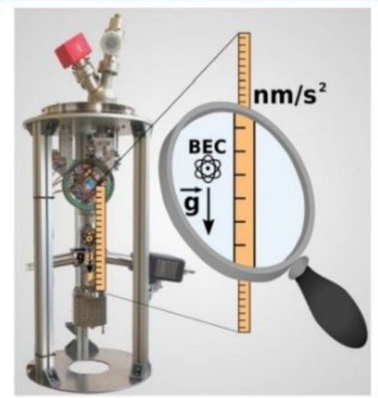
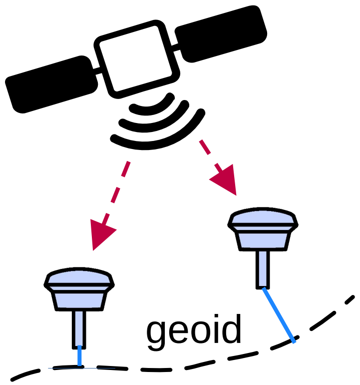
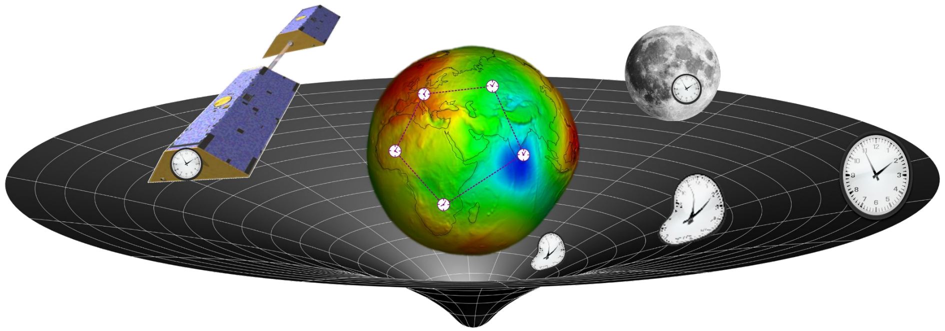






General Relativity

- Sets the stage for all measurements
- Completely new measurement approach:
  - clocks probe the curved spacetime geometry by their proper time



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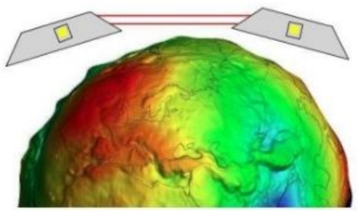
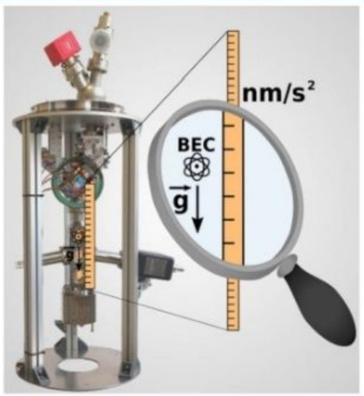
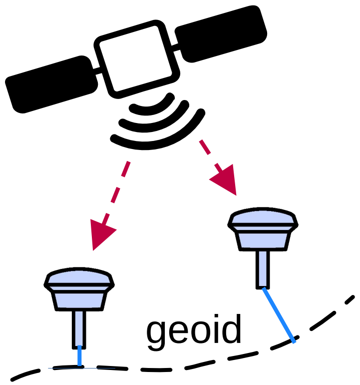
- Terrestial gravimetry
- GNSS station displacements
- **Clock networks**







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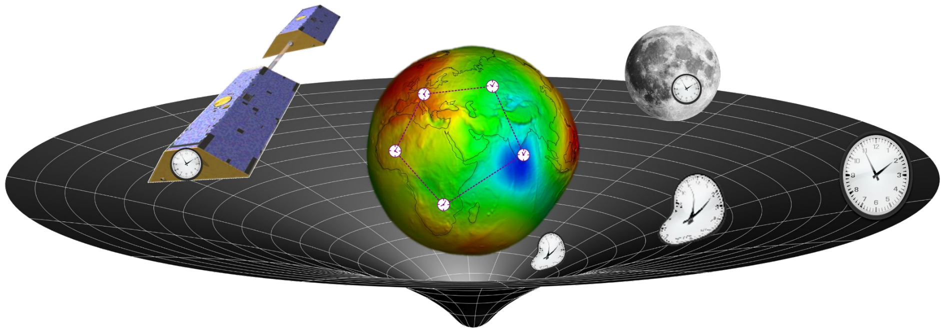
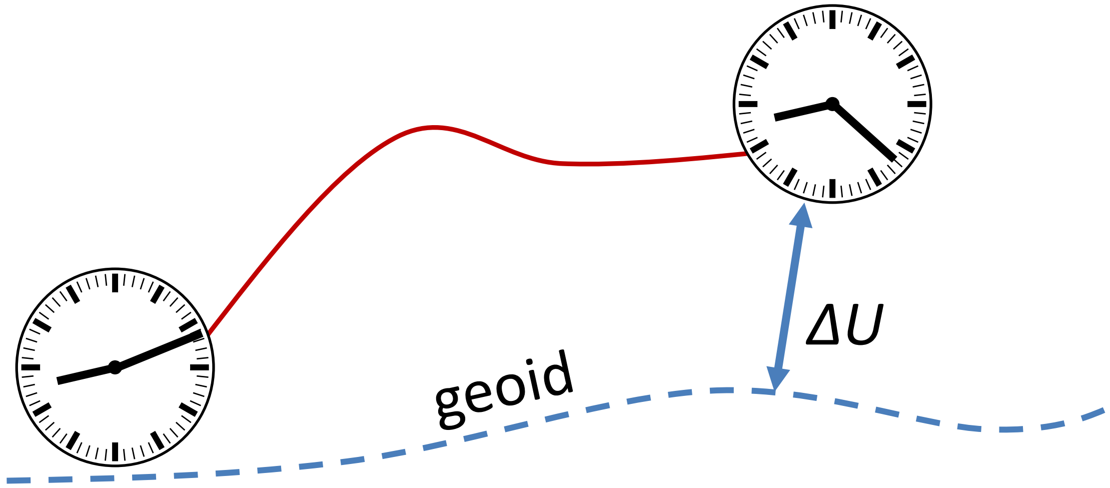




gravitational red shift:

$$\Delta\nu/\nu = \Delta U/c^2$$

M. Vermeer, Rep. of the Finnish Geod. Insti. (1983)
A. Bjerhammar, Bull. Geodesique (1985)

Clock comparison

- 1 cm-resolution within reach (1×10^{-18} optical clock)
- no error accumulation over distance
- high spatial resolution (atoms are small)
- fast measurements

