Solving Teukolksy equation to compute extreme mass ratio inspirals

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What are EMRIs?

- An Extreme Mass Ratio Inspiral (EMRI): inspiral of a stellar compact object of mass μ into a SuperMassive Black Hole SMBH $M \geq 10^5 M_\odot$, the mass ratio $q = \mu/M \gtrsim 10^{-4}$
- The inspiraling body traces the SMBHs spacetime background and emits information through gravitational waves.

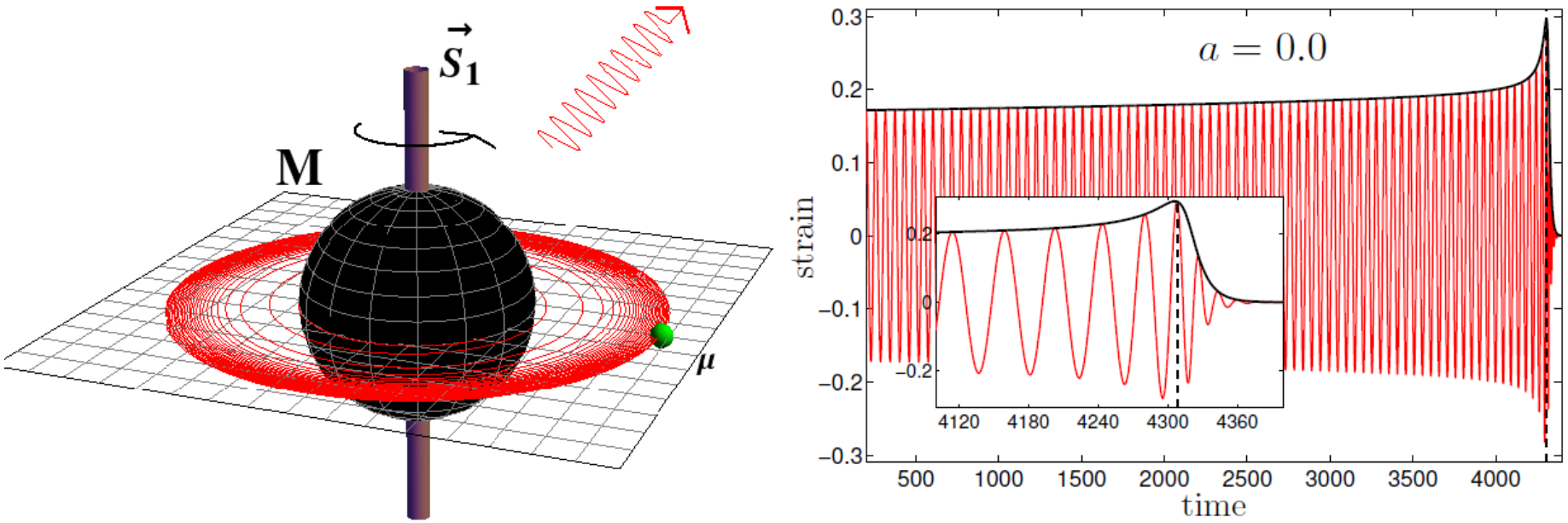

Figure: An EMRI on the equatorial plane on the left and on the right the respective waveform (Harms , 2016).

Why are EMRIs interesting?

- EMRIs scan thoroughly the spacetime background.
- By analyzing the GWs we can check the Kerrness of the primary black hole.
- Any deviation of the GW from the standard paradigm reveals deviation from GR.
- The rate of EMRIs has a direct impact on our population modeling.
- The perturbation modelling of EMRIs produces tools useful for other theoretical fields (Barack & Pound , 2019).

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Distance and mass ranges of two body systems

