

(HAMILTONIAN) CHARGES, BOUNDARY EFFECTS, AND GRAVITY



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based on

- GO, S. Speziale: **Brown-York charges with mixed boundary conditions**, JHEP 11 (2021) 224, 2109.02883
- GO, A. Rignon-Bret and S. Speziale: **Wald-Zoupas prescription with soft anomalies**, Phys. Rev. D 107 (2023), no. 8 084028, 2212.07947
- GO, A. Rignon-Bret and S. Speziale: **General gravitational charges on null hypersurfaces**, JHEP 12 (2023) 038, 2309.03854

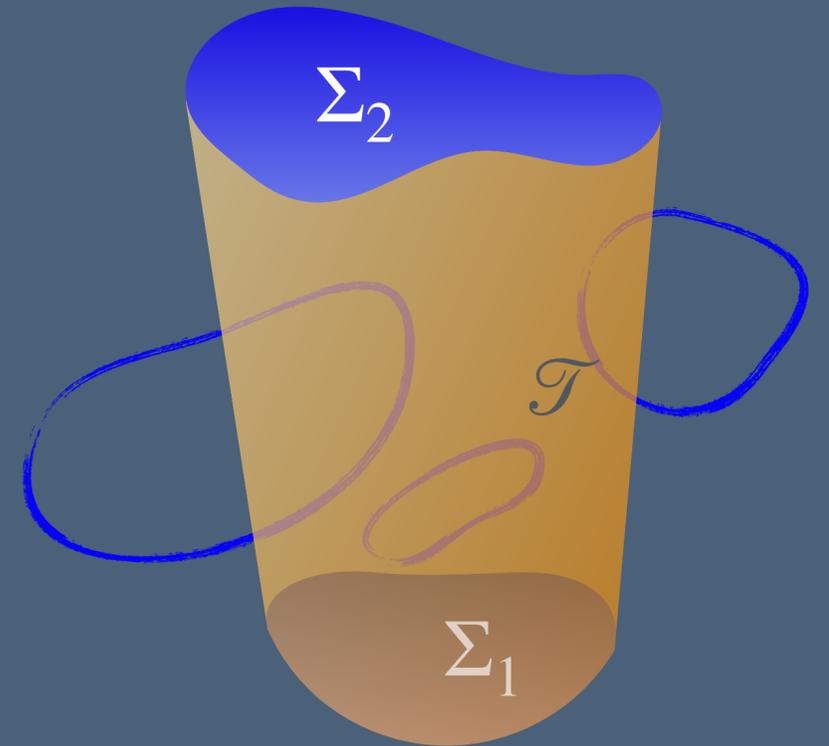
Boundaries

- gauge invariant observables are often described by Wilson loops



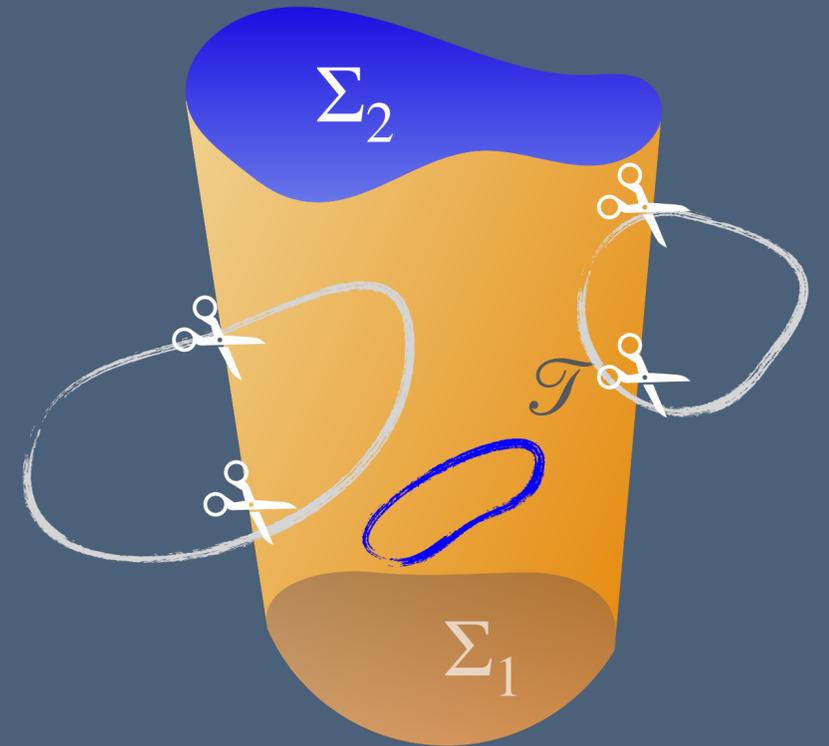
Boundaries

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- boundaries cut Wilson loops into Wilson lines — not gauge invariant