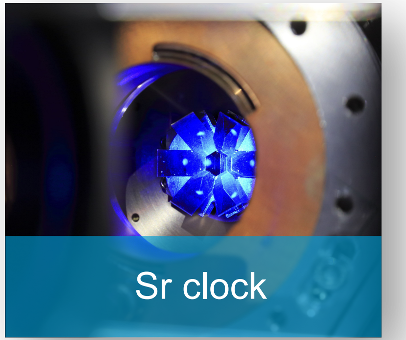
Requirements

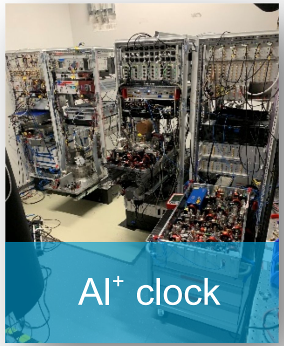
- two (transportable) optical clocks with 10^{-18} uncertainty
- link to compare them
- physical justification → GR

Chronometric levelling – state of the art

Transportable optical clocks

- 2^{nd} generation Sr lattice clock
  - most stable transportable clock laser
    Herbers *et al.* Opt. Lett. (2022)
  - uncertainty $<3\times10^{-18}$
    Lisdat *et al.* Phys. Rev. Res. (2021),
    Dörscher *et al.* Phys. Rev. Res. (2023)

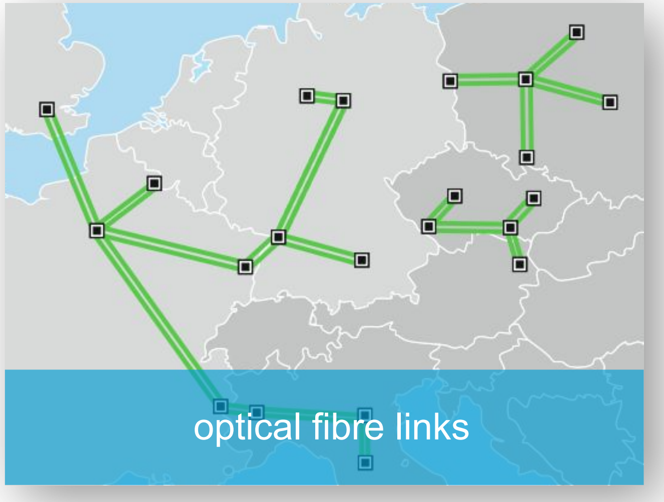




Frequency links

- Optical fibre links:

  $<10^{-18}$ instability in 100 s,
  $<10^{-19}$ offsets
  Schioppo *et al.*
  Nature Commun (2022),
  Koke *et al.* New J. Phys. (2019)

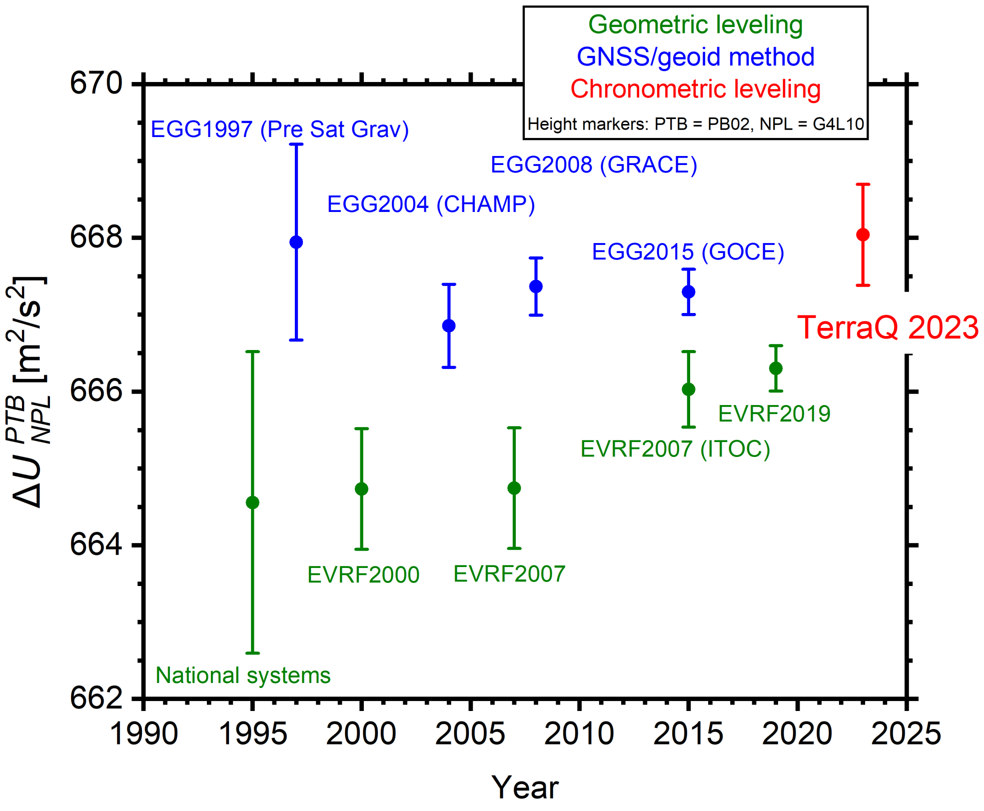


- GNSS frequency transfer
  - 5×10^{-17} instability on 50 km baseline
    Proc. 55th APTTISA Meeting 2024

Chronometric levelling campaigns

NPL - PTB

- Transportable & laboratory Sr clocks

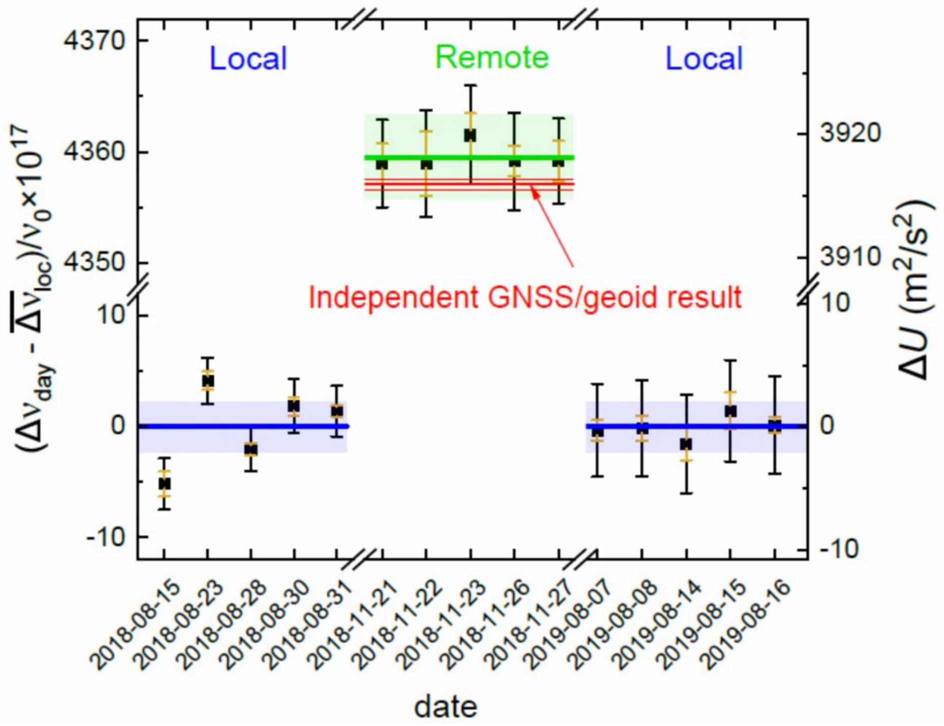


Munich - PTB

- 2018: uncertainty of 27 cm

  Dörscher *et al.* Phys. Rev. Res. (2023), Lisdat *et al.* Phys. Rev. Res. (2021)