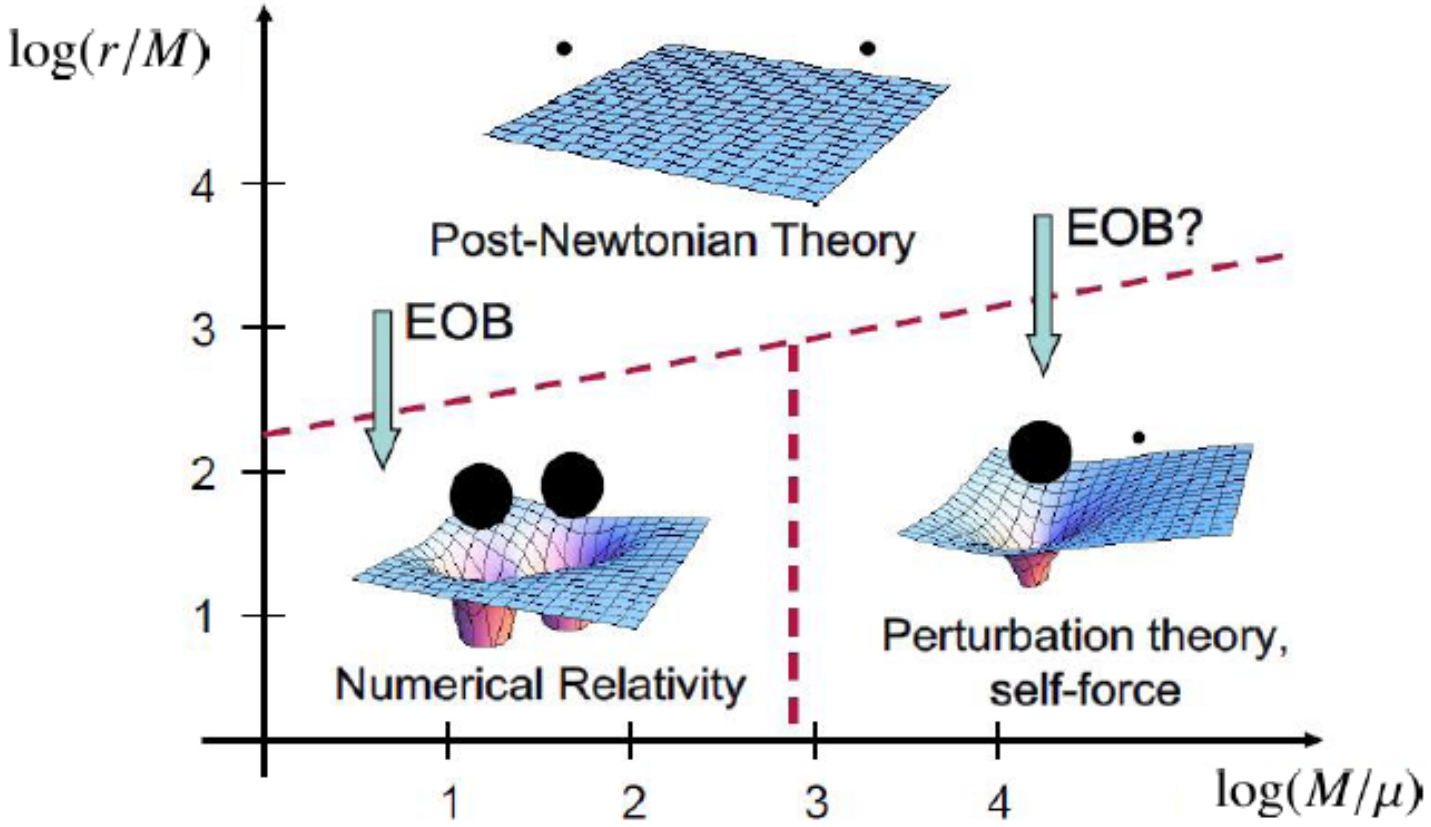

Figure: Leor Barack slide. M is the total mass, μ is the reduced mass, r is the distance between the bodies.

A very simple first approach to an EMRI

Assumptions:

- The inspiraling object is not spinning.
- The spacetime background is Kerr.
- The EMRI system is isolated from its surrounding environment.
- The loss of energy and angular momentum is adiabatic.

The above assumptions lead to

- a model in which the inspiraling object is shifting from geodesic to geodesic on a Kerr background.
- motion that the non-dissipative limit corresponds to is geodesic in Kerr, thus integrable (Carter, 1968).

Two time scale approximation

Basic Idea: The dissipation due to radiation reaction in an EMRI is very *slow* in comparison to the orbital motion of the secondary around the primary.

The **slow time scale** is concerned with the evolution of the constants of motion (*action variables*).

The **fast time scale** is concerned with the orbital phases (*angle variables*).

Action angle variables for EMRIs (Hinderer & Flanagan , 2008).