Boundaries

- gauge invariant observables are often described by Wilson loops
- boundaries cut Wilson loops into Wilson lines — not gauge invariant
- the presence of boundaries promotes gauge into physical symmetry with non-trivial charge



Boundary conditions

classification

- from theory of PDEs:
 - Dirichlet — fixing configuration variable — $\delta q = 0$
 - Neumann — fixing the (normal) derivatives — $\delta \dot{q} = 0$
 - Robin — any constant linear combination of the two — $a\delta q + \delta \dot{q} = 0$

Boundary conditions

in gravitational physics

- gravitational radiation makes the definition of energy ambiguous
- the ambiguity usually fixed by imposing Dirichlet boundary conditions
- analogy with thermodynamics :
 - isothermal — internal energy
 - adiabatic — free energy
- how about gravity? What happens if we use b.c. different than Dirichlet?1



Boundary conditions

from the action principle



- Variation of a general action $\mathcal{S} = \int_{\mathcal{M}} L(q, \dot{q})$ is always of form $\delta\mathcal{S} = \int_{\mathcal{M}} E\delta q + \int_{\partial\mathcal{M}} p\delta q \hat{=} \int_{\partial\mathcal{M}} p\delta q,$

$$E \hat{=} 0, p := \partial L / \partial \dot{q}$$

- Dirichlet boundary condition $\delta q = 0$ on $\partial\mathcal{M}$ makes the action stationary $\delta\mathcal{S} = 0$
- Neumann action is stationary for $\delta p = 0$ — it can be obtained from \mathcal{S} by adding a boundary

$$\text{Lagrangian } \mathcal{S}^N = \mathcal{S} + \int_{\partial\mathcal{M}} pq$$

- polarization of phase space can be encoded in a choice of coordinates q and can from this POV can be seen as equivalent to choosing boundary conditions

Boundary conditions

in general relativity



- Einstein-Hilbert lagrangian $L^{EH}[g_{\mu\nu}] = R\epsilon$, $\epsilon = \sqrt{-g}d^4x$
- arbitrary variation $\delta L^{EH} = G_{\mu\nu}\delta g^{\mu\nu}\epsilon + d \left[\left(K_{\mu\nu}\delta q^{\mu\nu} - 2\delta K \right) \epsilon_{\Sigma} \right]$
- Famously, L^{EH} can be amended with the **Gibbons-Hawking-York boundary Lagrangian** $\ell = 2K\sqrt{q}d^3x$ making it stationary after imposition of **Dirichlet** boundary conditions.
- What other choices do we have?

Boundary conditions

in general relativity



- Einstein-Hilbert lagrangian $L^{\text{EH}}[g_{\mu\nu}] = R\epsilon$, $\epsilon = \sqrt{-g}d^4x$
- arbitrary variation $\delta L^{\text{EH}} \approx d \left[q_{\mu\nu} \delta \Pi^{\mu\nu} \epsilon_{\Sigma} \right]$
- stationary with fixed gravitational momentum $\tilde{\Pi}_{\mu\nu} := \sqrt{q} \left(K_{\mu\nu} - K q_{\mu\nu} \right)$
- **No boundary Lagrangian** is required for **Neumann** boundary conditions.

Boundary conditions

in general relativity

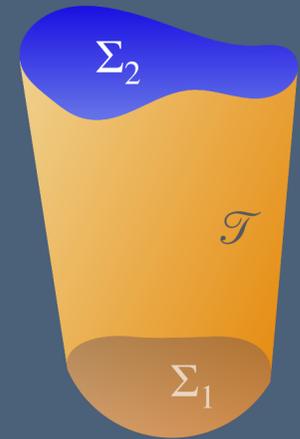


- Other choices?
- have 6 conditions to fix — can mix up Dirichlet and Neumann
- One particular choice of mixed b.c. that is geometrically motivated: Fixed conformal induced metric: $\hat{q}_{\mu\nu} := q^{-1/3} q_{\mu\nu}$ and the trace of extrinsic curvature $\delta\hat{q}_{\mu\nu} = 0 = \delta K$

[York '86]

$$\delta L^{EH} \approx d \left[-(P^{\mu\nu} \delta\hat{q}_{\mu\nu} + \frac{4}{3} \delta K) \epsilon_T - \frac{2}{3} \delta(K \delta q) \right]$$

- **York boundary Lagrangian** $\ell^Y = \frac{2}{3} K \epsilon_\Sigma$ is required for **conformal** boundary conditions.



$$\ell^b = bK\epsilon_\Sigma$$

$$L = L^{EH} + d\ell^b$$

<i>boundary conditions</i>	<i>quantity fixed on boundary</i>	<i>value of b</i>
Dirichlet	$q_{\mu\nu}$	2
York	$(\hat{q}_{\mu\nu}, K)$	2/3
Neumann	$\tilde{\Pi}^{\mu\nu}$	0