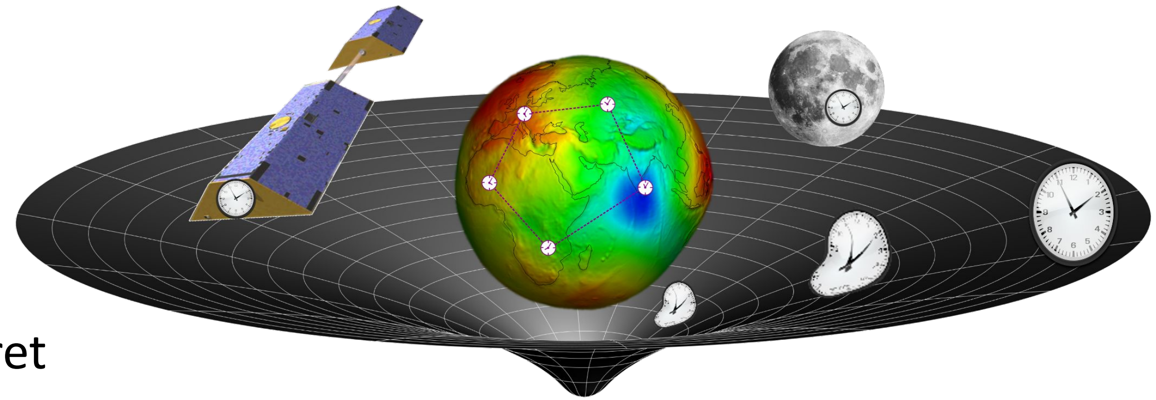
- 2024: 5 cm goal



General Relativity in Geodesy

- Geodesy is based on (post-)Newtonian notions
  - Reference systems and surfaces
  - Height definitions
  - Newtonian gravity potential only

➔ Develop consistent GR theoretical framework to interpret the observations



- Relativity in terrestial and satellite gravimetry
  - Terrestrial clock networks for realizing a global height system
  - Clocks for gravity field recovery (Earth, space, hybrid)
    - Needs also relativistic effects on satellite motion and on (quantum) sensor

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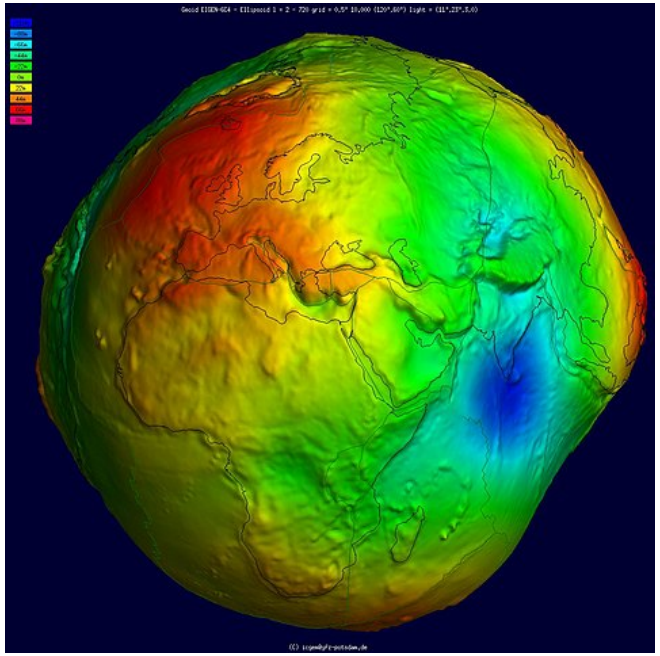
GR framework of Geodesy

Reference surfaces

- Geodesy uses a number of reference surfaces
- One of them is the geoid: the „mathematical figure of Earth“ (Gauss)
  - Equipotential surfaces of the Newtonian gravity potential $W = U + V$
  - One of them („mean sea level“) is singled out by convention $W = W_0$
- Based on measurement of acceleration (a-geoid)



Relativistic geoid

- “Surface where precise clocks run with the same speed“
  A. Bjerhammar, Bull. Geodesique (1985)
  → Surface where atomic clocks show vanishing redshift
- Based on measurement of time/frequency (u-geoid)
  M. Soffel et al, Manuscripta Geodaetica (1988), Kopeikin et al, Phys. Lett. A (2015)

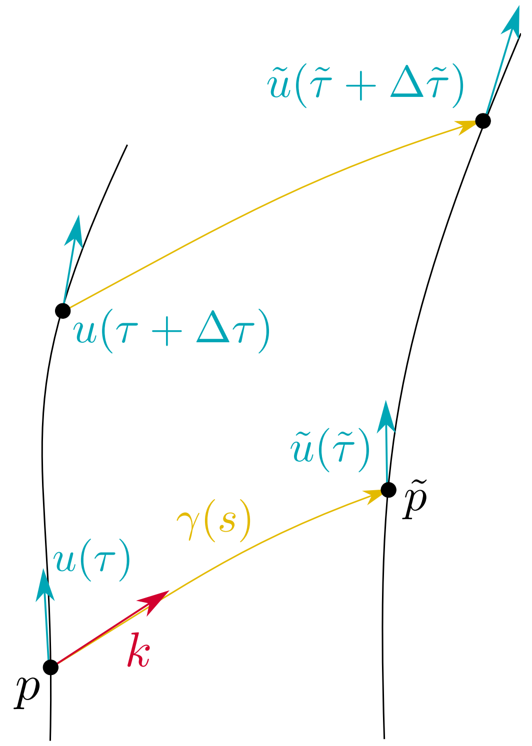
GR framework of Geodesy

The relativistic geoid Philipp et al, Phys. Rev. D (2017)

- Introduce congruence of timelike worldlines with four-velocity u
- The redshift z is

$$z+1=\frac{d\tilde{\tau}}{d\tau}=\frac{g_{\mu\nu}k^{\mu}u^{\nu}|_{p}}{g_{\rho\sigma}k^{\rho}u^{\sigma}|_{\tilde{p}}}$$

- Defines (dimensionless) redshift potential ϕ as $z+1=\exp\Delta\phi$
- ϕ is time independent iff $e^{\phi}u=\xi$ is a Killing vector field Hasse & Perlick, J. Math Phys (1988)
- Equipotential surfaces of ϕ are isochronometric: clocks have vanishing redshift