GW phase splitting

$$\varphi = \varphi_{\mathrm{A}} + \varphi_{\mathrm{res}} + \varphi_{\mathrm{PA1}} + \dots$$

- φ_{A} the adiabatic term contributes a phase $\mathcal{O}(q^{-1})$ to the inspiral.
- φ_{PA1} the post adiabatic term contributes $\mathcal{O}(1)$.
- $\varphi_{\mathrm{res}} \sim \dfrac{\sqrt{\epsilon}}{q}$.

For quasi-circular inspirals the GW phase φ is twice the ϕ orbital phase.

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Teukolsky equation

Basic Idea: Linearly perturb a field ${}_s\psi$ around Kerr black hole.

$$ {}_s\mathcal{O}\, {}_s\psi(x^\mu) = 4\pi\Sigma T \ , $$

where ${}_s\mathcal{O}$ is a second-order partial differential operator and T is a source term (Teukolsky , 1973).

For $s = -2$ we have a gravitational perturbation.

${}_{-2}\psi$ encodes the gravitational radiation emitted to infinity and infers the gravitational fluxes at the horizon of the primary as well.

The Teukolsky equation computed gravitational fluxes are equal to the averaged dissipative part of the first order self-force (adiabatic order).

Teukolsky equation solvers

Time domain

- Input just a time series.
- Slow less accurate (Skoupý & LG , 2021).
- Existing tested (2+1) code called Teukode uses horizon-penetrating hyperbolic coordinates (Harms , 2016) (v1), can handle a spinning secondary (missing Carter constant's flux) .
- Teukode v2 has been polished and improved by Anton Khirnov with the help of Viktor Skoupý.
- Fluxes to future null infinity.

Frequency domain

$$ {}_{-2}\psi = \sum_{l,m}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \, \psi_{lm\omega}(r) {}_{-2}S_{lm}^{a\omega}(z) e^{-i\omega t + im\phi} $$

- Need the characteristic frequencies of the system, i.e. the system has to be integrable.
- Fast accurate (Skoupý & LG , 2021).
- Code developed for generic orbits with a non-spinning secondary (Kerachian et al., 2023) and for a spinning secondary linearized in spin (Skoupý et al. , 2023) (missing Carter constant's flux).

Radial Teukolsky equation

$$\mathcal{D}_{lm\omega}\psi_{lm\omega}(r) = \mathcal{T}_{lm\omega}$$

$$k_{\mathcal{H}} = \omega - ma/(2Mr_+)$$

Asymptotic behavior

$$\psi_{lm\omega}(r) \approx C^{+}_{lm\omega} r^3 e^{i\omega r^*} \qquad r \to \infty\,,$$

$$\psi_{lm\omega}(r) \approx C^{-}_{lm\omega} \Delta e^{-ik_{\mathcal{H}} r^*} \qquad r \to r_+\,.$$

- Discrete spectrum:

$$C^{\pm}_{lm\omega} = \sum_{m,n,k,j} C^{\pm}_{lmnkj}\delta(\omega - \omega_{mnkj}) \quad \text{with} \quad \omega_{mnkj} = m\Omega_\phi + n\Omega_r + k\Omega_z + j\Omega_s\,,$$

$$C^{\pm}_{lmnkj} = \frac{1}{(2\pi)^2\Gamma}\int dw_r dw_z dw_s\; I^{\pm}_{lmnkj}(w_r, w_z, w_s) e^{i\omega\Delta t(w_r,w_z,w_s) - im\Delta\phi(w_r,w_z,w_s) + inw_r + ikw_z + ijw_s}\,,$$

Strain and Fluxes

- Strain

$$h = h_+ - i\ h_\times = -\frac{2}{r} \sum_{l,m,n,k,j} \frac{C^+_{lmnkj}}{\omega^2_{mnkj}} S_{lmnkj}(\theta) e^{-i\omega_{mnkj} t + im\phi} \,,$$

- Fluxes

$$\mathcal{F}^E = \sum_{l,m,n,k,j} \frac{|C^+_{lmnkj}|^2 + \alpha_{lmnkj} |C^-_{lmnkj}|^2}{4\pi\omega^2_{mnkj}} \,,$$

$$\mathcal{F}^{J_z} = m \sum_{l,m,n,k,j} \frac{|C^+_{lmnkj}|^2 + \alpha_{lmnkj} |C^-_{lmnkj}|^2}{4\pi\omega^3_{mnkj}} \,,$$