Noether's theorem

if the Lagrangian has a continuous symmetry, then there is a current which is conserved on-shell.

$$\delta_\epsilon L = dY_\epsilon \quad \Rightarrow \quad dj_\epsilon \approx 0, \quad j_\epsilon := I_\epsilon \theta - Y_\epsilon, \quad d\theta \approx \delta L$$

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defined only up to boundary terms

$\rightarrow j_\epsilon + da$ conserved as well

Noether's theorem

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$$\delta_\epsilon L = dY_\epsilon \quad \Rightarrow \quad dj_\epsilon \approx 0, \quad j_\epsilon := I_\epsilon \theta - Y_\epsilon, \quad d\theta \approx \delta L$$

$$j_\epsilon = C_\epsilon + dq_\epsilon$$

= constraint (global charge) + boundary term (surface charge)

example

scalar field with Neumann boundary conditions

- Consider the scalar field Lagrangian $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$
- spacetime translation symmetry: $x^\mu \rightarrow x^\mu + \epsilon^\mu$, $\delta_\epsilon \phi = -\epsilon^\mu \partial_\mu \phi$,
- $\delta \mathcal{L} = \square \phi \delta \phi + \partial_\mu (\partial^\mu \delta \phi) \rightarrow \delta \phi \Big|_{\partial M} = 0$, $\square \phi = 0$ Dirichlet b.c.,
 $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi \rightarrow E^D = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2$
- Neumann b.c.: $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \partial_\mu (\phi \partial^\mu \phi) \rightarrow \delta \partial_n \phi \Big|_{\partial M} = 0$, $\square \phi = 0$
 $E^N = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\nabla \phi)^2 - \partial_a (\phi \partial^a \phi) = E^D - \partial_a (\phi \partial^a \phi)$

Gravitational charges

Brown-York charges with different boundary conditions

Dirichlet (b=2) $q_{\xi}^{\text{BY}} = -2 \int_S n^{\mu \xi \nu} \left(\bar{K}_{\mu\nu} - \bar{q}_{\mu\nu} \bar{K} \right) \epsilon_S$

[Brown, York '93]

York (b=2/3) $q_{\xi}^{\text{Y}} = -2 \int_S n^{\mu \xi \nu} \left(\bar{K}_{\mu\nu} - \frac{1}{3} \bar{q}_{\mu\nu} \bar{K} \right) \epsilon_S$

Neumann (b=0) $q_{\xi}^{\text{N}} = -2 \int_S n^{\mu \xi \nu} \bar{K}_{\mu\nu} \epsilon_S$

<i>boundary conditions</i>	<i>quantity held fixed</i>	<i>value of b</i>	<i>quasi-local energy</i>	<i>Kerr (renormalized)</i>
Dirichlet	$q_{\mu\nu}$	2	k	M
York	$(\hat{q}_{\mu\nu}, K)$	2/3	$k - 2\bar{K}/3$	$2M/3$
Neumann	$\tilde{\Pi}^{\mu\nu}$	0	$k - \bar{K}$	$M/2$

[GO, Speziale '21]

Gravitational charges

Brown-York charges with different boundary conditions

these charges are physically distinct because they correspond to canonical generators for different ways of making the system conservative

<i>boundary conditions</i>	<i>quantity held fixed</i>	<i>value of b</i>	<i>quasi-local energy</i>	<i>Kerr (renormalized)</i>
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Gravitational charges

on null hypersurfaces

- Null surfaces are special because they allow radiation to leak through them only in one direction
- Due to their interesting geometry, they allow for easier extraction of physical degrees of freedom
- In non-null case, we restrict to conservative boundary conditions — to define energy we must first close the system
- Null boundaries allow for leakage — we can study the flow of gravitational waves
- Including different polarizations into our analysis allows for identification of various groups of symmetry of enclosed spacetimes each applicable in different physical scenarios
- A version of Bondi mass loss formula for each case

Gravitational charges

on null hypersurfaces [GO, Rignon-Bret, Speziale '23]

- canonical charges by Chandrasekaran-Flanagan-Prabhu '18 conserved on non-expanding horizons defined with Dirichlet polarization
- can use an analog of York's polarization on a null boundary — nice geometric properties
- this choice leads to charges conserved both on NEH and on Minkowski lightcones
- might have interesting implications for dynamical processes [Rignon-Bret '23; Wald, Zhang '23]

conclusions

- The dependence of the charges on boundary conditions was anticipated as early as Iyer&Wald '95
- analogy with thermodynamics: different notions of energy associated to different thermodynamic potentials
- null charges important for understanding GW at \mathcal{I} , event horizons, or general null surfaces
- conformal boundary conditions better behaved from IBVP POV, result in well behaving charges on horizons — relevant for computer simulations of spacetime?