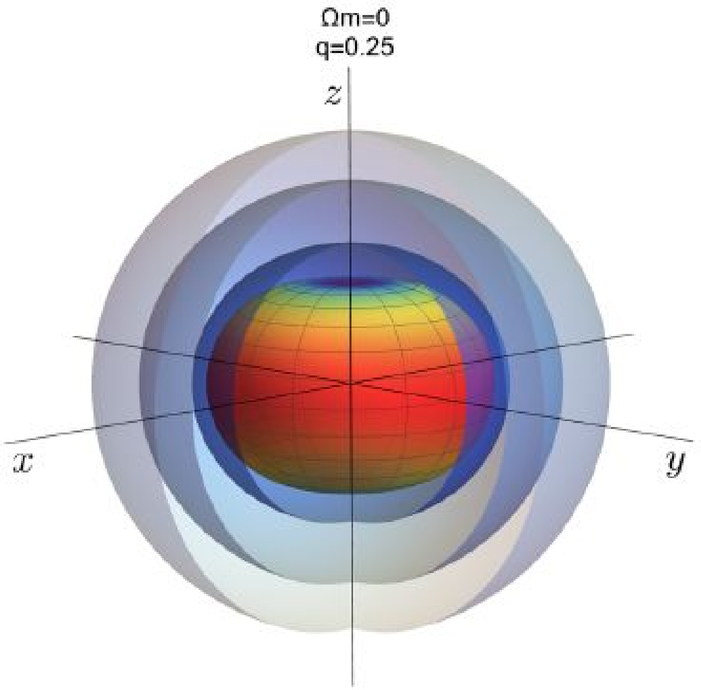
Equivalence to a-geoid

- We assume rigid rotation, constant angular velocity, no external forces
- allows to introduce an acceleration potential χ Ehlers 1961, Salzmann & Taub, Phys. Rev. (1954)
- We can show: $\phi=\chi$ → data fusion applies!

GR framework of Geodesy

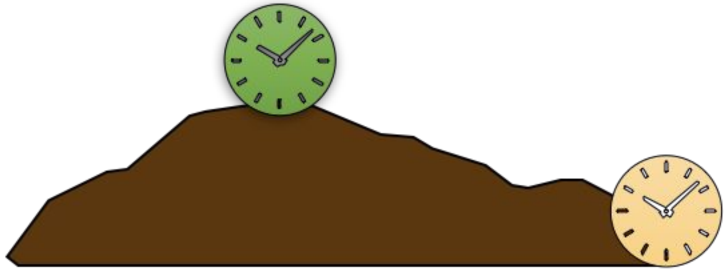
Relativistic gravity potential Philipp et al, Phys. Rev. D (2020)

- In adapted coordinates: $\exp(2\phi) = g_{00}$
- Introduce relativistic gravity potential $U^* = c^2(\sqrt{-g_{00}} - 1)$
- The relativistic geoid is then a level surface of U^*
- In ppN limit $U^* \approx W + \frac{U(\beta - 1/2)}{c^2}$
- The redshift $z + 1 = \frac{1 + U_2^*/c^2}{1 + U_1^*/c^2} \approx \frac{\Delta W}{c^2}$



Based on gravity potential

- The (relativistic) **potential numbers** $C_P^* := U_P^* - U_0^* = z(c^2 + U_0^*)$
- The (relativistic) chronometric **height** $H_P^* := C_P^*/\bar{a}$



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Terra

ZARM

Comparison relativistic vs Newtonian geoid

Setup

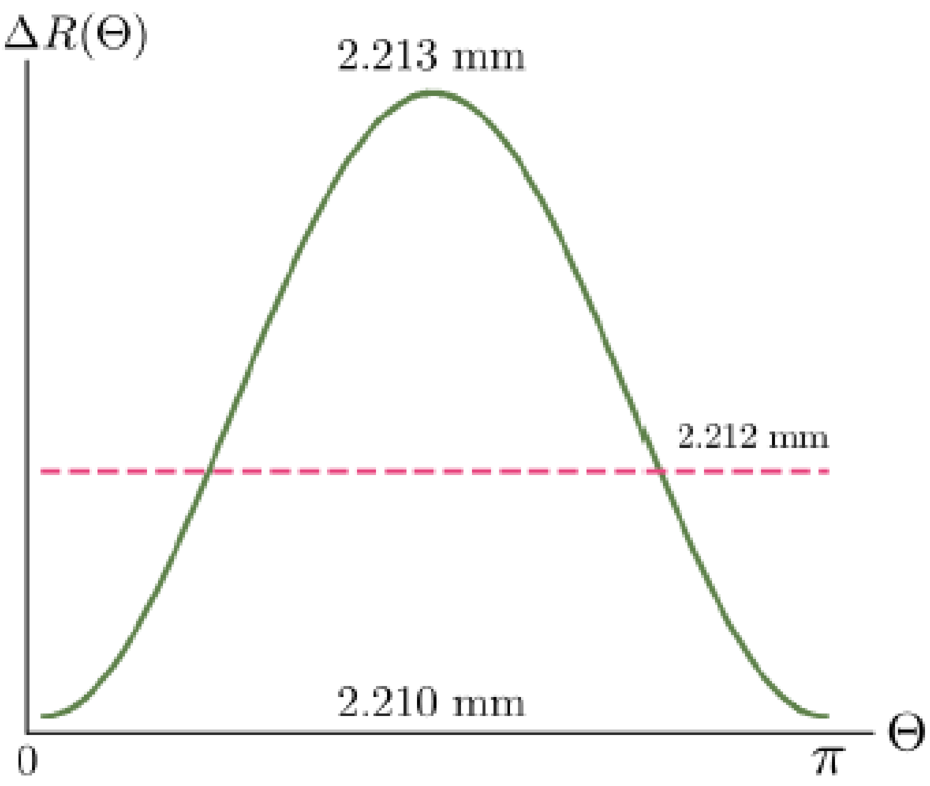
- We choose a simple quadrupolar model of the Earth
- Compare Newtonian geoid based on W
  with 1st-order post-Newtonian geoid based on U^*
- We expect
  - an overall spherical relativistic contribution due to the monopole
  - a latitude dependent correction (about three orders of magnitude smaller)

Comparison

- Coordinates cannot be directly compared
  → isometric embedding into Euclidian space
- How to choose U_0^* ?