Convergence of TD and FD and TD accuracy

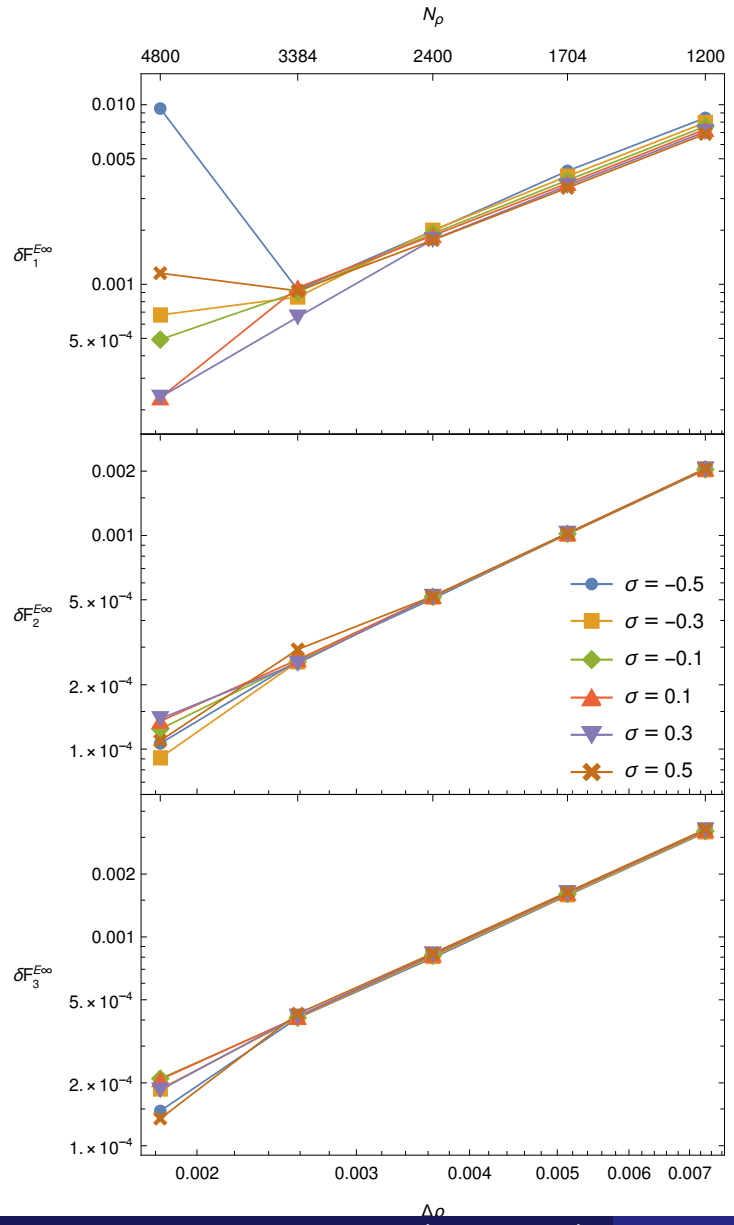

Figure: The relative difference is plotted for different values of the secondary spins σ. The Kerr parameter $\hat{a} = 0.9$, the semi-latus rectum $p = 12$ and the eccentricity 0.2 are kept fixed for all the cases. (Skoupý & LG , 2021)

p	e	$I/^\circ$	m	$\mathcal{F}^E_{S,m}$	$\Delta\mathcal{F}^E_{S,m}$
10	0.1	15	2	-2.8259×10^{-6}	1×10^{-3}
12	0.2	30	1	-1.1954×10^{-7}	2×10^{-5}
12	0.2	30	2	-1.0488×10^{-6}	1×10^{-3}
12	0.2	30	3	-1.4210×10^{-7}	3×10^{-3}
12	0.2	60	2	-8.0550×10^{-7}	5×10^{-4}
15	0.5	15	2	-4.2936×10^{-7}	2×10^{-3}

Table: Relative differences $\Delta\mathcal{F}^E_{S,m}$ of the linear in spin part of the energy flux $\mathcal{F}^E_{S,m}$ between frequency domain and time domain computations for given orbital parameters and azimuthal number m. All orbits have $a = 0.9M$. (Skoupý et al. , 2023)