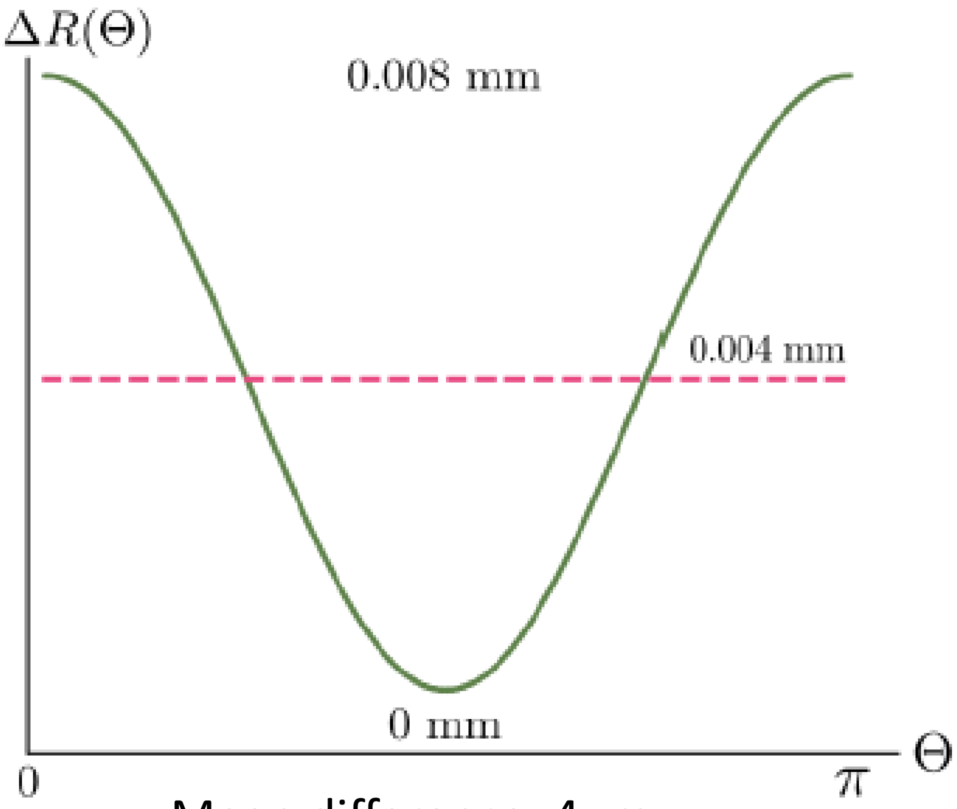
Remember: $U^* \approx W + U/(2c^2)$

$U_0^* = W_0$



Mean difference: 2.2mm
Latitudinal variations: 3µm

U_0^* such that geoids coincide at equator



Mean difference: 4µm
Latitudinal variations: 8µm

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Non-Newtonian gravitational degrees of freedom

Gravitomagnetic effects

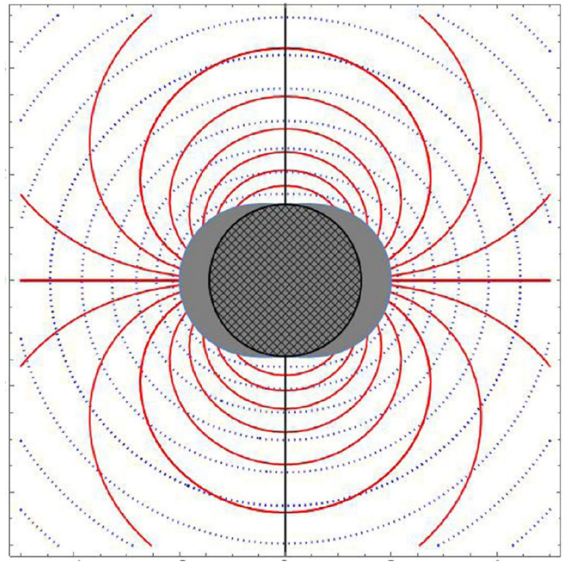
- Rotation also contributes to the gravitational field
  → frame dragging/gravitomagnetism

$$g = g_{00}c^2dt^2 + 2g_{0i}c\,dt\,dx^i + g_{ij}dx^i dx^j$$

- Proceed similar to redshift potential
  - Introduce congruence of timelike Killing observers ξ
  - Then the twist vector field is $\omega^\mu = \epsilon^{\mu\nu\rho\sigma}\xi_\nu\partial_\rho\xi_\sigma$
  - Einstein's vacuum equations imply the twist potential

$$\omega_\mu = \partial_\mu\omega$$

- One equipotential surface may be used as „rotoid"
- Redshift potential ϕ and twist potential ω
  → purely chronometric reference frame



Lämmerzahl & Perlick, Phys. Rev. D (2023)

Satellite gravimetry: GRACE-FO

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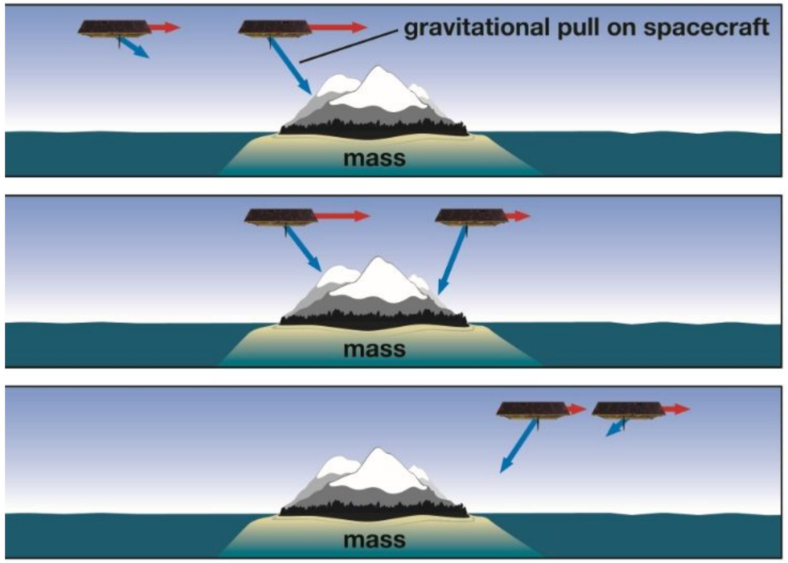
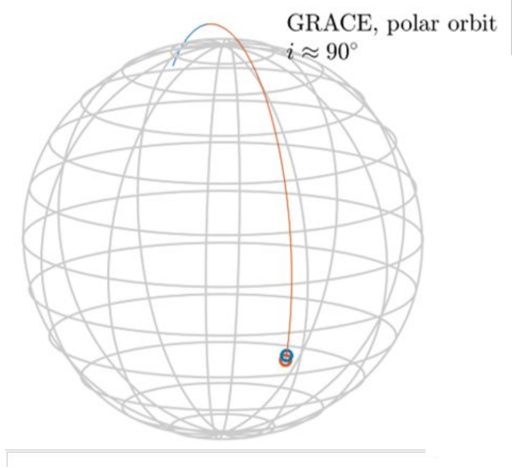




The mission

- 2 satellites about 220 km apart; Microwave and Laser link
- Range changes are measured on the nm level
- Monthly global gravity field solutions of ~300 km spatial resolution

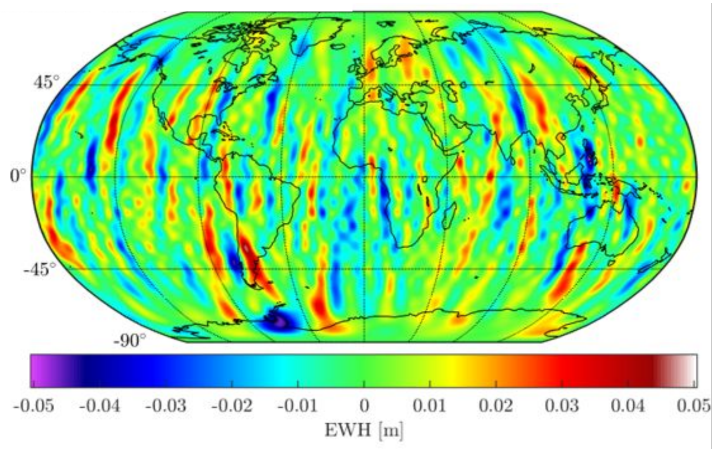
Gravity Field Recovery from space

Basic approach

- The Newtonian gravitational potential U is written as

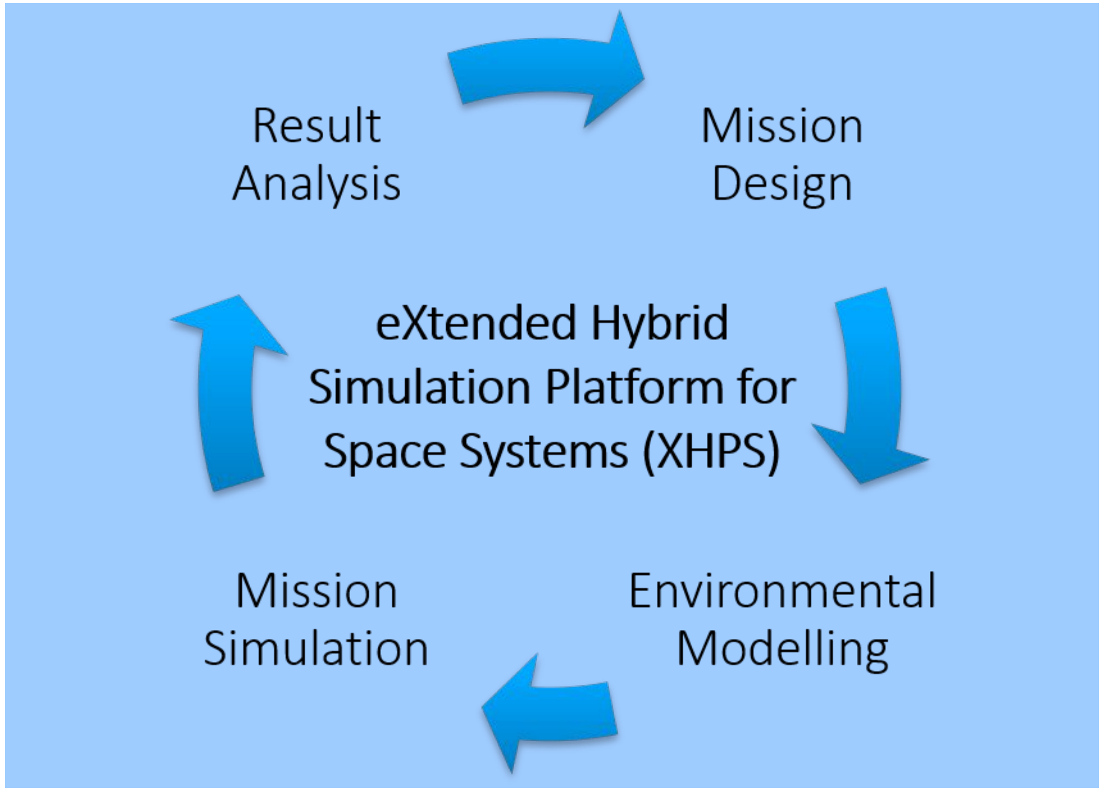
$$U = \frac{GM}{r} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{R_{\mathrm{ref}}}{r} \right)^{l} P_{lm}(\sin\theta) \left\{ C_{lm} \cos(m\varphi) + S_{lm} \sin(m\varphi) \right\}$$

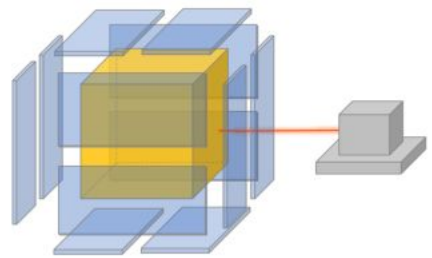
- Goal: determine the coefficients C_{lm} , S_{lm} from the data
  - Model the satellite trajectories
  - Model all kinds of perturbations
  - Model the sensors

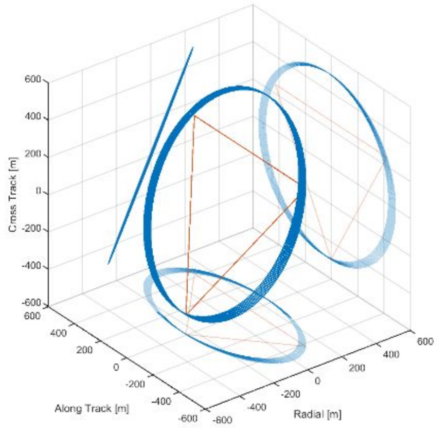


Simulator Tool XHPS

- model of optical-electrostatic accelerometers
- satellite swarm scenarios
- test mass dynamics and a generic drag-free approach
- 1st post-Newtonian order



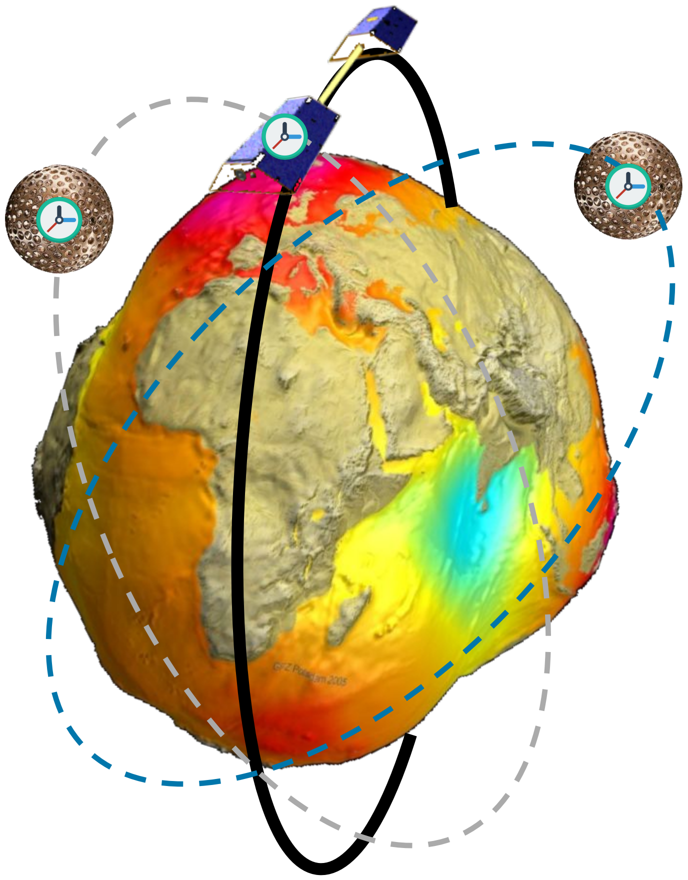




Clocks in space

Can clocks be used for
Gravity Field Recovery from space?