Grid and equatorial evolution

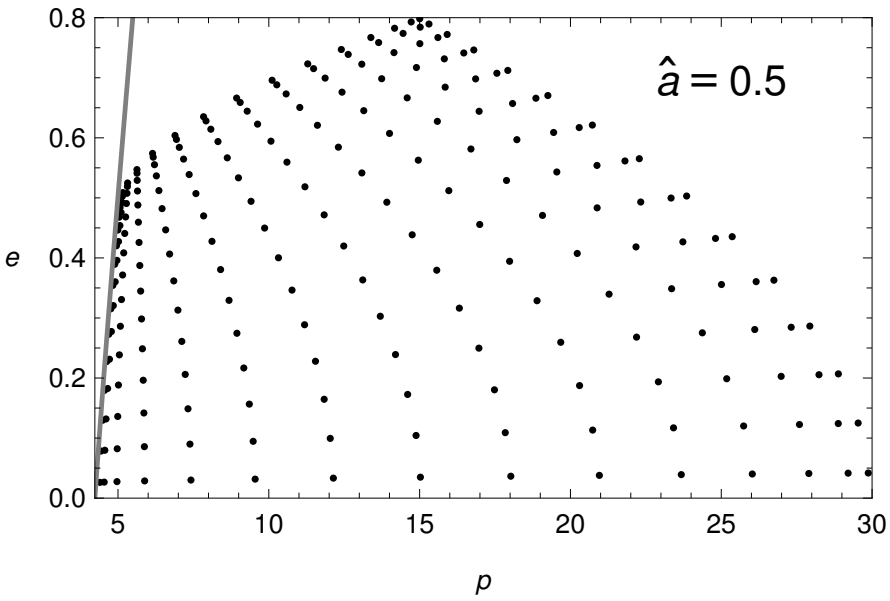

Figure: Grid for the interpolation in $p - e$ plane. The grid points are at Chebyshev nodes.

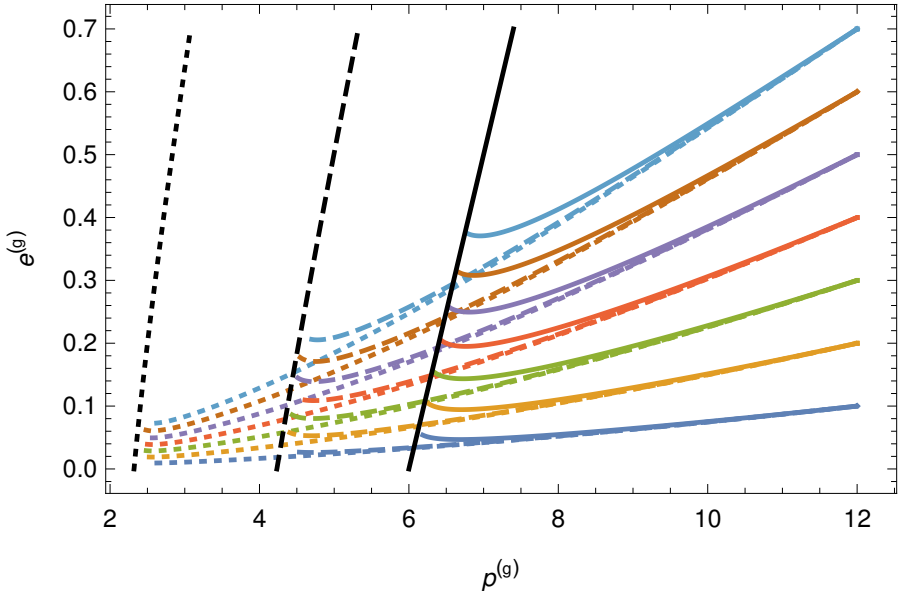

Figure: Adiabatic evolution of $p^{(g)}$ and $e^{(g)}$ for $\hat{a} = 0$ (solid line), $\hat{a} = 0.5$ (dashed line), and $\hat{a} = 0.9$ (dotted line), while the respective black lines denote the separatrices, where the evolution ends.

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Summary

- Fluxes computed by solving Teukolsky equation lead to EMRI modelling.
- TD computations are more flexible, more expensive and less accurate than FD ones.

Thank you for your attention!

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