- Challenge: separate the much larger special relativistic effects (satellites move with ~10 km/s)
- 1st order Doppler effect can be eliminated by using a two way link
- Uncertainty in 2nd order Doppler effect maps into gravitational potential determination
- Rough estimate:
  1 cm geoid corresponds to about 10 µm/s in LEO (Low Earth Orbit)
- This seems to be very challenging...



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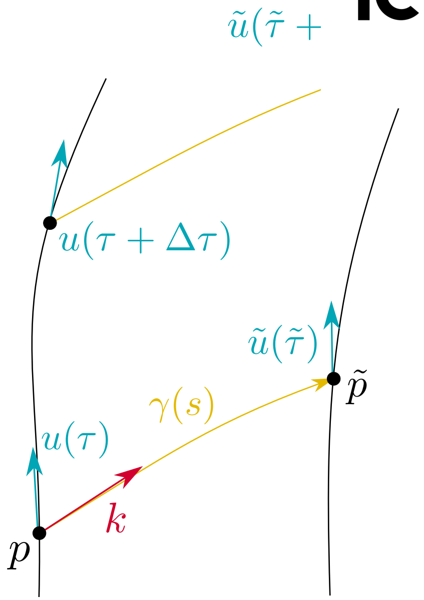
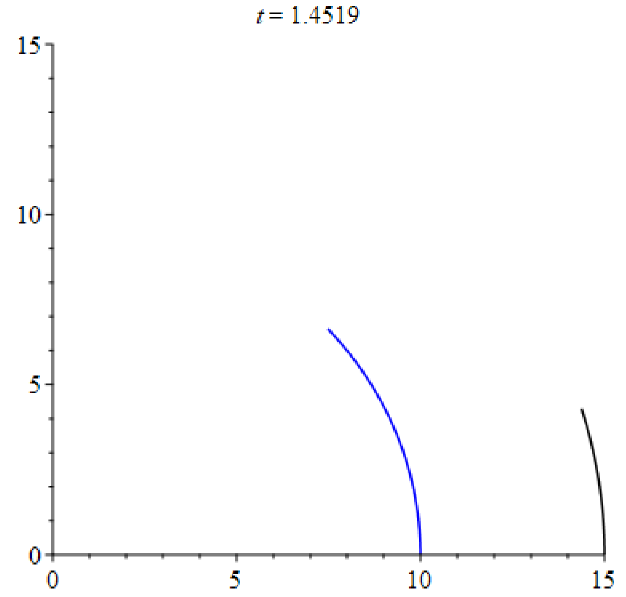
Clocks in space

Redshift calculation

- To determine redshift z we need the connecting light ray
  - General Relativistic Emitter-Observer Problem (EOP)
  - With moving boundaries
- Software GREOPy:
  - General Relativistic Emitter-Observer Python algorithm
  - Solution of EOP in arbitrary stationary spacetimes
  - Between two arbitrarily moving objects
  - So far only first order image

GREOPy link





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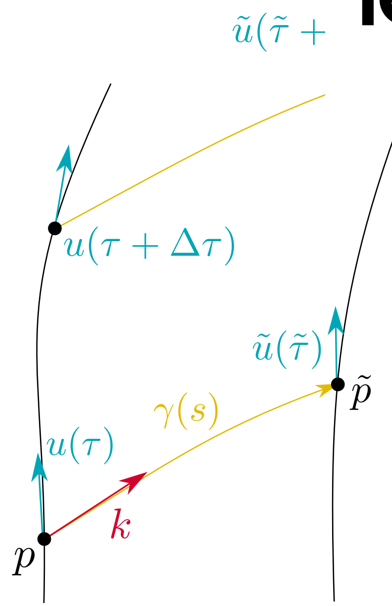

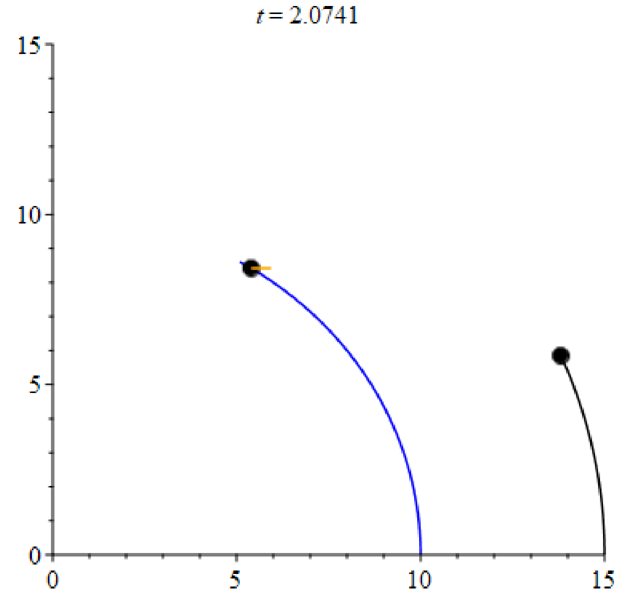

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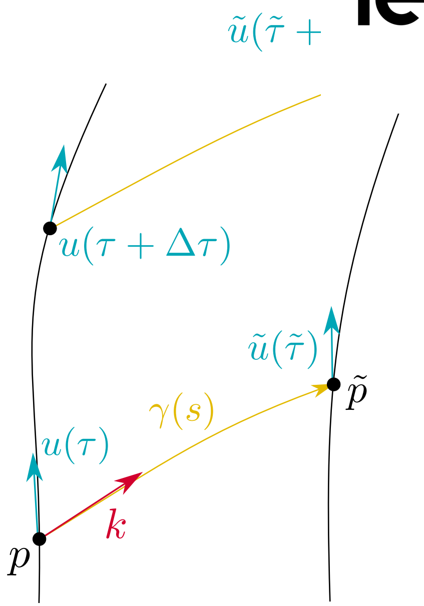

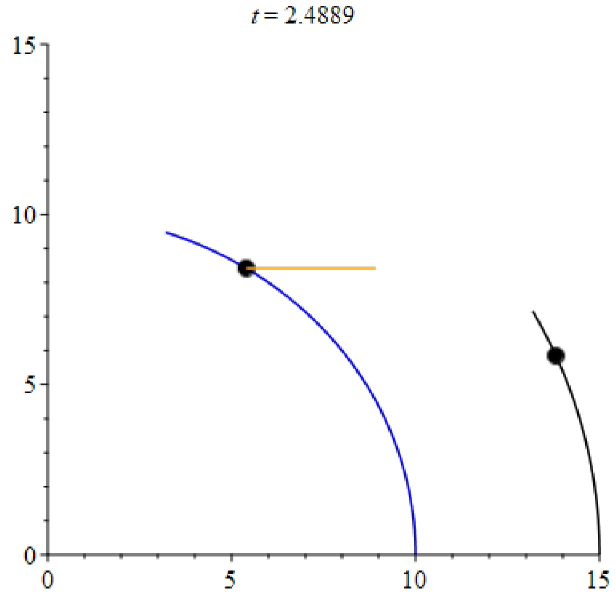

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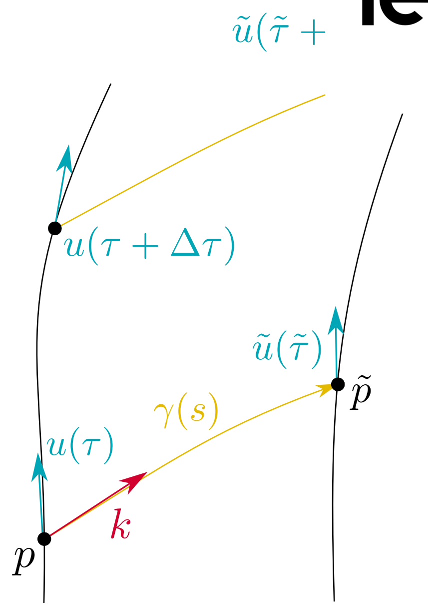
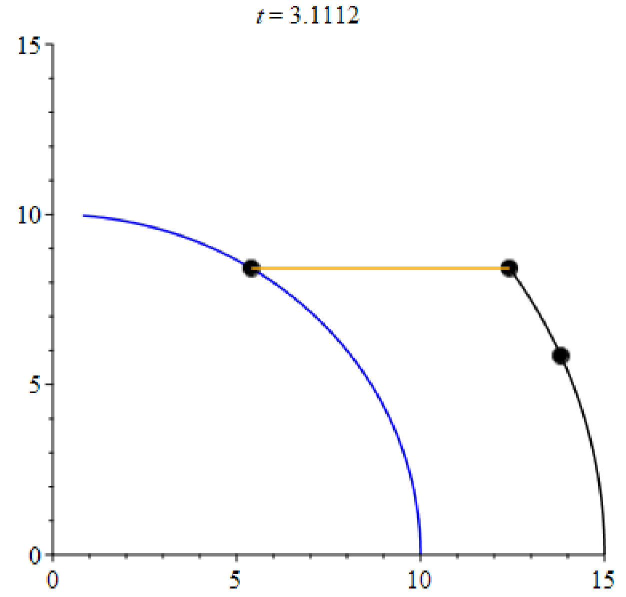
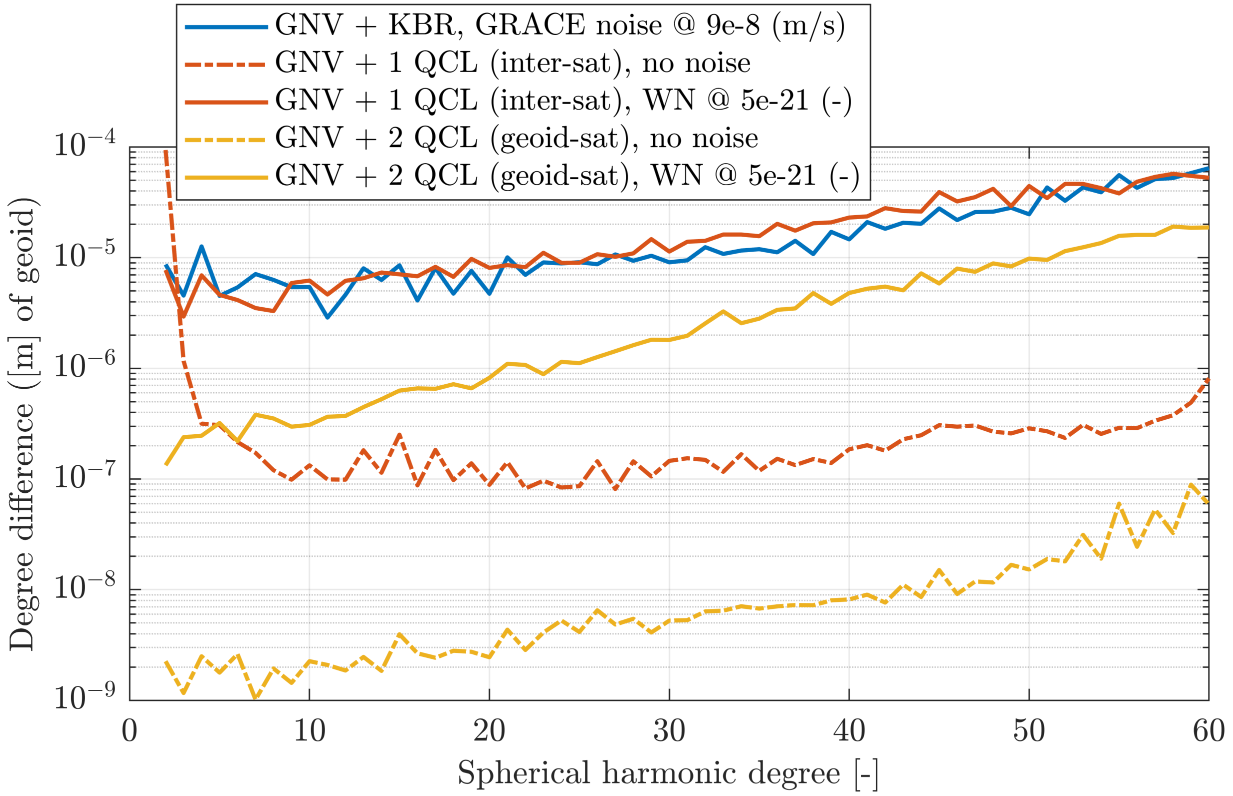




Clocks in space

Preliminary study

- Closed loop simulation with XHPS
- GRACE-like mission scenario
- Assume GNSS navigation (GNV) with 2 cm white noise
- Assume K-band ranging (KBR) with typical GRACE noise
- Assume extremely precise clocks (QCL)
- Estimate spherical harmonics up to degree and order 60



Gravity Field Recovery technically possible, but...

- GRACE-setup not well suited
- Clocks needs to be extremely precise

Take Home Messages

- Gravity field observations → climate variables
- Clocks are a new tool to observe Earth's gravity field
  - Height resolution on (sub-)cm level in the next years!
- General Relativistic theoretical framework of Geodesy under development
  - Role of non-Newtonian gravitational degrees of freedom?
- Clocks in space are technically feasible for global Gravity Field Recovery, but practically not precise enough